New perspectives in the equilibrium statistical mechanics approach to social and economic sciences

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Summary. In this work we review some recent development in the mathematical modeling of quantitative sociology by means of statistical mechanics. After a short pedagogical introduction to static and dynamic properties of many body systems, we develop a theory for particle (agents) interactions on random graph.

Our approach is based on describing a social network as a graph whose nodes represent agents and links between two of them stand for a reciprocal interaction. Each agent has to choose among a dichotomic option (i.e. agree or disagree) with respect to a given matter and he is driven by external influences (as media) and peer to peer interactions. These mimic the imitative behavior of the collectivity and may possibly be zero if the two nodes are disconnected.

For this scenario we work out both the dynamics and, given the validity of the detailed balance, the corresponding equilibria (statics). Once the two body theory is completely explored, we analyze, on the same random graph, a *diffusive strategic dynamics* with pairwise interactions, where detailed balance constraint is relaxed. The dynamic encodes some relevant processes which are expected to play a crucial role in the approach to equilibrium in social systems, i.e. diffusion of information and strategic choices. We observe numerically that such a dynamics reaches a well defined steady state that fulfills a *shift* property: the critical interaction strength for the canonical phase transition is lower with respect to the one expected from canonical equilibrium.

Finally, we show how the stationary states of this kind of dynamics can be described by statistical mechanics equilibria of a diluted p-spin model, for a suitable non-integer real $p > 2$. Several implications from a sociological perspective are discussed together with some general outlooks.

1 Introduction

Born as a microscopic foundation of thermodynamics, statistical mechanics provides nowadays a flexible approach to several scientific problems whose depth and wideness increases continuously. In the last decades, in fact, statistical mechanics has invaded fields as diverse as spin glasses [MPV87], neural networks [Ami92], protein folding [Hua07], immunological memory [Par90], and also made some attempt to describe social networks [CG07], theoretical economy [Coo05] and urban planning [CSS00].

In this paper we study statistical mechanics of imitative diluted systems, paying particular attention to its applications in social sciences.

After a review of the statistical mechanics methodology, we introduce the tools, both analytical and numerical, used for the investigation of many-body problems. We apply such a machinery to study at first the global behavior of a large amount of dichotomic agents (i.e. able to answer only 'yes/no" to a given question) whose decision making is driven both by a uniform external influence (as the media) and by pairwise imitative interactions among agent themselves. In general, the agents making up a community are not all contemporarily in contact with each other, namely, the network representing the social structure is not fully-connected but rather randomly diluted, hence mirroring acquaintances or family relationships.

Even though refined models as small-worlds graphs have recently been proposed, a standard one for such a network is provided by the famous Erdös-Renyi graph. The model turns out to be non trivial as, tuning the degree of connectivity and/or the strength of interaction, a cooperative state among the agents appears. Moreover, we show the existence of a region of the parameter space which is more convenient for the global behavior of the society, i.e. it corresponds to a minimum in the free energy.

As social systems do not need to obey Maxwell-Boltzmann distribution, detailed balance is not strictly required for their evolution, so, after having explored the "canonical" 2-body model in full detail, we introduce a more realistic description of its free temporal evolution by adopting a *diffusive strategic dynamics* [ABCV06a, ABCV06b]. This dynamic takes into account two crucial aspects, which are expected to be effective in the temporal evolution of social systems, i.e. diffusion of information and strategic choices. We implement it on the same Erdös-Renyi graph and study its equilibria. Each agent is selected through a diffusive rule, and a flip in its dichotomic status is not weighted "a la Glauber" [Ami92] but rather according to a strategic rule which produces the maximum energy gain. We stress that this operation involves more-than-two-body effective interactions, as the chosen agent interacts both with the first selected one as well as with its nearest neighbors, as a whole. This dynamics is shown to relax to a well defined steady state, where all the properties of stationarity are recovered [EM90], however the strength of the interactions at the critical line is lower, of a few percent, than the expected. The whole scenario suggests a "latent" many-body coupling influence, encoded into the particular rule for selecting the agents. This is also corroborated by further numerical analysis. As a consequence, we work out analytically a theory for the randomly diluted p-spin model so to fit an effective $p \in \mathbb{R}$, which turns out to be $p = 2.15$, in order to match the numerical data available by the dynamics. This result has implication both in market trends, as well as in quantitative sociology, where the effective interactions always play an important role in decision making [Dur99, McF01].

The paper is organized as follows: In section 2, for the sake of completeness, stochastic dynamics for discrete many body problems is outlined, section 3 deals with definitions and introduction to their equilibrium via statistical mechanics. In section 4, instead, the model we study is solved in full details with the aim of presenting both a scenario for these decision makers on random graphs as well as a general mathematical method which can be extended by the reader to other models. In section 5 our alternative dynamics is introduced and shortly discussed; then further numerical investigations toward a better understanding of a $p > 2$ behavior are presented.

In section 6 the randomly diluted p-spin model is defined and exploited in all details, both analytically (within the cavity field framework) as well as numerically (within a Monte Carlo approach). Full agreement is found among the two methods. At the end, the last section is left for conclusions: the effective interaction is found and its implications analyzed.

Furthermore, even though the paper is written within a theoretical physics approach, remarks concerning the application to quantitative sociology are scattered throughout the work. In particular, in the conclusion, a toy application of the outlined theory to trades in markets is shown.

2 A brief introduction to many-body dynamics

In this section we introduce the fundamental principles of stochastic dynamics used to simulate the relaxation to equilibrium of the systems we are interested in. Even though for discrete systems two kinds of dynamics are available, parallel and sequential, we are going to deepen only the latter as is the one we will implement thought the chapter. Although the topic is well known (see e.g. [LL80, Lig99]), for the sake of completeness and in order to offer to the reader a practical approach to these models, we present the underlying theory.

2.1 The model

Let us consider an ensemble of N agents labelled as $i = 1, .., N$. Each agent has two possible choices, with respect to a given situation, which are encoded into a variable $\sigma_i = \pm 1$, say $\sigma_i = +1$ is "agreement" and viceversa for -1. Each agent experiences an external influence (for example by media) which is taken into account by the one-body coupling $H_1(\sigma; \theta)$

$$H_1(\sigma; \theta) = -\sum_{i=1}^{N} \sigma_i(t)\theta_i(t), \tag{1}$$

where $\theta_i(t)$ is the stimulus acting on the ith agent at a given time t and $\sigma_i(t)$ is the opinion of the ith agent at the same time.

The interactions among the other agents are encoded into the $H_0(\sigma; \mathbf{J})$ term as follows

$$H_0(\sigma; \mathbf{J}) = -\sum_{i}^{N} \sum_{j>i}^{N} J_{ij}(\alpha)\sigma_i(t)\sigma_j(t). \tag{2}$$

For the moment there is no need to introduce explicitly the dilution of the underlying random network as the scheme applies in full generality and may be a benchmark for future development by the reader himself. We only stress that $J_{ij}(\alpha)$ is quenched, i.e. does not evolve with time, and can be thought of as a symmetric adjacency matrix in such a way that a zero entry $J_{ij} = 0$ means that the agents i and j are not in contact with each other, viceversa for $J_{ij} = 1$ there is a link between them. The ratio of connections $\sum_{i,j}^{N} J_{ij}/N^2$ is tuned by a parameter α, such that for $\alpha \to \infty$ the graph recovers the *fully connected* one, while for $\alpha = 0$ the graph is completely disconnected. Overall, the Hamiltonian defining the model is the sum of the two contributes, namely (with a little abuse of notation, thinking at J as $(\mathbf{J}, \theta)$) $H(\sigma; J) = H_0(\sigma; \mathbf{J}) + H_1(\sigma; \theta)$. With the signs as they are here, a positive value of J_{ij} makes the relevant spins want to line up together, i.e. to share the same opinion, and each spin also wants to be aligned with the corresponding external field.

The investigation of the properties displayed by systems described by this kind of Hamiltonian are both analytical and numerical. The former relies on series expansions, field theoretical methods, cavity and replica approaches. The latter are mainly based on Monte Carlo simulations where we directly simulate the temporal evolution of the system in such a way that an expectation value is calculated as a time average over the states that the system passes through. However, it must be underlined that the Hamiltonian contains no dynamical information, hence we have to *choose* a dynamic for our simulation, namely a rule for changing from one state to another during the simulation, which results in each state appearing with exactly the probability appropriate to it. Several possibility have been introduced in the past, ranging from deterministic, e.g. Q2R dynamics, to stochastic, e.g. Glauber algorithm and Wolff algorithm.

2.2 Transition rates and Markov process

In statistical mechanics, Maxwell-Boltzmann statistics (hereafter simply "Boltzmann statistics") describes the statistical distribution of material particles over various energy states in thermal equilibrium, when the temperature is high enough and density is low enough to render quantum effects negligible. Now, given the generic configuration $\sigma = \{\sigma_i\}_{i=1,\dots,N}$ according to Boltzmann statistics, the value of a thermodynamic observable $X(\beta; \mathbf{J})$ is given by

$$X(\beta; \mathbf{J}) = \langle X(\sigma; \mathbf{J}) \rangle_\beta = \frac{\sum_{\{\sigma_i\}} X(\sigma; \mathbf{J}) e^{-\beta H(\sigma; \mathbf{J})}}{\sum_{\{\sigma_i\}} e^{-\beta H(\sigma; \mathbf{J})}}, \tag{3}$$

where β represents the inverse of the temperature (that sometimes we call "noise"), i.e. $\beta \equiv (k_B T)^{-1}$, being k_B the Boltzmann constant, and the brackets are implicitly defined by the r.h.s. of eq.(3).

Monte Carlo techniques [NB01] work by choosing a subset of states $\tilde{S}$ at random from some probability distribution p_σ which we specify. Our best estimate of the quantity $X(\sigma; \mathbf{J})$ is then given by the so-called *estimator* $\langle X(\sigma; \mathbf{J}) \rangle_{\beta, \tilde{S}}$:

$$\langle X(\sigma; \mathbf{J}) \rangle_{\beta, \tilde{S}} = \frac{\sum_{\{\sigma \in \tilde{S}\}} p_\sigma^{-1} X(\sigma; \mathbf{J}) e^{-\beta H(\sigma; \mathbf{J})}}{\sum_{\{\sigma \in \tilde{S}\}} p_\sigma^{-1} e^{-\beta H(\sigma; \mathbf{J})}}. \tag{4}$$

The estimator has the property that, as the number of sampled states $|\tilde{S}|$ increases, it becomes a more and more accurate estimate of $X(\beta; \mathbf{J})$, and, as $|\tilde{S}| \to \infty$ we have $\langle X(\sigma; \mathbf{J}) \rangle_{\beta, \tilde{S}} = \langle X(\sigma; \mathbf{J}) \rangle_{\beta}$.

The choice made for p_σ is based on the following argument: when in equilibrium, the system is not sampling all states in S with equal probability, but according to the Boltzmann probability distribution. Hence, the strategy is this: instead of selecting the subspace $\tilde{S}$ in such a way that every state of the system is as likely to gets chosen as every other, we select them so that the probability that a particular state **s** is chosen is $p_\sigma = p_\sigma^{(eq)} = Z^{-1} e^{-\beta H(\sigma)}$, Z being a proper normalization factor by now.

The estimator then simplifies into a simple arithmetic average

$$\langle X(\sigma; \mathbf{J}) \rangle_{\beta, \tilde{S}} = \frac{1}{|\tilde{S}|} \sum_{\sigma \in \tilde{S}} X(\sigma; \mathbf{J}). \tag{5}$$

This definition for $\langle X(\sigma; \mathbf{J}) \rangle_{\beta, \tilde{S}}$ works much better than the one we would obtain from a uniform distribution for p_σ, especially when the system is spending the majority of its time in a small number of states. Indeed, the latter will be precisely the states sampled most often, and the relative frequency with which we select them will correspond to the amount of time the real system would spend in those states.

Therefore, we need to generate an appropriate random set of states, according to the Boltzmann weight $p_\sigma^{(eq)}$. In general, Monte Carlo schemes rely on *Markov processes* as the generating engine for the set of states to be used. Let us introduce a (normalized) transition probability $W[\sigma, \sigma']$ for any pair σ, σ' of configurations in the phase space S. Such a set of transition probabilities, together with the specification of an initial configuration, allows to construct a Markov chain of configurations, $\tilde{S}_\tau = (\sigma_1, \sigma_2, ..., \sigma_\tau)$. The Markov process is chosen in such a way that, when it is run for long enough, starting from any state of the system, it will eventually produce a succession of states which appear according to the canonical distribution. In order to achieve this, two conditions are sufficient: the condition of *ergodicity* and of *detailed balance*.

The former is the requirement that it must be possible, for the Markov process, to reach any state of the system from any other state, if it is run for long enough. Otherwise stated, $\forall \sigma, \sigma', \exists t : W^t[\sigma, \sigma']$ is non null, where $W^t[\sigma, \sigma']$ just represents the probability of reaching σ' from σ in t steps. The ergodic condition is also consistent with the fact that, in the Boltzmann distribution, every state σ appears with non-zero probability. On the other hand, notice that this condition does not require that $W(\sigma, \sigma') \neq 0, \forall \sigma, \sigma'$.

Conversely, the detailed balance condition ensures that, in the limit $\tau \to \infty$, a given configuration σ' appears in the Markov chain $\tilde{S}_\tau$ just according to the probability distribution $p_{\sigma'}^{(eq)}$. The detailed balance condition requires that the system is in equilibrium (the rate of transitions into and out of any state must be equal) and that no limit cycles are present. As a result, the detailed balance condition can be stated as

$$p_\sigma^{(eq)} W[\sigma, \sigma'] = p_{\sigma'}^{(eq)} W[\sigma', \sigma], \tag{6}$$

where the l.h.s. represents the overall rate at which transitions from σ to σ' occur in the system, while the r.h.s. is the overall rate for the reverse transition. This condition makes the system exhibit time-reversal symmetry at each move and it provides a sufficient (but not necessary) condition ensuring that the application of

these transition probabilities leads the system to an equilibrium distribution irrespective of the initial state. Now, since we wish the equilibrium distribution to be the Boltzmann one, we choose $p_\sigma^{(eq)} = Z^{-1} e^{-\beta H(\sigma; \mathbf{J})}$, obtaining

$$W[\sigma, \sigma'] = W[\sigma', \sigma] e^{-\beta[H(\sigma'; \mathbf{J}) - H(\sigma; \mathbf{J})]}. \tag{7}$$

The constraints introduced so far still leave a good deal of freedom over the definition of the transition probabilities. Indeed, the choice of a proper transition probability to apply to the system under study is crucial. In fact there is no kinetic information in the Hamiltonian given by eq. (1) and by eq. (2), as it only contains information about spin orientation and the spatial distribution of lattice sites. It is the transition probability which provides a dynamics, i.e. a rule according to which the system evolves.

In the following, we will be especially interested in the so called *single-spin-flip* dynamics, which means that the states involved in the transition only differ for the value of a single spin variable. More precisely, in this kind of dynamics, given the configuration σ, at each time step a single agent $i \in [1, ..., N]$, is randomly chosen among the N and updated with probability $W[\sigma, F_i\sigma]$ to give rise to the the configuration $F_i\sigma$, where the N spin-flip operators F_i is defined as $F_i\sigma \equiv F_i\{\sigma_1, ..., \sigma_i, ..., \sigma_N\} = \{\sigma_1, ..., -\sigma_i, ..., \sigma_N\}$.

The *Metropolis* and the *Glauber* dynamics [NB01] are examples of stochastic single-spin-flip dynamics. In particular, for the latter one has for the transition rates

$$W[\sigma; F_i\sigma] = \left(1 + \exp(\beta \Delta_i H(\sigma; \mathbf{J}))\right)^{-1}, \tag{8}$$

where $\Delta_i H(\sigma; \mathbf{J}) = H(F_i\sigma; \mathbf{J}) - H(\sigma; \mathbf{J})$.

3 Equilibrium behavior

In synthesis, thermodynamics describes all the macroscopic features of the system and statistical mechanics allows to obtain such a macroscopic description starting from its microscopic foundation, that is, obtaining the global society behavior by studying the single agent based dynamics, and then, using Probability Theory, for averaging over the ensemble with the weight encoded by the equilibrium distribution $p_\sigma^{(eq)}$. This scenario is fully derivable when both the internal energy density of the system $e(\beta, \alpha)$ and the entropy density $s(\beta, \alpha)$ are explicitly obtained (we are going to introduce such quantities hereafter). Then, the two prescription of minimizing the energy $e(\beta, \alpha)$ (minimum energy principle) and maximizing entropy $s(\beta, \alpha)$ (second law of thermodynamics) give the full macroscopic behavior of the system, expressed via suitably averages of its microscopic element dynamics. To fulfil this task the free energy $f(\beta, \alpha) = e(\beta, \alpha) - \beta^{-1} s(\beta, \alpha)$ turns out to be useful because, as it is straightforward to check, minimizing this quantity corresponds to both maximizing entropy and minimizing energy (at the given temperature), furthermore, and this is the key bridge, there is a deep relation among it and the equilibrium measure $p_\sigma^{(eq)}$, in fact

$$p_\sigma^{(eq)} \propto \exp(-\beta H(\sigma; \mathbf{J})),$$

$$f(\beta, \alpha) = \lim_{N\to\infty} f_N(\beta, \alpha) = \lim_{N\to\infty} \frac{-1}{\beta N} \log \sum_\sigma \exp(-\beta H_N(\sigma; \mathbf{J})), \tag{9}$$

$$e(\beta, \alpha) = \lim_{N\to\infty} e_N(\beta, \alpha) = \lim_{N\to\infty} -\partial_\beta(\beta f_N(\beta, \alpha)), \tag{10}$$

$$s(\beta, \alpha) = \lim_{N\to\infty} s_N(\beta, \alpha) = \lim_{N\to\infty} \left(f_N(\beta, \alpha) + \beta^{-1} \partial_\beta(\beta f_N(\beta, \alpha)) \right). \tag{11}$$

So, once explicitly obtained the free energy, equilibrium behavior is solved and the whole works once the equilibrium probability distribution is known, provided we use the Hamiltonian $H(\sigma; J)$.

Before proceeding in this derivation, we need some preliminary definitions:

At first, in the following, it will be convenient to deal with the pressure $A(\beta, \alpha)$, defined as

$$A(\beta, \alpha) = \lim_{N\to\infty} A_N(\beta, \alpha) = -\beta \lim_{N\to\infty} f_N(\beta, \alpha); \tag{12}$$

we stress that often we are going to consider results in the "thermodynamic limit" $N \to \infty$: such procedure allows us to use implicitly several theorem of convergence of random variables from Probability Theory and, when the number of agents is large enough, the agreement among results at finite N and results in the thermodynamic limit is excellent, as usually the two differ by factors $O(N^{-1})$ or at worse $O(N^{-1/2})$.

Let us now further introduce the partition function defined as

$$Z_N(\beta, \alpha) = \sum_{\sigma_N} e^{-\beta H_N(\sigma, \mathbf{J})} = \sum_\sigma p_\sigma^{(eq)}. \tag{13}$$

As we do not want a sample-dependent theory, using $\mathbb{E}$ for the average over the quenched variables (i.e. the connectivity), the quenched pressure can be written as

$$A_N(\beta, \alpha) = \frac{1}{N} \mathbf{E} \ln Z_N(\beta, \alpha),$$

the Boltzmann state is given by

$$\omega(g(\sigma, \mathbf{J})) = \frac{1}{Z_N(\beta, \alpha)} \sum_{\sigma_N} g(\sigma; \mathbf{J}) e^{-\beta H_N(\sigma; \mathbf{J})}, \tag{14}$$

with its replicated form on s replicas defined as

$$\Omega(g(\sigma; \mathbf{J})) = \prod_s \omega^{(s)}(g(\sigma^{(s)}; \mathbf{J})) \tag{15}$$

and the total average $\langle g \rangle$ as

$$\langle g \rangle = \mathbf{E}[\Omega(g(\sigma; \mathbf{J}))]. \tag{16}$$

Let us introduce further, as order parameters of the theory, the multi-overlaps

$$q_{1...n} = \frac{1}{N} \sum_{i=1}^N \sigma_i^{(1)} ... \sigma_i^{(n)}, \tag{17}$$

with a particular attention to the magnetization $m = q_1 = (1/N) \sum_{i=1}^N \sigma_i$ and to the two replica overlap $q_{12} = (1/N) \sum_{i=1}^N \sigma_i^1 \sigma_i^2$.

It is important to stress that the magnetization, which plays the role of the principal order parameter (able to recognize the different macroscopic phases displayed by the system), accounts for the averaged opinion into the social network, such that if $\langle m \rangle = 0$ there is no net preference in global decision, while for $\langle m \rangle \to 1$ there is a sharp preference toward the "yes" state and viceversa for $\langle m \rangle \to -1$. Analogously $\langle q \rangle$ accounts for similarity among two different "replicas" of the system (two independent realization of the adjacency matrix).

It is easy to check that when $\beta \to 0$ (or the interaction strength, that is always coupled with the noise-), details of the Hamiltonian are unfelt by the agents, which will be on average one half up (yes) and one half down (no), giving a null net contribution to $\langle m \rangle$.

At the contrary when the system is able to experience the rules encoded into the Hamiltonian it is easy to see that

$$-\frac{\partial f_N(\beta, a)}{\partial \theta_i} = \langle \sigma_i \rangle \neq 0, \quad -\frac{\partial f_N(\beta, a)}{\partial J_{ij}} = \langle \sigma_i \sigma_j \rangle \neq 0. \tag{18}$$

Averaging over the whole space of choices, we get the macroscopic response of the system in terms of the magnetization $\langle m \rangle$.

In the thermodynamic limit, further, self-averaging for this order parameter is expected to hold, which is expressed via

$$\lim_{N \to \infty} \langle m_N^2 \rangle = \lim_{N \to \infty} \langle m_N \rangle^2,$$

that means that the mean-value of the order parameter is not affected by the details of the microscopic structure in the $N \to \infty$ limit (it is an expression of the Central Limit Theorem in this framework).

From a purely thermodynamical viewpoint the equilibrium behavior (the phase diagram) is fundamental because it gives both the phase diagram and the critical scenario, so to say, the regions in the space of the tunable parameters β, α where the model displays a paramagnetic (independent agent viewpoint) or ferromagnetic (collective agent viewpoint) behavior, by which global decision on the whole society can not leave aside.

To obtain a clear picture of the equilibrium of the social network, we use standard techniques of statistical mechanics for positive valued interactions, namely the smooth cavity field technique [Bar06].

For simplicity, as conceptually this does not change the picture, we deal with the simpler case $\theta_i = \theta \ \forall i \in [1, N]$.

4 Equilibrium statistical mechanics of the "2-body" model

The "2-body" model has a long history in physics, having particular importance in interaction theories. In fact, from one side, (apart historical problems dealing with the deterministic dynamical evolution of the 3-body problem), for a long time the interaction in physics were thought of as particle scattering processes and in these events the probability of a more-than-2 bodies instantaneous interaction were effectively negligible. From the other side, the structure of the 2-body energy is

quadratic in its variables and this encodes several information: firstly, from a probabilistic viewpoint, the Maxwell-Boltzmann probability distribution assumes the form of a Gaussian, then, the forces (the derivatives of the energy with respect to its variable) are linear, such that superposition principle and linear response theory do hold and TLC is respected. However, as we will see later, neither of these properties of physical systems need to be strictly obeyed in social theories and the interest to $p > 2$ body interactions will arise.

In this section we consider diluted systems and systematically develop the interpolating cavity field method [Bar06] and use it to sketch the derivation of a free energy expansion: the higher the order of the expansion, the deeper we could go beyond the ergodic (agent independent) region. Within this framework we perform a detailed analysis of the scaling of magnetization (and susceptibility) at the critical line. The critical exponents turn out to be the same expected for a fully-connected system. Then, we perform extensive Monte Carlo (MC) simulations for different graph sizes and bond concentrations and we compare results with theory. Indeed, also numerically, we provide evidence that the universality class of this diluted Ising model is independent of dilution. In fact the critical exponents we measured are consistent with those pertaining to the Curie-Weiss model, in agreement with analytical results. The critical line is also well reproduced.

The section is organized as follows: after a detailed and technical introduction of the model, in Section (4.1) we introduce the cavity field technique, which constitutes the framework we are going to use in Section (4.2) to investigate the free energy of the system at general values of temperature and dilution. Section (4.3) deals with the criticality of the model; there we find the critical line and the critical behavior of the main order parameter, i.e. magnetization. Section (4.4) is devoted to numerical investigations, especially focused on criticality.

4.1 Interpolating with the cavity field

In this section, after a refined introduction of the model, we introduce further the cavity field technique on the line of [Bar06].

Given N points and families $\{i_\nu, j_\nu\}$ of i.i.d random variables uniformly distributed on these points, the (random) Hamiltonian of the diluted Curie-Weiss model is defined on Ising N-spin configurations $\sigma = (\sigma_1, \ldots, \sigma_N)$ through

$$H_N(\sigma, \alpha) = -\sum_{\nu=1}^{P_{\alpha N}} \sigma_{i_\nu} \sigma_{j_\nu} \ , \tag{19}$$

where P_ζ is a Poisson random variable with mean ζ and $\alpha > 1/2$ is the connectivity. The expectation with respect to all the *quenched* random variables defined so far will be denoted by $\mathbb{E}$, and is given by the composition of the Poissonian average with the uniform one performed over the families $\{i_\nu\}$

$$\mathbf{E}[\cdot] = E_P E_i[\cdot] = \sum_{k=0}^{\infty} \frac{e^{-\alpha N}(\alpha N)^k}{k! N^p} \sum_{i_\nu^1, \ldots, i_\nu^p}^{1,N} [\cdot]. \tag{20}$$

As they will be useful in our derivation, it is worth stressing the following properties of the Poisson distribution: Let us consider a function $g : \mathbb{N} \rightarrow \mathbb{R}$, and a Poisson

variable k with mean αN, whose expectation is denoted by $\mathbb{E}$.
It is easy to verify that

$$\mathbb{E}[kg(k)] = \alpha N \mathbb{E}[g(k-1)] \tag{21}$$

$$\partial_{\alpha N} \mathbb{E}[g(k)] = \mathbb{E}[g(k+1) - g(k)] \tag{22}$$

$$\partial^2_{(\alpha N)^2} \mathbb{E}[g(k)] = \mathbb{E}[g(k+2) - 2g(k+1) + g(k)]. \tag{23}$$

Turning to the cavity method, it works by expressing the Hamiltonian of a system made of $N+1$ spins through the Hamiltonian of N spins by scaling the connectivity degree α and neglecting vanishing terms in N as follows

$$H_{N+1}(\alpha) = - \sum_{\nu=1}^{P_{\alpha(N+1)}} \sigma_{i_\nu} \sigma_{j_\nu} \quad \sim \quad - \sum_{\nu=1}^{P_{\tilde{\alpha}N}} \sigma_{i_\nu} \sigma_{j_\nu} - \sum_{\nu=1}^{P_{2\tilde{\alpha}}} \sigma_{i_\nu} \sigma_{N+1} \tag{24}$$

such that we can use the more compact expression

$$H_{N+1}(\alpha) \sim H_N(\tilde{\alpha}) + \hat{H}_N(\tilde{\alpha}) \sigma_{N+1} \tag{25}$$

with

$$\tilde{\alpha} = \frac{N}{N+1}\alpha \overset{N\to\infty}{\longrightarrow} \alpha, \qquad \hat{H}_N(\tilde{\alpha}) = - \sum_{\nu=1}^{P_{2\tilde{\alpha}}} \sigma_{i_\nu}. \tag{26}$$

So we see that we can express the Hamiltonian of $N+1$ particles via the one of N

particles, paying two prices: the first is a rescaling in the connectivity (vanishing in the thermodynamic limit), the second being an added term, which will be encoded, at the level of the thermodynamics, by a suitably cavity function as follows: let us introduce an interpolating parameter $t \in [0, 1]$ and the cavity function $\Psi_N(\tilde{\alpha}, t)$ given by

$$\Psi(\tilde{\alpha}, \beta; t) = \lim_{N\to\infty} \Psi_N(\tilde{\alpha}, \beta; t) \tag{27}$$

$$= \lim_{N\to\infty} \mathbb{E}\Big[\ln \frac{\sum_{\{\sigma\}} e^{\beta \sum_{\nu=1}^{P_{\tilde{\alpha}N}} \sigma_{i_\nu} \sigma_{j_\nu} + \beta \sum_{\nu=1}^{P_{2\tilde{\alpha}t}} \sigma_{i_\nu}}}{\sum_\sigma e^{\beta \sum_{\nu=1}^{P_{\tilde{\alpha}N}} \sigma_{i_\nu} \sigma_{j_\nu}}} \Big] = \lim_{N\to\infty} \mathbb{E}\Big[\ln \frac{Z_{N,t}(\tilde{\alpha}, \beta)}{Z_N(\tilde{\alpha}, \beta)} \Big].$$

The three terms appearing in the decomposition (25) give rise to the structure of the following theorem which we prove by assuming the existence of the thermodynamic limit. (Actually we still do not have a rigorous proof of the existence of the thermodynamic limit but we will provide strong numerical evidences in Section 4.4).

Theorem 1. *In the $N \to \infty$ limit, the free energy per spin is allowed to assume the following representation*

$$A(\alpha, \beta) = \ln 2 - \alpha \frac{\partial A(\alpha, \beta)}{\partial \alpha} + \Psi(\alpha, \beta; t=1). \tag{28}$$

Proof
Consider the $N+1$ spin partition function $Z_{N+1}(\alpha, \beta)$ and let us split it as suggested by eq. (25)

$$Z_{N+1}\left(\alpha,\beta\right) = \sum_{\sigma_{N+1}} e^{-\beta H_{N+1}(\alpha)} \sim \sum_{\sigma_{N+1}} e^{-\beta H_N(\tilde{\alpha}) - \beta \hat{H}_N(\tilde{\alpha})\sigma_{N+1}} \qquad (29)$$

$$= \sum_{\sigma_{N+1}} e^{\beta \sum_{\nu=1}^{P_{\tilde{\alpha}N}} \sigma_{i\nu}\sigma_{j\nu} + \beta \sum_{\nu=1}^{P_{2\tilde{\alpha}}} \sigma_{i\nu}\sigma_{N+1}} = 2\sum_{\sigma_N} e^{\beta \sum_{\nu=1}^{P_{\tilde{\alpha}N}} \sigma_{i\nu}\sigma_{j\nu} + \beta \sum_{\nu=1}^{P_{2\tilde{\alpha}}} \sigma_{i\nu}}$$

where the factor two appears because of the sum over the hidden σ_{N+1} variable. Defining a perturbed Boltzmann state $\tilde{\omega}$ (and its replica product $\tilde{\Omega} = \tilde{\omega} \times ... \times \tilde{\omega}$) as

$$\tilde{\omega}(g(\sigma)) = \frac{\sum_{\{\sigma_N\}} g(\sigma) e^{-\beta H_N(\tilde{\alpha})}}{\sum_{\{\sigma_N\}} e^{-\beta H_N(\tilde{\alpha})}}, \qquad \tilde{\Omega}(g(\sigma)) = \prod_i \tilde{\omega}^{(i)}(g(\sigma^{(i)}))$$

where the tilde takes into account the shift in the connectivity $\alpha \to \tilde{\alpha}$ and multiplying and dividing the r.h.s. of eq.(29) by $Z_N(\tilde{\alpha},\beta)$, we obtain

$$Z_{N+1}(\alpha,\beta) = 2Z_N(\tilde{\alpha},\beta)\tilde{\omega}(e^{\beta \sum_{\nu=1}^{P_{2\tilde{\alpha}}}}). \qquad (30)$$

Taking now the logarithm of both sides of eq.(30), applying the average $\mathbb{E}$ and subtracting the quantity $[\ln Z_{N+1}(\tilde{\alpha},\beta)]$, we get

$$\mathbb{E}[\ln Z_{N+1}(\alpha,\beta)] - \mathbb{E}[\ln Z_{N+1}(\tilde{\alpha},\beta)] = \ln 2 + \mathbb{E}\left[\ln \frac{Z_N(\tilde{\alpha},\beta)}{Z_{N+1}(\tilde{\alpha},\beta)}\right] + \Psi_N(\tilde{\alpha},\beta; t=1) \tag{31}$$

in the large N limit the l.h.s. of eq.(31) becomes

$$\mathbb{E}[\ln Z_{N+1}(\alpha,\beta)] - \mathbb{E}[\ln Z_{N+1}(\tilde{\alpha},\beta)] = \tag{32}$$
$$(\alpha - \tilde{\alpha})\frac{\partial}{\partial \alpha}\mathbb{E}[\ln Z_{N+1}(\alpha,\beta)] = \alpha\frac{1}{N+1}\frac{\partial}{\partial \alpha}[\ln Z_{N+1}(\alpha,\beta)] = \alpha\frac{\partial A_{N+1}(\alpha,\beta)}{\partial \alpha}$$

then by considering the thermodynamic limit the thesis follows. $\square$

Hence, we can express the free energy via an energy-like term and the cavity function. While it is well known how to deal with the energy-like [ABC08, Gue95, GT04], the same can not be stated for the cavity function, and we want to develop its expansion via suitably chosen overlap monomials in a spirit close to the stochastic stability [AC98, CG05, Par00], such that, at the end, we will not have the analytical solution for the free energy in the whole (α,β) plane, but we will manage its expansion above and immediately below the critical line. To see how the machinery works, let us start by giving some definitions and proving some simple theorems:

Definition 1. *We define the t-dependent Boltzmann state $\tilde{\omega}_t$ as*

$$\tilde{\omega}_t(g(\sigma)) = \frac{1}{Z_{N,t}(\alpha,\beta)} \sum_{\{\sigma\}} g(\sigma) e^{\beta \sum_{\nu=1}^{P_{\tilde{\alpha}N}} \sigma_{i\nu}\sigma_{j\nu} + \beta \sum_{\nu=1}^{P_{2\tilde{\alpha}t}} \sigma_{i\nu}} \tag{33}$$

where $Z_{N,t}(\alpha,\beta)$ extends the classical partition function in the same spirit of the numerator of eq.(33).

As we will often deal with several overlap monomials let us divide them among two big categories:

Definition 2. *We can split the class of monomials of the order parameters in two families:*

- *We define "filled" or equivalently "stochastically stable" all the overlap monomials built by an even number of the same replicas (i.e. q_{12}^2, m^2, $q_{12}q_{34}q_{1234}$).*
- *We define "fillable" or equivalently "saturable" all the overlap monomials which are not stochastically stable (i.e. q_{12}, m, $q_{12}q_{34}$)*

We are going to show three theorems that will play a guiding role for our expansion: as this approach has been deeply developed in similar contexts (as fully connected Ising model [Bar08a] or fully connected spin glasses [Bar06] or diluted spin glasses [GT04], which are the *boundary models* of this subject) we will not show all the details of the proofs, but we sketch them as they are really intuitive. The interested reader can deepen this point by looking at the original works.

Theorem 2. *For large N, setting $t = 1$ we have*

$$\tilde{\omega}_{N,t}(\sigma_{i_1}\sigma_{i_2}...\sigma_{i_n}) = \tilde{\omega}_{N+1}(\sigma_{i_1}\sigma_{i_2}...\sigma_{i_n}\sigma_{N+1}^n) + O\left(\frac{1}{N}\right) \tag{34}$$

such that in the thermodynamic limit, if $t = 1$, the Boltzmann average of a fillable multi-overlap monomial turns out to be the Boltzmann average of the corresponding filled multi-overlap monomial.

Theorem 3. *Let Q_{2n} be a fillable monomial of the overlaps (this means that there exists a multi-overlap q_{2n} such that $q_{2n}Q_{2n}$ is filled). We have*

$$\lim_{N\to\infty}\lim_{t\to1}\langle Q_{2n}\rangle_t = \langle q_{2n}Q_{2n}\rangle \tag{35}$$

(examples: for $N \to \infty$ we get $\langle m_1\rangle_t \to \langle m_1^2\rangle$, $\langle q_{12}\rangle_t \to \langle q_{12}^2\rangle$, $\langle q_{12}q_{34}\rangle_t \to \langle q_{12}q_{34}q_{1234}\rangle$)

Theorem 4. *In the $N \to \infty$ limit the averages $\langle\cdot\rangle$ of the filled monomials are t-independent in β average.*

For the proofs of these theorems we refer to [ABC08].

It is now immediate to understand that the effect of Theorem (2) on a fillable overlap monomial is to multiply it by its missing part to be filled (Theorem (3)), while it has no effect if the overlap monomial is already filled (Theorem (4)) because of the Ising spins (i.e. $\sigma_{N+1}^{2n} \equiv 1 \ \forall n \in \mathbb{N}$).

Now the plan is as follows: We calculate the t-streaming of the Ψ function in order to derive it and then integrate it back once we have been able to express it as an expansion in power series of t with stochastically stable overlaps as coefficients. At the end we free the perturbed Boltzmann measure by setting $t = 1$ and in the thermodynamic limit we will have the expansion holding with the correct statistical mechanics weight.

$$\frac{\partial\Psi(\tilde{\alpha},\beta,t)}{\partial t} = \frac{\partial}{\partial t}\mathbb{E}[\ln\tilde{\omega}(e^{\beta\sum_{\nu=1}^{P_{2\tilde{\alpha}t}}\sigma_{i_\nu}})] \tag{36}$$

$$= 2\tilde{\alpha}\mathbb{E}[\ln\tilde{\omega}\ (e^{\beta\sum_{\nu=1}^{P_{2\tilde{\alpha}t}}\sigma_{i_\nu}+\beta\sigma_{i_0}})] - 2\tilde{\alpha}\mathbb{E}[\ln\tilde{\omega}(e^{\beta\sum_{\nu=1}^{P_{2\tilde{\alpha}t}}\sigma_{i_\nu}})] = 2\tilde{\alpha}\mathbb{E}[\ln\tilde{\omega}_t(e^{\beta\sigma_{i_0}})]$$

and by the equality $e^{\beta\sigma_{i_0}} = \cosh\beta + \sigma_{i_0}\sinh\beta$, we can write the r.h.s. of eq.(36) as

$$\frac{\partial\Psi(\tilde{\alpha},\beta,t)}{\partial t} = 2\tilde{\alpha}\mathbb{E}[\ln\tilde{\omega}_t(\cosh\beta + \sigma_{i_0}\sinh\beta)] = 2\tilde{\alpha}\log\cosh\beta - 2\tilde{\alpha}\mathbb{E}[\ln(1 + \tilde{\omega}_t(\sigma_{i_0})\theta)].$$

We can expand the function $\log(1 + \tilde{\omega}_t\theta)$ in powers of θ, obtaining

$$\frac{\partial\Psi(\tilde{\alpha},t)}{\partial t} = 2\tilde{\alpha}\ln\cosh\beta - 2\tilde{\alpha}\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\theta^n\langle q_{1,\ldots,n}\rangle_t. \tag{37}$$

We learn by looking at eq.(37) that the derivative of the cavity function is built by non-stochastically stable overlap monomials, and their averages do depend on t making their t-integration non trivial (we stress that all the fillable terms are zero when evaluated at $t = 0$ due to the gauge invariance of the model). We can escape this constraint by iterating them again and again (and then integrating them back too) because their derivative, systematically, will develop stochastically stable terms, which turn out to be independent by the interpolating parameter and their integration is straightforwardly polynomial. To this task we introduce the following

Proposition 1. *Let F_s be a function of s replicas. Then the following streaming equation holds*

$$\frac{\partial\langle F_s\rangle_{t,\tilde{\alpha}}}{\partial t} = 2\tilde{\alpha}\theta[\sum_{a=1}^{s}\langle F_s\sigma_{i_0}^a\rangle_{t,\tilde{\alpha}} - s\langle F_s\sigma_{i_0}^{s+1}\rangle_{t,\tilde{\alpha}}] \tag{38}$$

$$+ \quad 2\tilde{\alpha}\theta^2[\sum_{a<b}^{1,s}\langle F_s\sigma_{i_0}^a\sigma_{i_0}^b\rangle_{t,\tilde{\alpha}} - s\sum_{a=1}^{s}\langle F_s\sigma_{i_0}^a\sigma_{i_0}^{s+1}\rangle_{t,\tilde{\alpha}} + \frac{s(s+1)}{2!}\langle F_s\sigma_{i_0}^{s+1}\sigma_{i_0}^{s+2}\rangle_{t,\tilde{\alpha}}]$$

$$+ \quad 2\tilde{\alpha}\theta^3[\sum_{a<b<c}^{1,s}\langle F_s\sigma_{i_0}^a\sigma_{i_0}^b\sigma_{i_0}^c\rangle_{t,\tilde{\alpha}} - s\sum_{a<b}^{1,s}\langle F_s\sigma_{i_0}^a\sigma_{i_0}^b\sigma_{i_0}^{s+1}\rangle_{t,\tilde{\alpha}}$$

$$+ \frac{s(s+1)}{2!}\sum_{a=1}^{s}\langle F_s\sigma_{i_0}^a\sigma_{i_0}^{s+1}\sigma_{i_0}^{s+2}\rangle_{t,\tilde{\alpha}} \quad + \frac{s(s+1)(s+2)}{3!}\langle F_s\sigma_{i_0}^{s+1}\sigma_{i_0}^{s+2}\sigma_{i_0}^{s+3}\rangle_{t,\tilde{\alpha}}]$$

where we neglected terms $O(\theta^3)$.

For a complete proof of the Proposition we refer to [ABC08].

4.2 Free energy analysis

Now that we exploited the machinery we can start applying it to the free energy. Let us at first work out its streaming with respect to the plan (α,β):

$$\frac{\partial A(\alpha,\beta)}{\partial\beta} = -\frac{\langle H\rangle}{N} = \frac{1}{N}\mathbb{E}\Big(\frac{1}{Z_N}\sum_{\{\sigma\}}\sum_{\nu=1}^{P_{\alpha N}}\sigma_{i_\nu}\sigma_{j_\nu}e^{-\beta H_N(\alpha)}\Big) \tag{39}$$

$$= \sum_{k=1}^{\infty}k\pi(k-1,\alpha N)\frac{\mathbb{E}}{N}[\omega(\sigma_{i_k}\sigma_{j_k})_k] = \alpha\sum_{k=1}^{\infty}\pi(k-1,\alpha N)\mathbb{E}\Big[\frac{\omega(\sigma_{i_k}\sigma_{j_k}e^{\beta\sigma_{i_k}\sigma_{j_k}})_{k-1}}{\omega(e^{\beta\sigma_{i_k}\sigma_{j_k}})_{k-1}}\Big]$$

$$= \alpha\mathbb{E}\Big[\frac{\omega(\sigma_{i_k}\sigma_{j_k}(\cosh\beta + \sigma_{i_k}\sigma_{j_k}\sinh\beta))}{\omega(\cosh\beta + \sigma_{i_k}\sigma_{j_k}\sinh\beta)}\Big] = \alpha\mathbb{E}\Big[\frac{\omega(\sigma_{i_k}\sigma_{j_k}) + \theta}{1 + \omega(\sigma_{i_k}\sigma_{j_k})\theta}\Big]$$

by which we get (and with similar calculations for $\partial_\alpha A(\alpha, \beta)$ that we omit for the sake of simplicity):

$$\frac{\partial A(\alpha, \beta)}{\partial \beta} = \alpha\theta - \alpha \sum_{n=1}^{\infty} (-1)^n (1 - \theta^2) \theta^{n-1} \langle q_{1,..,n}^2 \rangle \tag{40}$$

$$\frac{\partial A(\alpha, \beta)}{\partial \alpha} = \ln \cosh \beta - \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \theta^n \langle q_{1,..,n}^2 \rangle \tag{41}$$

Now remembering Theorem (1) and assuming critical behavior (that we will verify

a fortiori in sec. (4.3)) we move for a different formulation of the free energy by considering the cavity function as the integral of its derivative. In a nutshell the idea is as follows: Due to the second order nature of the phase transition for this model (i.e. criticality that so far is assumed) we can expand the free energy in terms of the whole series of order parameters. Of course it is impossible to manage all these infinite overlap correlation functions to get a full solution of the model in the whole (α, β) plane but it is possible to show by rigorous bounds that close to the critical line (that we are going to find soon) higher order overlaps scale with higher order critical exponents so we are allowed to neglect them close to this line and we can investigate deeply criticality, which is the topic of the section.
To this task let us expand the cavity functions as

$$\Psi(\tilde{\alpha}, \beta, t) = \int_0^t \frac{\partial \Psi}{\partial t'} dt' \tag{42}$$

$$= 2\tilde{\alpha}t \log \cosh \beta + \tilde{\beta} \int_0^t \langle m \rangle_{t',\tilde{\alpha}} dt' - \frac{1}{2}\tilde{\beta}\theta \int_0^t \langle q_{12} \rangle_{t',\tilde{\alpha}} dt' + O(\theta^3)$$

where $\tilde{\beta} = 2\tilde{\alpha}\theta \to \beta' = 2\alpha\theta$ for $N \to \infty$. Now using the streaming equation as dictated by Proposition (1) we can write the overlaps appearing in the expression of Ψ as polynomials of higher order filled overlaps so to obtain a straightforward polynomial back-integration for the Ψ as they no longer will depend on the interpolating parameter t thanks to Theorem (1).
For the sake of simplicity the $\tilde{\alpha}$-dependence of the overlaps will be omitted keeping in mind that our results are all taken in the thermodynamic limit and so we can quietly exchange $\tilde{\alpha}$ with α in these passages. The first equation we deal with is:

$$\frac{d\langle m \rangle_t}{dt} = \tilde{\beta}[\langle m^2 \rangle - \langle m_1 m_2 \rangle_t] \tag{43}$$

where $\langle m_1 m_2 \rangle$ is not filled and so we have to go further in the procedure and derive

it in order to obtain filled monomials:

$$\frac{d\langle m_1 m_2 \rangle_t}{dt} = \tag{44}$$
$$= 2\tilde{\beta}[\langle m_1^2 m_2 \rangle_t - \langle m_1 m_2 m_3 \rangle_t] + \tilde{\beta}\theta[\langle m_1 m_2 q_{12} \rangle - 4\langle m_1 m_2 q_{13} \rangle_t + 3\langle m_1 m_2 q_{34} \rangle_t].$$

In this expression we stress the presence of the filled overlap $\langle m_1 m_2 q_{12} \rangle$ and of $\langle m_1^2 m_2 \rangle_t$ which can be saturated in just one derivation. Wishing to have an expansion

for $\langle m \rangle_t$ up to the third order in θ, it is easy to check that the saturation of the other overlaps in the last derivative would carry terms of higher order and so we can stop the procedure at the next step

$$\frac{d\langle m_1^2 m_2 \rangle_t}{dt} = \tilde{\beta}[\langle m_1^2 m_2^2 \rangle] + \tilde{\beta}[\text{unfilled terms}] + O(\theta^2) \tag{45}$$

from which integrating back in t

$$\langle m_1^2 m_2 \rangle_t = \tilde{\beta}[\langle m_1^2 m_2^2 \rangle]t \tag{46}$$

inserting now this result in the expression (44) and integrating again in t we find

$$\langle m_1 m_2 \rangle_t = \tilde{\beta}\theta\langle m_1 m_2 q_{12} \rangle t + \tilde{\beta}^2 \langle m_1^2 m_2^2 \rangle t^2 \tag{47}$$

and coming back to $\langle m \rangle_t$ we get

$$\langle m \rangle_t = \tilde{\beta}\langle m^2 \rangle t - \frac{\tilde{\beta}^2 \theta}{2}\langle m_1 m_2 q_{12} \rangle t^2 - \frac{\tilde{\beta}^3}{3}\langle m_1^2 m_2^2 \rangle t^3 \tag{48}$$

which is the attempted result. Let us move our attention to $\langle q_{12} \rangle_t$, analogously we can write

$$\frac{d\langle q_{12} \rangle_t}{dt} = 2\tilde{\beta}[\langle m_1 q_{12} \rangle_t - \langle m_3 q_{12} \rangle_t] + \tilde{\beta}\theta[\langle q_{12}^2 \rangle - 4\langle q_{12} q_{13} \rangle_t + 3\langle q_{12} q_{34} \rangle_t] \tag{49}$$

and consequently obtain

$$\langle q_{12} \rangle_t = \tilde{\beta}\theta\langle q_{12}^2 \rangle t + \tilde{\beta}^2 \langle m_1 m_2 q_{12} \rangle t^2 + O(\theta^4). \tag{50}$$

With the two expansion above, in the $N \to \infty$ limit, putting $t = 1$ and neglecting terms of order higher than θ^4, we have

$$\Psi(\alpha, \beta, t = 1) = 2\alpha \ln \cosh \beta + \frac{\beta'}{2}\langle m^2 \rangle - \frac{\beta'^4}{12}\langle m_1^2 m_2^2 \rangle - \frac{\beta'^2 \theta^2}{4}\langle q_{12}^2 \rangle - \frac{\beta'^3 \theta}{3}\langle m_1 m_2 q_{12} \rangle \tag{51}$$

At this point we have all the ingredients to write down the polynomial expansion for the free energy function as stated in the next:

Proposition 2. *A general expansion via stochastically stable terms for the free energy of the random two body interacting imitative agent model can be written as*

$$A(\alpha, \beta) = \ln 2 + \alpha \ln \cosh \beta + \frac{\beta'}{2}(\beta' - 1)\langle m_1^2 \rangle + \tag{52}$$
$$- \frac{\beta'^4}{12}\langle m_1^2 m_2^2 \rangle - \frac{\beta'^2}{8\alpha}\left(\frac{\beta'^2}{2\alpha} - 1\right)\langle q_{12}^2 \rangle - \frac{\beta'^4}{6\alpha}\langle m_1 m_2 q_{12} \rangle + O(\theta^6).$$

It is immediate to check that the above expression, in the ergodic region where the averages of all the order parameters vanish, reduces to the well known high-temperature (or high connectivity) solution [ABC08] (i.e. $A(\alpha, \beta) = \ln 2 + \alpha \log \cosh \beta$). Of course we are neglecting θ^6 higher order terms because we are interested in an expansion holding close to the critical line, but we are not allowed to truncate the series for a general point in the phase space far beyond the ergodic region.

4.3 Critical behavior and phase transition

Now we want to analyze the critical behavior of the model: we find the critical line where the ergodicity breaks, we obtain the critical exponent of the magnetization and the susceptibility χ, which is defined as $\langle\chi\rangle = \beta N[\langle m^2\rangle - \langle m\rangle^2]$.

Let us firstly define the rescaled magnetization ξ_N as $\xi_N = \sqrt{N}m_N$. By applying the gauge transformation $\sigma_i \to \sigma_i\sigma_{N+1}$ in the expression for the quenched average of the magnetization (eq. (48)) and multiplying it times N so to switch to ξ_N^2, setting $t = 1$ and sending $N \to \infty$ we obtain

$$\langle\xi_1^2\rangle = \frac{\beta'^3}{3(\beta'-1)}\langle\xi_1\xi_2 m_1 m_2\rangle + \frac{\beta'^2\theta}{2(\beta'-1)}\langle\xi_1\xi_2 q_{12}\rangle + O(\frac{\theta^5}{\beta'-1}) \qquad (53)$$

by which we see (again remembering criticality and so forgetting higher order terms)

that the only possible divergence of the (centered and rescaled) fluctuations of the magnetization happens at the value $\beta' = 1$ which gives $2\alpha\theta = 1$ as the critical line, in perfect agreement with the expected Landau-like behavior. The same critical

line can be found more easily by simply looking at the expression (52) as follows: remembering that in the ergodic phase the minimum of the free energy corresponds to a zero order parameter (i.e. $\sqrt{\langle m^2\rangle} = 0$), this implies the coefficient of the second order $a(\beta') = \frac{\beta'}{2}(\beta'-1)$ to be positive. Anyway immediately below the critical line values of the magnetization different from zero must be allowed (by definition otherwise we were not crossing a critical line) and this can be possible if and only if $a(\beta') \leq 0$. Consequently (and using once more the second order nature of the transition) on the critical line we must have $a(\beta') = 0$ and this gives again $2\alpha\theta = 1$.

Now let us move to the critical exponents. Critical exponents are needed to characterize singularities of the observables at the critical line and, for us, these indexes are the ones related to the magnetization $\langle m\rangle$ and to the susceptibility $\langle\chi\rangle$.
We define $\tau = (2\alpha\tanh\beta - 1)$ and we write $\langle m(\tau)\rangle \sim G_0 \cdot \tau^\delta$ and $\langle\chi(\tau)\rangle \sim G_0 \cdot \tau^\gamma$, where the symbol $\sim$ has the meaning that the term at the second member is the dominant but there are corrections of higher order.
Remembering the expansion of the squared magnetization that we rewrite for completeness

$$\langle m^2\rangle = \frac{\beta'^3}{3(\beta'-1)}\langle m_1^2 m_2^2\rangle + \frac{\beta'^2\theta}{2(\beta'-1)}\langle m_1 m_2 q_{12}\rangle + O(\frac{\theta^5}{\beta'-1}) \qquad (54)$$

and considering that using the same gauge transformation $\sigma_i \to \sigma_i\sigma_{N+1}$ on (eq.(50)) we have for the two replica overlap the following representation

$$\langle q_{12}^2\rangle = -\frac{\beta'^2}{(\beta'\theta-1)}\langle m_1 m_2 q_{12}\rangle + O(\theta^6) \qquad (55)$$

we can arrive by simple algebraic calculations to write down the free energy, of course close to the critical line, depending only by the two parameters $\langle m^2\rangle$ and $\langle q_{12}^2\rangle$

$$A(\alpha,\beta) = \ln 2 + \alpha\ln\cosh\beta + \frac{\beta'}{4}(\beta'-1)\langle m_1^2\rangle - \frac{\beta'^2}{48\alpha}\left(\frac{\beta'^2}{2\alpha}-1\right)\langle q_{12}^2\rangle + O(\theta^6). \quad (56)$$
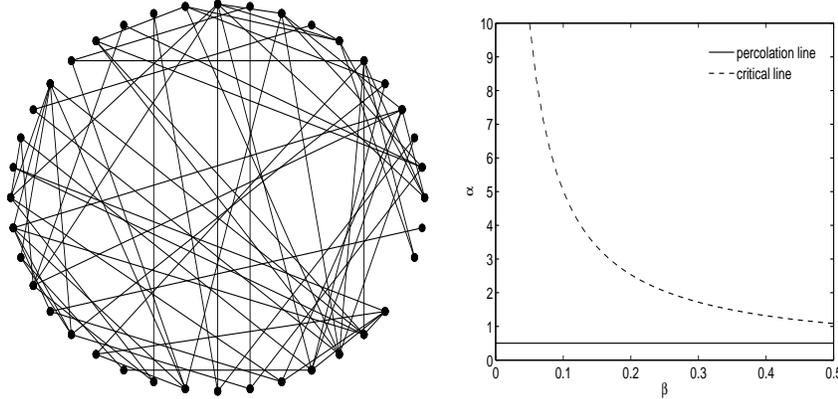
Fig. 1. Left panel: Example of Erdös-Renyi random graph made up of 40 nodes and with average degree equal to 4. Right panel: Phase diagram: below $\alpha_c = 0.5$ there is no giant component in the graph, α_c defines the percolation threshold. Above α_c at left of the critical line the system behaves ergodically, conversely on the right ergodicity is broken and the system displays magnetization.

By a comparison of the formula obtained by deriving $A(\alpha, \beta)$ as expressed by eq.(56) and the expression we have previously found (eq. 41), it is immediate to see that we have

$$\frac{\partial}{\partial \alpha}\left[\frac{\beta'}{4}(\beta' - 1)\langle m_1^2 \rangle\right] = \theta \langle m_1^2 \rangle. \tag{57}$$

Close to the value $\beta' = 1$, making a change of variable $\tau = \beta' - 1$ with $\partial_\alpha = 2\theta\partial_\tau$, we get

$$\frac{\partial}{\partial \alpha}\left[\frac{\beta'}{4}(\beta' - 1)\langle m_1^2 \rangle\right] \sim \frac{\theta}{2}\frac{\partial}{\partial \tau}[\tau\langle m_1^2 \rangle] = \frac{\theta}{2}\langle m_1^2 \rangle + \frac{\theta\tau}{2}\frac{\partial\langle m_1^2 \rangle}{\partial \tau} = \theta\langle m_1^2 \rangle, \tag{58}$$

by which we easily obtain

$$\frac{\partial\langle m_1^2 \rangle}{\langle m_1^2 \rangle} = \frac{\partial\tau}{\tau} \qquad \Rightarrow \langle m_1^2 \rangle \sim \tau \qquad \Rightarrow \sqrt{\langle m_1^2 \rangle} \sim \tau^{\frac{1}{2}} = \tau^\delta \tag{59}$$

Therefore we get the critical exponent for the magnetization, $\delta = 1/2$, which turns out to be the same as in the fully connected counterpart [Bar08a], in agreement with the disordered extension of this model [Bar06].
Again, by simple direct calculations, once we get the critical exponent for the magnetization it is straightforward to show that the susceptibility $\langle \chi \rangle$ obeys

$$\langle \chi \rangle \sim |\tau|^{-1} = \tau^\gamma \tag{60}$$

close to the critical line, by which we find its critical exponent to be once again in agreement with the classical fully connected counterpart [Ami92], i.e. $\gamma = -1$.

4.4 Numerics

In this section, by performing extensive Monte Carlo simulations with the Glauber algorithm [NB01], we analyze, from the numerical point of view, the 2-body diluted imitative system previously introduced.

The Erdös-Renyi random graph is constructed by taking N sites and introducing a bond between each pair of sites with probability $p = \bar{\alpha}/(N-1)$, in such a way that the average coordination number per node is just $\bar{\alpha}$. Clearly, when $p = 1$ the complete graph is recovered.

The simplest version of the diluted Curie-Weiss Hamiltonian has a Poisson variable per bond as $H_N = -\sum_{ij} \sum_{\nu=0}^{P_{\bar{\alpha}/N}} \sigma_{i_\nu} \sigma_{j_\nu}$, and it is the easiest approach when dealing with numerics.

For the analytical investigation we choose a slightly changed version (see eq.(19)): each link gets a bond with probability close to α/N for large N; the probabilities of getting two, three bonds scale as $1/N^2, 1/N^3$ therefore negligible in the thermodynamic limit.

Working with directed links (as we do in the analytical framework) the probability of having a bond on any undirected link is twice the probability for directed link (i.e. $2\alpha/N$). Hence, for large N, each site has average connectivity 2α. Finally in this way we allow self-loop but they add just σ-independent constant to H_N and are irrelevant, but we take the advantage of dealing with an H_N which is the sum of independent identically distributed random variables, that is useful for analytical investigation.

When comparing with numerics consequently we must keep in mind that $\bar{\alpha} = 2\alpha$.

In the simulation, once the network has been diluted, we place a spin σ_i on each node i and allow it to interact with its nearest-neighbors. Once the external parameter β is fixed, the system is driven by the single-spin dynamics and it eventually relaxes to a stationary state characterized by well-defined properties. More precisely, after a suitable time lapse t_0 and for sufficiently large systems, measurements of a (specific) physical observable $x(\sigma, \bar{\alpha}, \beta)$ fluctuate around an average value only depending on the external parameters β^{-1} and $\bar{\alpha}$.

Moreover, for a system $(\bar{\alpha}, \beta)$ of a given finite size N the extent of such fluctuations scales as $N^{-\frac{1}{2}}$ with the size of the system. The estimate of the thermodynamic observables $\langle x \rangle$ is therefore obtained as an average over a suitable number of (uncorrelated) measurements performed when the system is reasonably close to the equilibrium regime.

The estimate is further improved by averaging over different realizations of the same system $(\bar{\alpha}, \beta)$. In summary,

$$\langle x(\sigma, \bar{\alpha}, \beta) \rangle = \mathbb{E}\left[\frac{1}{M} \sum_{n=1}^{M} x(\sigma(t_n)) \right], \;\; t_n = t_0 + n\mathcal{T}$$

where $\sigma(t)$ denotes the configuration of the magnetic system at time step t and $\mathcal{T}$ is the decorrelation parameter (i.e. the time, in units of spin flips, needed to decorrelate a given magnetic arrangement). In general, statistical errors during a MC run in a given sample result to be significantly smaller than those arising from the ensemble averaging.

Figure (2) shows the dependence of the macroscopic observables $\langle m \rangle$ and $\langle e \rangle$ from the size of the system; values are obtained starting from a ferromagnetic ar-
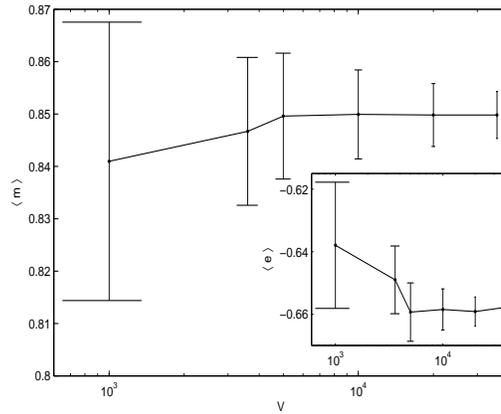
Fig. 2. Finite size scaling for the magnetization and the internal energy (inset) for $\bar{\alpha} = 10$ and $\frac{\beta}{\bar{\alpha}} = 1.67$. All the measurements were carried out in the stationary regime and the error bars represent the fluctuations about the average values. We find good indication of the convergence of the quantities on the size of the system and thus of the existence of the thermodynamic limit.

rangement, at the normalized inverse temperature $\beta/\bar{\alpha} = 1.67$. Notice that at this temperature the system composed of $N = 10^4$ is already very close to the asymptotic regime. Analogous results are found for different systems $(\bar{\alpha}, \beta)$, with β far enough from β_c.

In the following we focus on systems of sufficiently large size so to discard finite size effects. For a wide range of temperatures and dilutions, we measure the average magnetization $\langle m \rangle$ and energy $\langle e \rangle$, as well as the magnetic susceptibility $\langle \chi \rangle$

Their profiles display the typical behavior expected for a ferromagnet (i.e. imitative coupling) and, consistently with the theory, highlight a phase transition at well defined temperatures $\beta_c(\alpha)$.

Now, we investigate in more detail the critical behavior of the system. We collect accurate data of magnetization and susceptibility, for different values of $\bar{\alpha}$ and for temperatures approaching the critical one. These data are used to estimate both the critical temperature and the critical exponents for the magnetization and susceptibility. In Fig. (3) we show data as a function of the reduced temperature $\tau = (|\beta - \beta_c|/\beta_c)^{-1}$ for $\bar{\alpha} = 10$ and $\bar{\alpha} = 20$. The best fit for observables is the power law

$$\langle m \rangle \sim \tau^\delta, \ \beta > \beta_c \tag{61}$$

$$\langle \chi \rangle \sim \tau^\gamma. \tag{62}$$

We obtain estimates for $\beta_c(\bar{\alpha})$, $\delta(\bar{\alpha})$ and $\gamma(\bar{\alpha})$ by means of a fitting procedure. Results are gathered in Tab. 1. Within the errors ($\leq 2\%$ for β_c and $\leq 5\%$ for the exponents), estimates for different values of $\bar{\alpha}$ agree and they are also consistent with the analytical results exposed in Sec. (4.3)
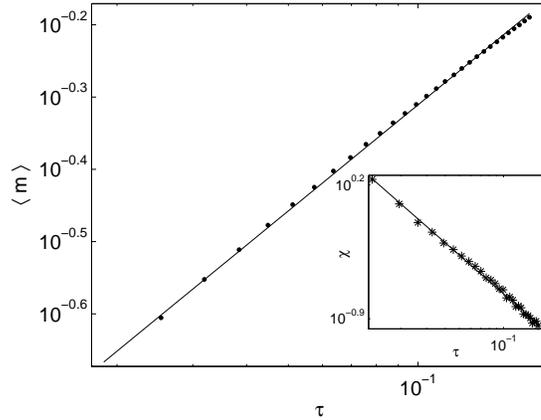
Fig. 3. Log-log scale plot of magnetization (main figure) and susceptibility (inset) versus the reduced temperature $\tau = (|\beta - \beta_c|/\beta_c)^{-1}$ for $\bar{\alpha} = 10$. Symbols represents data from numerical simulations performed on systems of size $N = 36000$, while lines represent the best fit.

$\bar{\alpha}$	β_c^{-1}	δ	γ
10	9.93	0.48	−0.97
20	19.92	0.49	−1.04
30	29.98	0.48	−1.04
40	39.59	0.50	−1.02

Table 1. Estimates for the critical temperature and the critical exponents δ and γ obtained by a fitting procedure on data from numerical simulations concerning Ising systems of size $N = 36000$ and different dilutions (we stress that analytically we get $\delta = 0.5$ and $\gamma = -1$). Errors on temperatures are $< 2\%$, while for exponents are within 5%.

We also checked the critical line for the ergodicity breaking, again finding optimal agreement with the criticality investigated by means of analytical tools.

5 Beyond detailed balance: Diffusive strategic dynamics

As we saw in Section 2, a dynamics which obeys detailed balance is constrained to reach the Maxwell-Boltzmann equilibrium. In social context however, detailed balance actually lacks a clear interpretation, or a data comparison. In this section we figure out a possible and more realistic dynamics which aims to mimic opinion spreading and strategic choices on random networks, without any a priori constraint based on detailed balance.

5.1 The algorithm

In order to simulate the dynamical evolution of Ising-like system as our one several different algorithms have been introduced. In particular, a well established one is the so-called single-flip algorithm, which makes the system evolve by means of successive spin-flips, where we call "flip" on the node j the transformation $\sigma_j \to -\sigma_j$ [Lig99].

More precisely, the generic single-flip algorithm is made up of two parts: first we need a rule according to which we select a spin to be updated, then we need a probability distribution which states how likely the spin-flip is. As for the latter, following the Glauber rule, given a configuration σ, the probability for the spin-flip on the j-th node reads off as

$$p(\sigma, j, \mathbf{J}) = \frac{1}{1 + e^{\beta \Delta H(\sigma, j, \mathbf{J})}}, \tag{63}$$

where $\Delta H(\sigma, j, \mathbf{J}) = 2\sigma_i \sum_j J_{ij}\sigma_j$ is the variation in the cost function due to the flip $\sigma_j \to -\sigma_j$. Hence, for single-flip dynamics the cost variation ΔH, consequent to a flip, only depends on the spin of a few sites, viz. the j-th one undergoing the flipping process and its α_j nearest-neighbors.
As for the selection rule according to which sites are extracted, there exist several different choices, ranging from purely random to deterministic. In several contexts (condensed-matter physics [BBCV02], sociology [ABC09], etc.) unless no peculiar mechanisms or strategies are at work, the random updating seems to be the most plausible.

In a social context a spin-flip can occur as a result of a direct interaction (phone call, mail exchange, etc.) between two neighbors and if agent i has just undergone an opinion-flip he will, in turn, prompt one out of his α_i neighbors to change opinion where, we recall, in social context opinion plays the role of the spin orientation in material systems.

These aspects are neglected by traditional dynamics and can not be described by a random updating rule. A different relaxation dynamics, introduced and developed in [ABCV05, ABCV06a, ABCV06b], is able to take into account these aspects, namely:
i. the selection rule exhibits a *diffusive character*: The sequence of sites selected for the updating can be thought of as the path of a random walk moving on the underlying structure.
ii. the diffusion is *biased*: The α_i neighbors are not equally likely to be chosen but, amongst the α_i neighbors, the most likely to be selected is also the most likely to undergo a spin-flip, namely the one which minimizes $\Delta H(\sigma, j, \mathbf{J})$.

Let us now formalize how the dynamics works. Our MC simulations are made up of successive steps [ABC09]:
- Being i the newest updated spin/agent (at the very first step i is extracted randomly from the whole set of agents), we consider the corresponding set of nearest-neighbors defined as $\mathcal{N}_i = \{i_1, i_2, ..., i_{\alpha_i}\}$; we possibly consider also the subset $\tilde{\mathcal{N}}_i \subseteq \mathcal{N}_i$ whose elements are nearest- neighbors of i not sharing the same orientation/opinion: $j \in \tilde{\mathcal{N}}_i \Leftrightarrow j \in \mathcal{N}_i \wedge \sigma_i \sigma_j = -1$. Now, for any $j \in \mathcal{N}_i$ we compute the cost function variation $\Delta H(\sigma, j, \mathbf{J})$, which would result if the flip $\sigma_j \to -\sigma_j$ occurred; notice that $\Delta H(\sigma, j, \mathbf{J})$ involves not only the nearest-neighbors of i.

- We calculate the probability of opinion-flip for all the nodes in $\mathcal{N}_i$, hence obtaining $p(\sigma, i_1, \mathbf{J}), p(\sigma, i_2, \mathbf{J}), ..., p(\sigma, i_{\alpha_i}, \mathbf{J})$, where $p(\sigma, \sigma'_j, \mathbf{J})$ (see eq. 63), is the probability that the current configuration σ changes due to a flip on the j-th site.

- We calculate the probability $\mathcal{P}^{\mathcal{S}}(\sigma; i, j; \mathbf{J})$ that among all possible α_i opinion-flips considered just the j-th one is realized; this is obtained by properly normalizing the $p(\sigma, j, \mathbf{J})$:

$$\mathcal{P}^{\mathcal{S}}(\sigma; i, j; \mathbf{J}) = \frac{p(\sigma, j, \mathbf{J})}{\sum_{k \in \mathcal{N}_i} p(\sigma, k, \mathbf{J})}. \tag{64}$$

We can possibly restrict the choice just to the set $\tilde{\mathcal{N}}_i$, hence defining $\tilde{\mathcal{P}}^{\mathcal{S}}(\sigma; i, j; \mathbf{J}) = p(\sigma, j, \mathbf{J})/\sum_{k \in \tilde{\mathcal{N}}_i} p(\sigma, k, \mathbf{J})$.

- According to the normalized probability $\mathcal{P}^{\mathcal{S}}$ (see eq. 64), we extract randomly the node $j \in \mathcal{N}_i$ and realize the opinion flip $\sigma_j \rightarrow -\sigma_j$.

- We set $j \equiv i$ and we iterate the procedure.

Finally, it should be underlined that in this dynamics detailed balance is explicitly violated [BBCV02, ABCV05]; indeed, its purpose is not to recover a canonical Boltzmann equilibrium but rather to model possible mechanism making the system evolve, and ultimately, to describe, at an effective level, the statics reached by a "socially plausible" dynamics for opinion spreading [ABC09].

5.2 Equilibrium behavior

The diffusive dynamics was shown to be able to lead the system toward a well defined steady state and to recover the expected phase transition, although the critical temperature revealed was larger than the expected one [BBCV02]. Such results were also shown to be robust with respect to the the spin magnitude [ABCV05] and the underlying topology [ABC09]. More precisely, after a suitable time lapse t_0 and for sufficiently large systems, measurements of a (specific) physical observable $X(\sigma, \beta)$ fluctuate around an average value only depending on the external parameter β and on the geometry of the underlying structures (in particular, for diluted systems, on α). It was also verified that, for a system of a given finite size N, the extent of such fluctuations scales as $N^{-\frac{1}{2}}$ (see also [BBCV02, ABCV05]), as indicated by standard statistical mechanics for a system in equilibrium. The estimate of the a given observable $\langle X(\beta) \rangle$ can therefore be obtained as an average over a suitable number of (uncorrelated) measurements performed when the system is reasonably close to the equilibrium regime.

Moreover, the final state obtained with the diffusive dynamics is stable, well defined and, in particular, it does not depend on the initial conditions, thus it displays all the properties of a stationary state. Similar to what happens with the usual dynamics, the relaxation time needed to drive the system sufficiently close to the equilibrium situation is found to depend on the temperature. More precisely, we experience the so called 'critical slowing down": the closer T to its critical value, the longer the relaxation time.

In particular, it was evidenced that there exists a critical value of the parameter $\beta_c^{\mathcal{S}}$ below which the system is spontaneously ordered. However, $\beta_c^{\mathcal{S}}$ was found to be appreciably smaller than the critical value $\beta_c(\alpha)$ expected for the canonical Ising model on a Erdös-Renyi random graph. For example, we found $\beta_c^{\mathcal{S}} \approx 0.07$ and
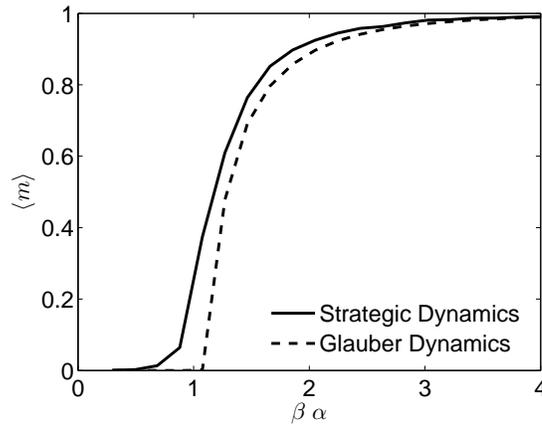
Fig. 4. Critical behavior of the magnetization for the two dynamics (diffusive and standard Glauber) as a function of β and fixed $\alpha = 10$. The former dynamics gives rise to a critical point higher with respect to the latter.

$\beta_c^{\mathcal{S}} \approx 0.016$ for $\bar{\alpha} = 10$ and $\bar{\alpha} = 45$, respectively, versus $\beta_c(\bar{\alpha}) = \tanh^{-1}(1/\bar{\alpha})$, yielding $\beta_c(10) \approx 0.10$ and $\beta_c(45) = 0.022$, (see Fig.4). Interestingly, it is not possible describe the system subjected to the diffusive dynamics by introducing an effective Hamiltonian obtained from eq. (19) by a trivial rescaling. In fact, calling E the numerical energies (to separate them from the analytical e), we consider the dependence on the magnetization displayed by the energies $E^{\mathcal{S}}(m)$ and $E(m)$, measured for system evolving according to the diffusive dynamics and to a traditional dynamics, respectively. As for the latter, from eq. (19) it is easy to see that $E \propto m^2$. As for $E^{\mathcal{S}}(m)$, we found that $E^{\mathcal{S}} < E$ for $0 < m < 1$, while $E^{\mathcal{S}} = E$ for $m = 0$ and $m = 1$. This is compatible with a power law behaviour $E^{\mathcal{S}} \sim m^{2+\epsilon}$. In order to obtain an estimate for ϵ we measured the ratio $E^{\mathcal{S}}/E$ as a function of m; data are shown in the log-log scale plot of Fig. (5) and fitting procedures suggest that $\epsilon \approx 0.15$. In the next section we will show that this result can be interpreted as a consequence of an effective p-agent interaction, being $p > 2$.

6 Statics of many body interactions

So far, summing the discussion starting the sections 4 and 5, we understood that interactions in social networks may involve more than only couple exchanges. Consequently, corresponding to this observation the need for a many-body imitative-behavior Hamiltonian on a random graph appears. In this section, we introduce such a p-spin model and study its thermodynamics.
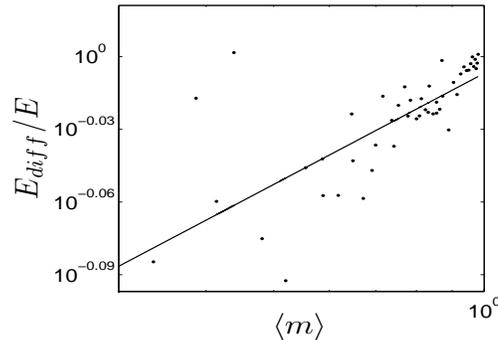
Fig. 5. We extrapolate from $m(\beta)$ and $E(\beta)$ the plot $E(m)$ for both the dynamics. It is worth noting that the diffusive dynamics deals to a curve living always "below" the one obtained by Glauber dynamics results. This strongly suggest the cooperation of $p > 2$ spins each interaction.

6.1 The diluted p-agent imitative behavior model

We start exploiting the properties of a diluted even p-spin imitative model: we restrict ourselves only to even values of p for mathematical convenience (the investigation with the cavities is much simpler), but, due to monotonicity of all the observables in p, there is no need to see this as a real restriction (as simulations on odd values of p confirm and we will see later on).
First of all, we define a suitable Hamiltonian acting on a Erdos-Renyi random graph, with connectivity α, made up by N agents $\sigma_i = \pm 1$, $i \in [1, N]$.
Introducing p families $\{i_\nu^1\}, \{i_\nu^2\}, ..., \{i_\nu^p\}$ of i.i.d. random variables uniformly distributed on the previous interval, the Hamiltonian is given by the following expression

$$H_N(\sigma, \gamma(\alpha)) = - \sum_{\nu=1}^{k_{\gamma(\alpha)N}} \sigma_{i_\nu^1} \sigma_{i_\nu^2} ... \sigma_{i_\nu^p} \qquad (65)$$

where, reflecting the underlying network, k is a Poisson distributed random variable with mean value $\gamma(\alpha)N$. The relation among the coordination number α and γ is $\gamma \propto \alpha^{p-1}$: this will be easily understood a few line later by a normalization argument coupled with the high connectivity limit of this mean field model.

The quenched expectation of the model is given by the composition of the Poissonian average with the uniform one performed over the families $\{i_\nu\}$

$$\mathbf{E}[\cdot] = E_P E_i[\cdot] = \sum_{k=0}^{\infty} \frac{e^{-\gamma(\alpha)N}(\gamma(\alpha)N)^k}{k!N^p} \sum_{i_\nu^1, ..., i_\nu^p}^{1,N} [\cdot]. \qquad (66)$$

The Hamiltonian (66), as written, has the advantage that it is the sum of (a random number of) i.i.d. terms. To see the connection to a more familiar Hamiltonian wrote in terms of adjacency tensor $A_{i_1,...,i_p}$, we note that the Poisson-distributed

total number of bonds obeys $P_{\gamma N} = \gamma N + O(\sqrt{N})$ for large N. As there are N^p ordered spin p-plets $(i_1, ..., i_p)$, each gets a bond with probability $\sim \alpha/N$ for large N. The probabilities of getting two, three (and so on) bonds scale as $1/N^2, 1/N^3, ...$ so can be neglected. The probability of having a bond between any unordered p-plet of spins is $p!$ as large, i.e. $2\alpha/N$ for $p = 2$.

It is possible to show that our version of the Hamiltonian in fact is thermodynamically equivalent with the more familiar involving the explicit adjacency tensor $A_{i_1, ..., i_p}$, by recall at first the latter model too:

$$-H_N(\sigma; k) \sim -\hat{H}_N(\sigma) = \sum_{i_1, ..., i_p}^{N} A_{i_1, ..., i_p} \sigma_{i_1} ... \sigma_{i_p}, \tag{67}$$

where k is a Poisson variable with mean $\gamma N \sim \alpha^{p-1} N$ and $A_{i_1, ..., i_k}$ are all independent Poisson variables of mean $\gamma/N^{p-1} \sim (\alpha/N)^{p-1}$.

Then, it is enough to consider the streaming of the following interpolating free energy (whose structure proves the statement a priori by its thermodynamics meaning), depending on the real parameter $t \in [0, 1]$

$$\phi(t) = \frac{\mathbb{E}}{N} \ln \sum_{\sigma} e^{\beta(\sum_{\nu=1}^{k} \sigma_{i_\nu^1} ... \sigma_{i_\nu^p} + \sum_{i_1, ..., i_p}^{N} A_{i_1, ..., i_p} \sigma_{i_1} ... \sigma_{i_p})},$$

where k is a Poisson random variable with mean $\gamma N t$ and $A_{i_1, ..., i_p}$ are random Poisson variables of mean $(1-t)\gamma/N^{p-1}$. Note that the two models alone are recovered in the two extremals of the interpolation (for $t = 0, 1$). By computing the t-derivative, we get

$$\frac{1}{\gamma} \frac{d\phi(t)}{dt} = \mathbb{E} \ln(1 + \Omega(\sigma_{i_0^1} ... \sigma_{i_0^p}) \tanh(\beta)) \tag{68}$$

$$- \frac{1}{N^p} \sum_{i_1, ..., i_p}^{N} \ln(1 + \Omega(\sigma_{i_1} ... \sigma_{i_p}) \tanh(\beta)) = 0,$$

where the label 0 in i_0^k stands for a new spin, born in the derivative, accordingly to the Poisson property (22); as the i_0's are independent of the random site indices in the t-dependent Ω measure, the equivalence is proved.

Following a statistical mechanics approach, we know that the macroscopic behavior, versus the connectivity α and the inverse temperature β, is described by the following free energy density

$$A(\alpha, \beta) = \lim_{N \to \infty} A_N(\alpha, \beta) \tag{69}$$

$$= \lim_{N \to \infty} \frac{1}{N} \mathbb{E} \ln \sum_{\sigma} \exp(-\beta H_N(\sigma, \gamma(\alpha))).$$

The normalization constant can be checked by performing the expectation value of the cost function:

$$\mathbf{E}[H] = -\gamma N m^p$$

$$\mathbf{E}[H^2] - \mathbf{E}^2[H] = \gamma^2 N^2 \Big[(q_{12}^p - m^p) + O(\frac{1}{N}) \Big], \tag{70}$$

by which it is easy to see that the model is well defined, in particular it is linearly extensive in the volume. Then, in the high connectivity limit each agent interacts with all the others ($\alpha \sim N$) and, in the thermodynamic limit, $\alpha \to \infty$. Now, if $p = 2$ the amount of couples in the summation scales as $N(N-1)/2$ and, with $\gamma = 2\alpha$, a linear divergence of α (desired to get a finite ratio α/N for each coupling) provides the right scaling; if $p = 3$ the amount of triples scales as $N(N-1)(N-2)/3!$ and, with $\gamma = 3!\alpha^2$, again we find the right connectivity behavior. The generalization to every finite $p < N$ is straightforward.

6.2 Properties of the random diluted p-spin model

Before starting our free energy analysis, we want to point out also the connection among this diluted version and the fully connected counterpart (where each agents interact with the whole community, not just a fraction as in the random network). Let us remember that the Hamiltonian of the fully connected p-spin model (FC) can be written as [Bar08b]

$$H_N^{FC}(\sigma) = \frac{p!}{2N^{p-1}} \sum_{1 \leq i_1 < ... < i_p \leq N} \sigma_{i_1} \sigma_{i_2} ... \sigma_{i_p}, \tag{71}$$

and let us consider the trial function $\hat{A}(t)$ defined as follows

$$\hat{A}(t) = \frac{1}{N} \mathbb{E} \ln \sum_{\sigma} \exp \left[\beta \sum_{\nu}^{P_{\gamma N t}} \sigma_{i_\nu^1} \sigma_{i_\nu^2} ... \sigma_{i_\nu^p} + (1-t) \frac{\beta' N}{2} m^p \right], \tag{72}$$

which interpolates between the fully connected p-spin model and the diluted one, such that for $t = 0$ only the fully connected survives, while the opposite happens for $t = 1$. Let us work out the derivative with respect to t to obtain

$$\partial_t \hat{A}(t) = (p-1)\alpha^{p-1} \ln \cosh(\beta) \tag{73}$$
$$- (p-1)\alpha^{p-1} \sum_n \frac{-1^n}{n} \theta^n \langle q_n^p \rangle - \frac{\beta'}{2} \langle m^p \rangle,$$

by which we see that the correct scaling, in order to recover the proper infinite connectivity model, is obtained when $\alpha \to \infty$, $\beta \to 0$ and $\beta' = 2(p-1)\alpha^{p-1} \tanh(\beta)$ is held constant.

Remark 1. It is worth noting that in social modeling, usually, the role of the temperature is left, or at least coupled together, to the interaction strength J. As a consequence, in order to keep β' fixed, on different network dilution, the strength must be rescaled accordingly to

$$J = \tanh^{-1} \left(\frac{\beta'}{2(p-1)\alpha^{p-1}} \right),$$

while, if present, an external field remains unchanged as it is a one-body term, like $h \sum_i^N \sigma_i$, unaffected by dilution.

Remark 2. The dilute p-spin model reduces to the fully connected one, in the infinite connectivity limit, uniformly in the size of the system.

6.3 Properties of the free energy via the smooth cavity approach

In this section we want to show some features of the free energy corresponding to this model, which is investigated by extending the previous method (the smooth cavity approach) to the many body Hamiltonian.
As the generalization is simple and immediate to be achieved by the reader, we skip the proofs in this section.
The starting point is always the representation theorem

Theorem 5. *In the thermodynamic limit, the quenched pressure of the even p-spin diluted ferromagnetic model is given by the following expression*

$$A(\alpha, \beta) = \ln 2 - \frac{\alpha}{p-1} \frac{d}{d\alpha} A(\alpha, \beta) + \Psi(\alpha, \beta, t = 1), \tag{74}$$

where the cavity function $\Psi(t, \alpha, \beta)$ is introduced as

$$\mathbf{E}\left[\ln \frac{\sum_{\{\sigma\}} e^{\beta \sum_{\nu=1}^{k_{\tilde{\gamma} N}} \sigma_{i_\nu^1} \sigma_{i_\nu^2} \dots \sigma_{i_\nu^p}} e^{\beta \sum_{\nu=1}^{k_{2\tilde{\gamma} t}} \sigma_{i_\nu^1} \sigma_{i_\nu^2} \dots \sigma_{i_\nu^{p-1}}}}{\sum_{\{\sigma\}} e^{\beta \sum_{\nu=1}^{k_{\tilde{\gamma} N}} \sigma_{i_\nu^1} \sigma_{i_\nu^2} \dots \sigma_{i_\nu^p}}} \right] =$$

$$\mathbf{E}\left[\ln \frac{Z_{N,t}(\tilde{\gamma}, \beta)}{Z_N(\tilde{\gamma}, \beta)} \right] = \Psi_N(\tilde{\gamma}, \beta, t), \tag{75}$$

with $\lim_{N \to \infty} \tilde{\gamma} = \gamma$ as for $\tilde{\alpha}$ and α in the two body model (see eq. 26)

$$\Psi(\gamma, \beta, t) = \lim_{N \to \infty} \Psi_N(\tilde{\gamma}, \beta, t). \tag{76}$$

Thanks to the previous theorem, it is possible to figure out an expression for the pressure by studying the properties of the cavity function $\Psi(\alpha, \beta)$ and the connectivity shift $\partial_\alpha A(\alpha, \beta)$.
Let us notice that

$$\frac{d}{d\alpha} A(\alpha, \beta) = (p-1)\alpha^{p-2} \ln \cosh \beta - \tag{77}$$

$$- (p-1)\alpha^{p-2} \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \theta^n \langle q_{1,\dots,n}^p \rangle,$$

$$\frac{d}{dt} \Psi(\tilde{\alpha}, \beta, t) = 2\tilde{\alpha}^{p-1} \ln \cosh \beta - \tag{78}$$

$$- 2\tilde{\alpha}^{p-1} \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \theta^n \langle q_{1,\dots,n}^{p-1} \rangle_{\tilde{\alpha},t}.$$

So we can understand the properties of the free energy by analyzing the properties of the order parameters: magnetization and overlaps, weighted in their extended Boltzmann state $\tilde{\omega}_t$.
Further, as we expect the order parameters being able to describe thermodynamics even in the true Boltzmann states ω, Ω, accordingly to the earlier Definition (2), we are going to recall that *filled* order parameters (the ones involving even numbers of replicas) are stochastically stable, or in other words, are independent by the t-perturbation in the thermodynamic limit, while the others, not filled, become filled, again in this limit (such that for them $\omega_t \to \omega$ in the high N limit and thermodynamics is recovered).

Theorem 6. *In the thermodynamic limit and setting $t = 1$ we have*

$$\tilde{\omega}_{N,t}(\sigma_{i_1}\sigma_{i_2}...\sigma_{i_n}) = \tilde{\omega}_{N+1}(\sigma_{i_1}\sigma_{i_2}...\sigma_{i_n}\sigma_{N+1}^n). \tag{79}$$

Theorem 7. *Let Q_{ab} be a not-filled monomial of the overlaps (this means that $q_{ab}Q_{ab}$ is filled). We have*

$$\lim_{N\to\infty}\lim_{t\to 1}\langle Q_{ab}\rangle_t = \langle q_{ab}Q_{ab}\rangle, \tag{80}$$

(examples:
for $N \to \infty$ we get $\langle m_1\rangle_t \to \langle m_1^2\rangle$, $\langle q_{12}\rangle_t \to \langle q_{12}^2\rangle$).

Theorem 8. *In the $N \to \infty$ limit, the averages $\langle \cdot \rangle$ of the filled polynomials are t-independent in β average.*

With the following definition

$$\tilde{\beta} = 2(p-1)\tilde{\alpha}^{p-1}\theta \tag{81}$$
$$= 2(p-1)\alpha^{p-1}\frac{N}{N+1}\theta \quad \stackrel{N\to\infty}{\longrightarrow} \quad 2(p-1)\alpha^{p-1}\theta = \beta',$$

we show the streaming of replica functions, by which not filled multi-overlaps can be expressed via filled ones.

Proposition 3. *Let F_s be a function of s replicas. Then the following streaming equation holds*

$$\frac{\partial\langle F_s\rangle_{t,\tilde{\alpha}}}{\partial t} = \tilde{\beta}\Big[\sum_{a=1}^{s}\langle F_s m_a^{p-1}\rangle_{t,\tilde{\alpha}} - s\langle F_s m_{s+1}^{p-1}\rangle_{t,\tilde{\alpha}}\Big] \tag{82}$$

$$+ \tilde{\beta}\theta\Big[\sum_{a<b}^{1,s}\langle F_s q_{a,b}^{p-1}\rangle_{t,\tilde{\alpha}} - s\sum_{a=1}^{s}\langle F_s q_{a,s+1}^{p-1}\rangle_{t,\tilde{\alpha}}$$

$$+ \frac{s(s+1)}{2!}\langle F_s q_{s+1,s+2}^{p-1}\rangle_{t,\tilde{\alpha}}\Big] + O(\theta^3).$$

Remark 3. We stress that, at the first two level of approximation presented here, the streaming has has the structure of a θ-weighted linear sum of the Curie-Weiss streaming (θ^0 term) [Bar08a] and the Sherrington-Kirkpatrick streaming (θ^1 term) [Bar06], conferring a certain degree of independence by the kind of quenched noise (frustration or dilution) to mathematical structures of disordered systems.

Overall the result we were looking for, a polynomial form of the free energy, reads off as

$$A(\alpha,\beta) = \ln 2 \, + \, \alpha^{p-1}\ln\cosh\beta + \tag{83}$$

$$+ \frac{\beta'}{2}\Big(\beta'\langle m^{2(p-1)}\rangle - \langle m^p\rangle\Big) +$$

$$+ \frac{\beta'\theta}{4}\Big(\beta'\theta\langle q_{12}^{2(p-1)}\rangle - \langle q_{12}^p\rangle\Big) + O(\theta^5).$$

Now, several conclusions can be addressed from the expression (83):

Remark 4. At first let us note that, by constraining the interaction to be pairwise, critical behavior should arise [LL80]. Coherently, we see that for $p = 2$ we can write the free energy expansion as

$$A(\alpha, \beta)_{p=2} = \ln 2 + \alpha \ln \cosh(\beta) - \frac{\beta'}{2}(1 - \beta')\langle m^2 \rangle - \frac{\beta'\theta}{4}\langle q_2^2 \rangle,$$

which coincides with the one of the diluted two-body model (eq. 52) and displays criticality at $2\alpha\theta = 1$, where the coefficient of the second order term vanishes, in agreement with previous results (sec. 4.3).

Remark 5. The free energy density of the fully connected p-spin model is [Bar08b] $A(\beta') = \ln 2 + \ln \cosh(\beta m^{p-1}) - (\beta/2)m^p$, which coincides with the expansion (83) in the limit of $\alpha \to \infty$ and $\beta \to 0$ with $\beta' = 2(p - 1)\alpha^{p-1}\theta$ held constant.

Remark 6. It is worth noting that the connectivity no longer plays a linear role in contributing to the free energy density, as it does happen for the diluted two body models [ABC08, GT04], but, in complete generality as $p - 1$. This is interesting in social networks, where, for high values of coordination number it may be interesting developing strategies with more than one exchange [NB01].

6.4 Numerics

We now analyze the system lastly described, from the numerical point of view by performing extensive Monte Carlo simulations. Within this approach it is more convenient to use the second Hamiltonian introduced (see eq.(67)):

$$\hat{H}_N(\sigma) = -\sum_{i_i}^{N} \sigma_{i_1} \sum_{i_2 < i_3 < ... < i_p = 1}^{N} A_{i_1,...,i_p}\sigma_{i_2}\sigma_{i_3}...\sigma_{i_p}. \tag{84}$$

The product between the elements of the adjacency tensor ensures that the $p - 1$ spins considered in the second sum are joined by a link with i_1.

The evolution of the magnetic system is realized by means of a single spin flip dynamics based on the Metropolis algorithm [NB01]. At each time step a spin is randomly extracted and updated whenever its coordination number is larger than $p - 1$. For α large enough (at least above the percolation threshold, as obviously it is the case for the results found previously) and $p = 3, 4$ this condition is generally verified. The updating procedure for a spin σ_i works as follows: Firstly we calculate the energy variation Δe_i due to a possible spin flip, which for $p = 3$ and $p = 4$ reads respectively

$$\Delta e_i = 2\sigma_i \sum_{j < k = 1}^{N} A_{i,j} A_{i,k}\sigma_j\sigma_k, \tag{85}$$

$$\Delta e_i = 2\sigma_i \sum_{j < k < w = 1}^{N} A_{i,j} A_{i,k} A_{i,w}\sigma_j\sigma_k\sigma_w. \tag{86}$$

Now, if $\Delta e_i < 0$, the spin-flip $\sigma_i \to -\sigma_i$ is realized with probability 1, otherwise it is realized with probability $e^{-\beta\Delta e}$.
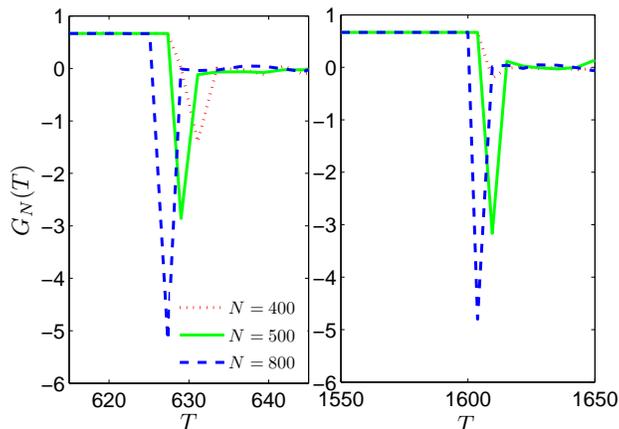
Fig. 6. Binder cumulants $G_L(T)$ for systems of different size N, as shown in the legend, and connectivity $\bar{\alpha} = 50$ (left panel) and $\bar{\alpha} = 80$ (right panel).

The case $p = 3$ has been studied in details and some insight is provided also for the case $p = 4$, while for $p = 2$ we refer to Sec. (4.4). Our investigations concern two main issues:

- the existence of a phase transition and its nature
- the existence of a proper scaling for the temperature as the parameter α is tuned.

As for the first point, we measured the so-called Binder cumulants defined as follows:

$$G_N(T(\alpha)) \equiv 1 - \frac{\langle m^4 \rangle_N}{3\langle m^2 \rangle_N^2}, \tag{87}$$

where $\langle \cdot \rangle_N$ indicates the average obtained for a system of size N [Bin97]. The study of Binder cumulants is particularly useful to locate and catalogue the phase transition. In fact, at any given connectivity (above the percolation threshold), in the case of continuous phase transitions, $G_N(T)$ takes a universal positive value at the critical point T_c, namely all the curves obtained for different system sizes N cross each other. On the other hand, for a first-order transition $G_N(T)$ exhibits a minimum at T_{min}, whose magnitude diverges as N. Moreover, a crossing point at T_{cross} can be as well detected when curves pertaining to different sizes N are considered. Now, T_{min} and T_{cross} scale as $T_{min} - T_c \propto N^{-1}$ and $T_{cross} - T_c \propto N^{-2}$, respectively.

In Fig. 6 we show data for $G_N(T)$ obtained for systems of different sizes ($N = 400$, $N = 500$, and $N = 800$) but equal connectivity ($\alpha = 50$ and $\alpha = 80$, respectively) as a function of the temperature T. The existence of a minimum is clear and it occurs for $T \approx 625$ and $T \approx 1600$. Similar results are found also for $p = 4$ and they all highlight the existence of a first-order phase transition at a temperature which depends on the connectivity α.

In order to deepen the role of connectivity in the evolution of the system we measure the macroscopic observable $\langle m \rangle$ and its (normalized) fluctuations $\langle m^2 \rangle -$
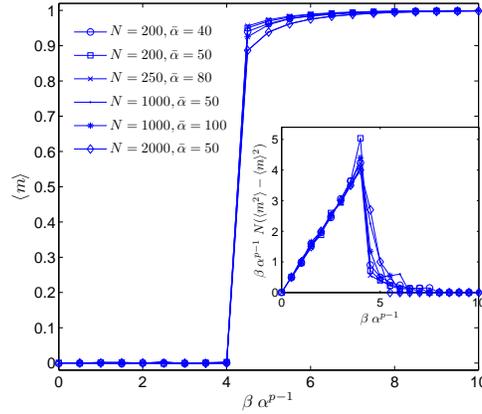
Fig. 7. Magnetization (main figure) and its normalized fluctuations (inset) for systems of different sizes and different dilution as a function of $\beta\,\alpha^{p-1}$. The collapse of all the curves provides a strong evidence for the scaling of the temperature.

$\langle m \rangle^2$, studying their dependence on the temperature β and on the dilution α. Data for different choices of size and dilution are shown in Figure 7.

The profile of the magnetization, with an abrupt jump, and the correspondent peak found for its fluctuations confirm the existence of a first order phase transition at a well defined temperature T_c whose value depends on the dilution α. More precisely, by properly normalizing the temperature in agreement with analytical results, namely $\tilde{\beta} \equiv \beta\,\bar{\alpha}^{p-1}$ we found a very good collapse of all the curves considered. Hence, we can confirm that the temperature scales like α^{p-1}.

6.5 Diffusive dynamics revisited

In the previous section we showed that the diffusive dynamics introduced give rise to a non-trivial thermodynamic which can not be explained by, say, a temperature rescaling. The crucial point is that such a dynamics intrinsically yields effective interactions which are more-than-two bodies. This idea is supported by two important evidences (see 5.2): a larger "critical" temperature $\beta_c^{\mathcal{S}}$ and the anomalous power-law behavior $E^{\mathcal{S}} \sim m^{2.15}$, which suggests the effective p to be 2.15. Now, while p spans from two to infinity the critical temperature raises accordingly, hence we want to check that there exist a suitable real value of p that matches the critical temperature found numerically and that is compatible with the plots of $E^{\mathcal{S}}(m)$ and $E(m)$. When $p = 2$ and close to criticality, the temperature for the phase transition is given by $\beta_c = \tanh^{-1}(1/2\alpha^{p-1}) = \tanh^{-1}(1/2\alpha)$. For $p = 2.15$ this expression becomes $\beta_c \sim \tanh^{-1}(1/2\alpha^{p-1}) = \tanh^{-1}(1/2\alpha^{1.15})$: The ratio among the two expressions, when evaluated for $\alpha = 10$ gets approximately 1.4, in agreement with data depicted in Fig. 4.

Before concluding we notice that for $p > 2, p \in \mathbb{N}$, ferromagnetic transitions are no longer critical phenomena. At the critical line the magnetization is discontinuous and a latent heat does exist. However, if p is thought of as real, for p slightly bigger than two, as suggested by our data, the "jump" in the magnetization is expected to be small and to approach zero whenever $p \to 2$.

7 A simple application to trading in markets

An appealing application of the whole theory developed concerns trading among agents: Suppose we represent a market society only with couple exchanges ($p = 2$), then there are just sellers and buyers and they interact only pairwise. In this case if the buyer i has money ($\sigma_i = +1$) and the seller j has the product ($\sigma_j = +1$), or if the buyer has no money and the seller has no products ($\sigma_i = \sigma_j = -1$), the two merge their will and the imitative cost function (19) reaches the minimum. Otherwise, if the seller has the product but the buyer has no money (or viceversa), their two states are different ($\pm$) and the cost function is not minimized. In this scenario, the random graph connects on average each agent to α acquaintances and this simply increases linearly the possibility that each agent is satisfied. In fact, the higher the number of "neighbors", the larger the possibility of trading.

When switching to the case $p = 3$, other strategies are available: for example the buyer may not have the money, but he may have a valuable good which can be offered to a third agent, who takes it and, in change, gives to the seller the money, so that the buyer can obtain his target by using a barter-like approach. In this case the two frustrated configurations from the previous sketch can be avoided by multiplying by a factor $\sigma_k = -1$ given by the third contributor k, or everything can remain the same of course if the latter does not agree ($\sigma_k = +1$). Interestingly, we find that in this case ($p = 3$), the amount of acquaintances one is in touch with (strictly speaking, the degree of connectivity α) does not contribute linearly as for $p = 2$, but quadratically: this seems to suggest that if a society deals primarily with direct exchanges, no particular effort should be done to connect people, while, if barter-like approaches are allowed, then the more connected the society is, the larger is the satisfaction reached on average by each agent in his specific goal. The above scenario, intuitively, seems to match the contrast among the classical barter-like approach of villages, where, thanks to the small amount of citizens, their degree of reciprocal knowledge is quite high and the money-mediated one of citizens in big metropolis, where a real reciprocal knowledge is missing.

8 Conclusions and Outlooks

The idea to apply statistical mechanics methods to social and economical sciences has appeared several years ago in the history of science. The main drawback of the approach is the lack of a proper measure criteria for the utility function. Although the comparison may appear somehow risky one can say that the current approach is at the same stage of the pre-thermodynamic epoch when it wasn't clear at all that heat was a form of energy and both the first and second principle of thermodynamics were still to be identified. This parallel was indeed pointed out already by Poincaré

in reply to the Walras theory of Economics and Mechanics [Wal09]. Poincaré said moreover that the lack of ability to measure the utility function is not a severe obstacle at a preliminary stage. What instead has proved to be a serious problem in the development of mathematical approaches to economics and social sciences has been the choice of axioms before phenomenology had been studied properly and quantitative data had been extensively analyzed (see [Bou08]). The attempt that are made nowadays, and we proposed an instance of statistical mechanics nature, are to use mathematical models to mimic microscopic realistic dynamics and reproduce to some extent the typical macroscopic observed behavior. Further refinements of the models will be necessary once the collection of data will start to provide a fit with the free parameters introduced and hopefully new principles and axioms will emerge after that stage.

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