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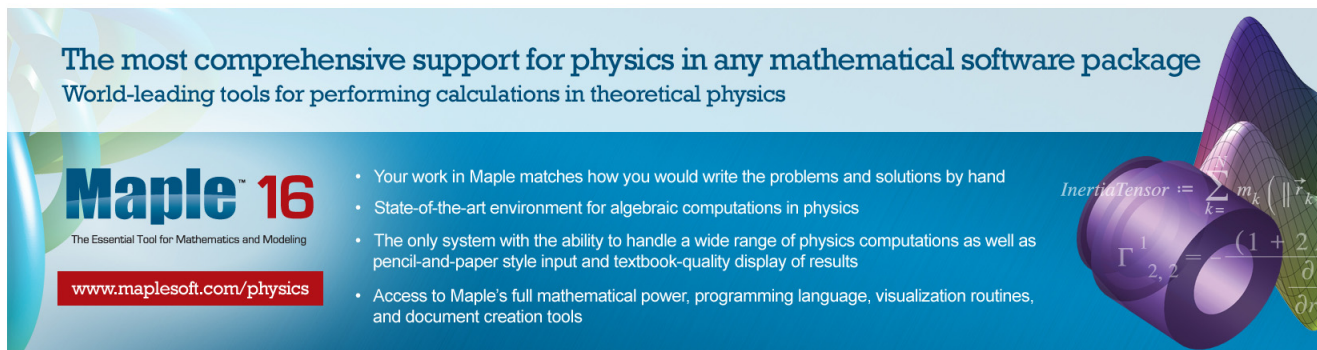
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Notes on the p-spin glass studied via Hamilton-Jacobi and smooth-cavity techniques

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In these notes, we continue our investigation of classical toy models of disordered statistical mechanics, through techniques recently developed and tested mainly on the paradigmatic Sherrington-Kirkpatrick spin glass. Here, we consider the p-spin-glass model with Ising spins and interactions drawn from a normal distribution $\mathcal{N}[0, 1]$. After a general presentation of its properties (e.g., self-averaging of the free energy, existence of a suitable thermodynamic limit), we study its equilibrium behavior within the Hamilton-Jacobi framework and the smooth cavity approach. Through the former we find both the RS and the 1-RSB expressions for the free-energy, coupled with their self-consistent relations for the overlaps. Through the latter, we recover these results as irreducible expression, and we study the generalization of the overlap polynomial identities suitable for this model; a discussion on their deep connection with the structure of the internal energy and the entropy closes the investigation. © *2012 American Institute of Physics.* [http://dx.doi.org/10.1063/1.4729233]

I. INTRODUCTION

In these notes, we continue our investigation on the mathematical methods and the physics underlying many body interactions, namely, we adapt recent mathematical techniques to the study of equilibrium statistical mechanics of p-spin glasses. In the past, we analyzed p-spin systems with the simpler ferromagnetic couplings[8] and p-spin systems with diluted coupling,[2] while now we turn to p-spin systems with frustrated couplings, which are termed p-spin glasses.[14, 17]

We first introduce the model, with all the necessary definitions stemmed from statistical mechanics, and then we adapt the Hamilton-Jacobi technique (developed for the Sherrington-Kirkpatrick model by Guerra[18] and later enlarged to a broad validity[7, 10, 15, 23]) to these systems, so to be able to solve the model (in some physical approximation, that is, replica symmetric and one-step of broken replica symmetry, as discussed later), without any relation with the original statistical mechanics framework.

This has two advantages: the development of a clear and powerful mathematical alternative to solve the thermodynamics of these many body systems, and a further rigorous confirmation of results raised in the theoretical physics scenario.

Then, we adapt the method of the smooth cavity to the same problem to obtain another series of results: in particular, after recovering a clear picture of the thermodynamics in perfect agreement with the previous part of the work and with existing results, we focus on the polynomial identities often called Aizenman-Contucci[4] and Ghirlanda-Guerra[16] relations. We will show how to prove their validity even for the p-spin glasses considered here and we will try to revise their deep physical meaning ultimately offering a unifying framework where cavity fields[21] and stochastic stability[11] merge to work synergically.[9] Furthermore, comparison among the results obtained with both the

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methods will provide the reader with a deeper understanding of the techniques themselves as well as of the physical properties of these models.

In order to be comprehensible for both the communities of theoretical physicists and of mathematical physicists, the two methods are exposed with a slightly different approach. In the former (closer to the first community), results are presented in form of a theorem following the related proof, which is never explicitly expressed as a "proof," while in the latter (closer to the second community) results are first declared and then proved. Finally, in Sec. VI we discuss results and possible outlooks.

II. THE MODEL AND THE RELATED STATISTICAL MECHANICS PACKAGE

The p-spin glass is the model for a system of spins σ, i.e., dichotomic variables which can take the values ± 1, interacting together in p-tuples with random couplings $J_{i_1 \ldots i_p}$, and, possibly, with an external field h. The Hamiltonian is the function which defines the model and physically speaking represents the extensive energy associated with a given configuration of the spins, for a certain value of the couplings and of the external field.

Definition 1: Given a system of N spins σ_i, $i = 1, \ldots, N$, the Hamiltonian associated with a configuration $\sigma = \{\sigma_1, \ldots, \sigma_N\}$ of the spins, interacting in p-tuples and with an external uniform magnetic field h, is defined as follows:

$$H_N(\sigma, J, h) = -\sqrt{\frac{p!}{2N^{p-1}}} \sum_{i_1 < \cdots < i_p}^{1,N} J_{i_1 \ldots i_p} \sigma_{i_1} \ldots \sigma_{i_p} - h \sum_{i=1}^{N} \sigma_i. \tag{1}$$

The first summation is taken over all the possible choices of indices $1 \leq i_1 < \cdots < i_p \leq N$ and the couplings J are independent standard Gaussian random variables. This can be considered as a generalization of the well-known Sherrington-Kirkpatrick model (SK) and its interest lays in the fact that its low temperature behavior is much simpler than in the SK model. The normalization factor preceding the first sum ensures that the Hamiltonian is an extensive quantity (i.e., proportional to the number of spins N) and the 2 at the denominator allows recovering the SK definition when $p = 2$.

For the sake of simplicity, we only consider the case of an even number p of interacting spins. In this case, the system has a gauge symmetry when the external field h is set equal to zero: it is left invariant under the transformation $\sigma_{i_k} \to \sigma_{i_k} \sigma_{i_{p+1}}$ for all $k = 1, 2, \ldots, p$. Moreover, we assume that the external field vanishes, thus we neglect the second term: in fact, this is a one-body term, which is simple to deal with.[1,3] In the following, $H_N(\sigma, J)$ has to be interpreted as $H_N(\sigma, J, 0)$.

For this model, the investigation of the free energy and its decomposition via Hamilton-Jacobi technique or in terms of a cavity function and the energy can still be performed, but the simple mathematical treatment of the SK, ultimately due to the second order nature of its phase transition allowing expansions in small overlaps, is lost whenever $p > 2$ because the transition becomes first order.

This is an interesting remark because, when using the replica trick, the p-spin models are always thought of as simpler cases. This has a deep physical counter-part: the covariance of the Hamiltonian is given by the overlap to the power p, so, for example, the SK Hamiltonian has covariance $\sim Nq^2$, while a generic p-spin model has a covariance $\sim Nq^p$. Of course, as the overlap is bounded by one, this means that by increasing the order of interactions p, these correlations become more and more negligible until, in the limit $p \to \infty$, one recovers an uncorrelated model, i.e., the random energy model.[13] The latter is analytically solvable without either replica tricks or cavity field techniques.

Through a direct calculation (by applying Wick theorem), we can check that the normalization of the Hamiltonian ensures a correct volume scaling for the energy such that $\lim_{N \to \infty} \langle -H_N(\sigma, J)/N \rangle \leq c \in \mathbb{R}$.

All physical information is encoded in the free energy density $f(\beta) = \lim_{N \to \infty} f_N(\beta)$.

Definition 2: The free energy density $f_N(\beta)$ at finite volume N, which is a function of the inverse temperature $\beta = 1/T$, is defined as

$$f_N(\beta) \equiv -\frac{1}{\beta N} \mathbb{E} \log Z_N(\beta, J) \equiv -\frac{1}{\beta N} \mathbb{E} \log \sum_\sigma e^{-\beta H_N(\sigma, J)}, \tag{2}$$

where Z_N is called the partition function and $\mathbb{E}$ stands for the expected value with respect to all the J's. As usual, the sum is over the 2^N configurations $\sigma = \{\sigma_1, \sigma_2, \ldots, \sigma_N\}$ of the spins. Sometimes it is more convenient to deal with the "pressure" $\alpha(\beta) = \lim_{N\to\infty} \alpha_N(\beta) = \lim_{N\to\infty} -\beta f_N(\beta)$. These are the so-called *quenched* free energy/pressure, where the disorder is "frozen" and which are more difficult to compute than the *annealed* ones, where the expectation is taken directly in the partition function. Using the property $\mathbb{E} \exp \lambda z = \exp \lambda^2/2$ valid for a standard random variable z, the computation of the *annealed* free energy density $f_A(\beta)$ is in fact straightforward.

Lemma 1: The annealed free energy density is given by

$$-\beta f_A(\beta) \equiv \lim_{N\to\infty} \frac{1}{N} \log \mathbb{E} Z_N(\beta, J) = \log 2 + \beta^2/4. \tag{3}$$

We notice that when β is sufficiently small, namely, at high temperature, this result coincides with that obtained for the quenched average. Physically speaking, when the temperature is high enough, spins are no longer correlated and averaging the disorder directly in the partition function (which in some way means that it participates to thermodynamic equilibrium) and then taking the logarithm is the same as averaging $\log Z_N$.

Definition 3: If $F(\sigma)$ is a (real-valued) physical observable, we denote the Boltzmann average with

$$\omega(F(\sigma)) = (1/Z_N(\beta, J)) \sum_\sigma F(\sigma) \exp(-\beta H_N(\sigma, J)). \tag{4}$$

This can be generalized by considering two or more independent replicas of the system with the same disorder, so that if $F(\sigma, \sigma')$ is an observable depending on the configuration of two replicas σ, σ', its Boltzmann average is $\Omega(F(\sigma, \sigma')) \equiv (1/Z_N^2(\beta, J)) \sum_\sigma \sum_{\sigma'} F(\sigma, \sigma') \exp(-\beta H(\sigma) - \beta H(\sigma'))$. Notice that, even if we did not write it explicitly, ω depends on the disorder J, too. We denote the average over the disorder with brackets: $\langle F(\sigma, \sigma') \rangle \equiv \mathbb{E} \Omega(F(\sigma, \sigma'))$.

III. THERMODYNAMIC LIMIT

The quantity one is typically interested in is actually the thermodynamical limit of the quenched free energy

$$f(\beta) = \lim_{N\to\infty} f_N(\beta). \tag{5}$$

Guerra and Toninelli first were able to find out a mathematical strategy to prove the existence of the thermodynamic limit for these frustrated systems,[19,20] which, for the sake of completeness, we briefly outline the following.

Theorem 1: *The thermodynamic limit of the free energy density exists and it is equal to its infimum*

$$\lim_{N\to\infty} f_N(\beta) = \inf_N \left(-\frac{1}{\beta N} \mathbb{E} \log Z_N(\beta, J) \right). \tag{6}$$

Proof: Let us consider two separated systems, one constituted by N elements and the other one by two independent subsystems (labeled by 1 and 2) with $N = N_1 + N_2$ elements. The Hamiltonian and free energy density for the first system correspond to expressions (1), (2), while for the second system, indicating with $\sigma^{(1)}$ and $\sigma^{(2)}$ the two subsets $\{\sigma_1, \ldots, \sigma_{N_1}\}$ and $\{\sigma_{N_1+1}, \ldots, \sigma_N\}$, we have a

Hamiltonian

$$H_{N_1}(\sigma^{(1)}, J') + H_{N_2}(\sigma^{(2)}, J'') = -\sqrt{\frac{p!}{2N_1^{p-1}}} \sum_{1 \leq i_1 < \cdots < i_p \leq N_1} J'_{i_1 \ldots i_p} \sigma_{i_1} \ldots \sigma_{i_p}, \tag{7}$$

$$-\sqrt{\frac{p!}{2N_2^{p-1}}} \sum_{N_1 < i_1 < \cdots < i_p \leq N} J''_{i_1 \ldots i_p} \sigma_{i_1} \ldots \sigma_{i_p}, \tag{8}$$

where the J' and J'' are distributed as the J, and an extensive free energy given by

$$\mathbb{E} \log \sum_{\sigma^{(1)}} \exp(-\beta H_{N_1}(\sigma^{(1)}, J')) + \mathbb{E} \log \sum_{\sigma^{(2)}} \exp(-\beta H_{N_2}(\sigma^{(2)}, J'')). \tag{9}$$

Let us introduce a new fundamental quantity, called overlap, which measures the correspondence between two configurations of spins belonging to different replicas of the system

Definition 4: The overlap $q_{\sigma\sigma'}$ between two configurations σ and σ' is defined as

$$q_{\sigma\sigma'} \equiv \frac{1}{N} \sum_{i=1}^{N} \sigma_i \sigma_i'. \tag{10}$$

In the same way, we define overlaps for the two subsystems 1 and 2 making up the second system as

$$q_{\sigma\sigma'}^{(1)} = \frac{1}{N_1} \sum_{i=1}^{N_1} \sigma_i \sigma_i', \tag{11}$$

$$q_{\sigma\sigma'}^{(2)} = \frac{1}{N_2} \sum_{i=N_1+1}^{N} \sigma_i \sigma_i'. \tag{12}$$

Choosing a proper free energy, which for $t \in [0, 1]$ interpolates between the free energies of the two systems presented before,

$$\frac{1}{N} \mathbb{E} \log Z_N(t) = \frac{1}{N} \mathbb{E} \log \sum_{\sigma} \exp\left[\beta \sqrt{t} H_N(\sigma, J) + \beta \sqrt{1-t} (H_{N_1}(\sigma^{(1)}, J') + H_{N_2}(\sigma^{(2)}, J')) \right], \tag{13}$$

we can easily compute its derivative with respect to the parameter t,

$$\frac{d}{dt} \frac{1}{N} \mathbb{E} \log Z_N(t) = -\frac{\beta^2}{4} \left(\langle q_{12}^p \rangle_t - \frac{N_1}{N} \langle (q_{12}^{(1)})^p \rangle_t - \frac{N_2}{N} \langle (q_{12}^{(2)})^p \rangle_t \right), \tag{14}$$

where $\langle . \rangle_t$ is the average over all the disorder J, J', J'' of the generalized interpolating Boltzmann state. Since the function $q \to q^p$ is convex for even p, and

$$q_{\sigma\sigma'} = \frac{N_1}{N} q_{\sigma\sigma'}^{(1)} + \frac{N_2}{N} q_{\sigma\sigma'}^{(2)} \tag{15}$$

the derivative of the interpolating free energy is always non-negative

$$\frac{d}{dt} \frac{1}{N} \mathbb{E} \log Z_N(t) \geq 0. \tag{16}$$

Integrating this equation between 0 and 1, it is straightforward to show that the thermodynamic pressure is superadditive

$$N \alpha_N(\beta) \geq N_1 \alpha_{N_1}(\beta) + N_2 \alpha_{N_2}(\beta).$$

Hence, being $\alpha(\beta) = -\beta f(\beta)$, the quenched free energy is sub-additive in the system size. By noticing that it is also limited, e.g., by its annealed value $\alpha(\beta) \leq \log 2 + \beta^2/4$ (see Eq. (3)), the existence of its thermodynamic limit is shown, mirroring the original scheme by Guerra and Toninelli.[20]   □

IV. FIRST APPROACH: THE HAMILTON-JACOBI TECHNIQUE

We now consider the analogy between the p-spin glass model and the a proper mechanical system, obeying a certain Hamilton-Jacobi equation. Interestingly, the potential in this equation is related to the fluctuations of the order parameter for the corresponding thermodynamic system. As we will see, neglecting this potential we will be able to reconstruct the free energy density for the original model.

To this aim, let us consider the interpolating partition function, depending on the non-negative parameters t and x (which symbolically may be thought of as a fictitious space-time continuum),

$$Z_N(t, x) = \sum_\sigma \exp\left(\sqrt{\frac{tp!}{2N^{p-1}}} \sum_{i_1 < \cdots < i_p}^{1,N} J_{i_1 \ldots i_p} \sigma_{i_1} \ldots \sigma_{i_p} + \sqrt{x} \left(\frac{p}{2} Q^{p-2}(\beta) \right)^{1/4} \sum_i J_i \sigma_i \right). \quad (17)$$

The J_i's are independent random variables, with the same distribution as the $J_{i_1 \ldots i_p}$, and represent an external random field, while $Q(\beta)$ is a regular function of β, which we will later identify with the average overlap between two replicas of the system endowed with the same disorder. Note that we omitted to write explicitly the dependence of Z_N on β and on the J's and we will refer to the free energy both at finite size and in the infinite volume limit when there is no danger of confusion.

We may consequently define an interpolating free energy as the following.

Definition 5: The interpolating free energy density is defined as

$$\alpha_N(t, x) \equiv \frac{1}{N} \mathbb{E} \log Z_N(t, x), \quad (18)$$

where the expectation $\mathbb{E}$ is taken with respect to all the J's, that is with respect to the mutual interactions between spins as well as on the external random fields. It is immediate to see that the true physical free energy is obtained by taking $t = \beta^2$ and $x = 0$, so our strategy will consist in computing the interpolating free energy (18) and obtaining the statistical mechanics by choosing the right values of the parameters t, x.

We may now proceed to compute the derivatives of α with respect to the parameters. With an integration by parts, and neglecting terms which are unimportant in the thermodynamic limit, we obtain the following.

Lemma 2: The derivatives of $\alpha(t, x)$ with respect to the parameters t, x are

$$\partial_t \alpha(t, x) = \frac{1}{4} \left(1 - \langle q_{\sigma\sigma'}^p \rangle_{t,x} \right), \quad (19)$$

$$\partial_x \alpha(t, x) = \frac{1}{2} \left(\frac{p}{2} Q^{p-2}(\beta) \right)^{1/2} \left(1 - \langle q_{\sigma\sigma'} \rangle_{t,x} \right), \quad (20)$$

where the generalized brackets $\langle . \rangle_{t,x}$ are meant to weight the observable with the generalized Boltzmann factor implicitly defined in Eq. (17).

We then define a new function, which will play the role of the bridge with a "mechanical" description.

Definition 6: The Hamilton principal function $S(t, x)$ is defined as

$$S(t, x) \equiv 2\alpha(t, x) - x \left[\frac{p}{2} Q^{p-2}(\beta) \right]^{1/2} - \frac{t}{2} \left[1 + \left(\frac{p}{2} - 1 \right) Q^p(\beta) \right]. \quad (21)$$

The derivatives of $S(t, x)$ are immediately deduced by (19) and (20),

$$\partial_t S(t, x) = -\frac{1}{2} \langle q_{\sigma\sigma'}^p \rangle_{t,x} - \left(\frac{p}{4} - \frac{1}{2} \right) Q^p(\beta), \quad (22)$$

$$\partial_x S(t, x) = -\left(\frac{p}{2} Q^{p-2}(\beta) \right)^{1/2} \langle q_{\sigma\sigma'} \rangle_{t,x}. \quad (23)$$

Lastly, we introduce a proper potential.

Definition 7: The potential V(t, x) for the mechanical problem is defined as

$$V(t,x) \equiv \frac{1}{2}\left(\langle q_{\sigma\sigma'}^p \rangle_{t,x} - Q^p(\beta)\right) + \frac{p}{4}\left(Q^p(\beta) - Q^{p-2}(\beta)\langle q_{\sigma\sigma'}\rangle_{t,x}^2\right). \tag{24}$$

With these definitions, we are now able to formulate our problem (solving the thermodynamics of the p-spin model) as a suitable mechanical model.

Proposition 1: The Hamilton principal function S(t, x), together with the potential V(t, x) satisfies the Hamilton-Jacobi equation

$$\partial_t S(t,x) + \frac{1}{2}(\partial_x S(t,x))^2 + V(t,x) = 0. \tag{25}$$

Now we assume that the variance of the generalized overlap vanishes

$$\langle q_{\sigma\sigma'}^2 \rangle_{t,x} = \langle q_{\sigma\sigma'}\rangle_{t,x}^2 \tag{26}$$

and make the identification

$$\langle q_{\sigma\sigma'}\rangle_{t,x} = Q(\beta). \tag{27}$$

These assumptions are very important as they imply, in statistical mechanics, the self-averaging property for the order parameter. Despite we assume them and not prove them, we simply note that, in order to keep finite the potential $V(t, x)$, even in the $p \to \infty$ limit (which is the interesting case of the random energy model), the expression in the brackets of the second term at the rhs of Eq. (24) must vanish, hence recovering our assumption. Under these hypotheses, within the mechanical analogy we are developing, the two terms of the potential $V(t, x)$ vanish allowing the system to a free motion, and the corresponding solution $\bar{S}(t, x)$ is related to the so-called replica-symmetric (RS) free-energy, which is the approximation of the free energy density $f_N(\beta)$ obtained by neglecting overlap fluctuations.

This phenomenology, as it is based on free-field propagation, gives straight lines as equations of motion

$$x(t) = x_0 - \left(\frac{p}{2}Q^p(\beta)\right)^{\frac{1}{2}} t, \tag{28}$$

where x_0 is the starting point. When $x = 0$ and $t = \beta^2$ (namely, in the point recovering the standard statistical mechanics framework), we get

$$x_0 = \beta^2 \left(\frac{p}{2}Q^p(\beta)\right)^{1/2}. \tag{29}$$

The trajectories (28) do not intersect, as stated in the following theorem.

Theorem 2: *Given a generic point (x, t) with $x \geq 0$, $t \geq 0$, there exists a unique $x_0(x, t)$ such that*

$$x = x_0(x,t) - \langle q_{\sigma\sigma'}\rangle_{0,x_0(x,t)}\, t, \tag{30}$$

and a unique $\bar{q}(x, t) = \langle q_{\sigma\sigma'}\rangle_{0,x_0(x,t)}$ such that

$$\bar{q}(x,t) = \int \frac{dz}{\sqrt{2\pi}} e^{-z^2/2} \tanh^2\left[z\left(\frac{p}{2}Q^{p-2}\right)^{1/4}\sqrt{x + \bar{q}(x,t)t}\right]. \tag{31}$$

The proof is based on the fact that the point $t(x_0)$ at which the free trajectory intersects the t-axis is a monotonous function of the starting point x_0 and can be found in Ref. 18, where the SK case is studied in detail.

The Hamilton-Jacobi equation admits both a Hamiltonian $H(t, x)$ and a Lagrangian $L(t, x)$ description, being, respectively,

$$H(t, x) = \frac{1}{2}\left(\frac{dS(t, x)}{dx}\right)^2 + V(t, x), \tag{32}$$

$$L(t, x) = \frac{1}{2}\left(\frac{dS(t, x)}{dx}\right)^2 - V(t, x). \tag{33}$$

As we are working in the assumption of zero potential, they both correspond to the kinetic energy only.

Definition 8: The kinetic energy $T(t, x)$ is given by

$$T(t, x) \equiv \frac{1}{2}\left(\partial_x S(t, x)\right)^2 = \frac{p}{4}Q^p(\beta). \tag{34}$$

This definition allows the following proposition.

Proposition 2: The solution $\bar{S}(t, x)$ of the Hamilton-Jacobi problem (25) for $V(t, x) = 0$ is obtained by taking the function $S(t, x)$ in one point (e.g., at time $t = 0$ and space $x = x_0$) and adding the Lagrangian times t (strictly speaking it should be times $(t - t_0)$ but we choose $t_0 = 0$),

$$\bar{S}(t, x) = S(0, x_0) + L(t, x)t = S(0, x_0) + T(t, x)t. \tag{35}$$

Remark 1: The freedom in the assignation of the Cauchy problem plays an important role as, by choosing $t_0 = 0$, we are left with a one-body problem in the calculation of the starting point and all the technical difficulties are left in the propagator which, at the replica symmetric level (e.g., $V(t, x) = 0$), simply reduces to the kinetic energy times time.

From (35) and (21), we obtain the corresponding expression for the generalized free energy $\bar{\alpha}(t, x)$ in the replica symmetric (RS) approximation

$$\bar{\alpha}(t, x) = \alpha(0, x_0) - \frac{1}{2}x_0\left(\frac{p}{2}Q^{p-2}(\beta)\right)^{1/2} + \frac{p}{8}Q^p(\beta)t + \frac{1}{2}x\,Q^{\frac{p-2}{2}}(\beta) + \frac{t}{4}\left[1 + \left(\frac{p}{2} - 1\right)Q^p(\beta)\right]. \tag{36}$$

Now it is easy to obtain the physical free energy, since the free energy for $t = 0$ does not contain the interaction and may be computed straightforwardly

$$\alpha(0, x_0) = \log 2 + \int \frac{dz}{\sqrt{2\pi}}e^{-z^2/2}\log\cosh\left[\left(\frac{p}{2}Q^{p-2}(\beta)\right)^{\frac{1}{4}}\sqrt{x_0}z,\right], \tag{37}$$

so that, using (29), we finally find the expression for the physical (RS) free energy as stated by the next theorem.

Theorem 3: *The replica symmetric free energy $\bar{\alpha}(\beta)$ of the p-spin model, obtained under the assumption of zero potential $V(t, x)$ in the mechanical analogy, is encoded in the following formula (which must be extremized over the order-parameter),*

$$\bar{\alpha}(\beta) = \log 2 + \int \frac{dz}{\sqrt{2\pi}}e^{-z^2/2}\log\cosh\left[\beta\left(\frac{p}{2}Q^{p-1}(\beta)\right)^{\frac{1}{2}}z\right] + \frac{\beta^2}{4}\left[1 + (p - 1)Q^p(\beta) - pQ^{p-1}(\beta)\right]. \tag{38}$$

This represents the RS free energy, which corresponds to the true free energy only for sufficiently small values of β.[21] In fact, assuming a vanishing potential corresponds to neglect overlap

fluctuations, and the overlap may be identified with a single value (RS approximation) only for high temperatures.

Proposition 3: Dealing with the overlap, which is related to the initial velocity of the mechanical system, we obtain the following viscous Burger equation which encodes the standard self-consistency procedure of the statistical mechanics counterpart

$$\langle q_{\sigma\sigma'}\rangle_{0,x_0} = Q(\beta) = \int \frac{dz}{\sqrt{2\pi}} e^{-z^2/2} \tanh^2\left[\beta\left(\frac{p}{2}Q^{p-1}(\beta)\right)^{1/2} z\right]. \tag{39}$$

Note that the correct SK replica symmetric free energy and self-consistence equation are recovered for $p = 2$, and both equations predict in this case a phase transition for $\beta = \beta_c = 1$. Above this value the replica symmetric solution ceases to be valid.[14]

It will be useful for a comparison among results gained within this technique and the next one, to have a polynomial expansion through $Q(\beta)$ of the expression (38), hence getting

$$\bar{\alpha}(\beta) \sim \log 2 + \frac{\beta^2}{4} + \frac{\beta^2}{4}(p-1)Q^p(\beta) - \frac{\beta^4}{8}pQ^{2(p-1)} + O(Q^{2(p-1)}). \tag{40}$$

A. Extension to the broken replica symmetry scenario

We now extend the technique presented before to the case of one step of broken replica symmetry, which is known to broaden the correctness of the solution to values of β higher than those required by the previous approximation.[14] In general, it is possible to consider even several steps of broken symmetry, and in fact in the case of the SK model the free energy for $\beta > \beta_c = 1$ is obtained in the limit of infinite iterative steps (this is the so-called full RSB or ∞-RSB scheme[21]). For higher β, a broken replica phase is the correct solution even in the case of $p > 2$, so we want to investigate deeply even the mathematical architecture beyond the preserved replica symmetry. Following the approach of Refs. 10 and 18, we see that in order to account for breaking of this symmetry in our mechanical analogy, we have to enlarge our fictitious space-time by one extra spatial dimension for each step of replica symmetry breaking that we want to consider.

To this task, let us introduce the recursive generalized partition function $\tilde{Z}_N(t; x_1, \ldots, x_K)$, depending on the non-negative real parameters t and $x_1, \ldots, x_K$,

$$\tilde{Z}_N(t; x_1, \ldots, x_K) \equiv \sum_{\sigma} \exp\left[\sqrt{\frac{tp!}{2N^{p-1}}} \sum_{1 \leq i_1 < \cdots < i_p \leq N} J_{i_1, \ldots i_p} \sigma_{i_1} \ldots \sigma_{i_p}\right.$$

$$\left. + \sum_{a=1}^{K} \sqrt{x_a}\left(\frac{p}{2}Q_a^{p-2}\right)^{1/4} \sum_{i=1}^{N} J_i^a \sigma_i\right]. \tag{41}$$

Here, as before, the J_i^a are independent random Gaussian variables with zero mean and unitary variance, and we denote by $\mathbb{E}_a$ the expectation with respect to all the J_i^a for $i = 1, \ldots N$. The $Q_a(\beta)$ are regular functions of β which may be identified with the values around which the overlap distribution accumulates, and they are ordered in the interval [0, 1],

$$0 \equiv Q_0(\beta) < Q_1(\beta) < \cdots < Q_K(\beta) < 1. \tag{42}$$

We denote the Boltzmann-Gibbs state associated with this partition function with $\tilde{\omega}(.)$, and observe that the physical model is recovered by choosing $t = \beta^2$ and $x_a = 0$ for $a = 1, \ldots, K$.

Given the $K + 1$ ordered real numbers within the interval [0, 1], the typical nested structure of the broken replica symmetry is encoded in the generalized partition functions Z_a, defined recursively as

$$Z_a = \left(\mathbb{E}_{a+1} Z_{a+1}^{m_{a+1}}\right)^{1/m_{a+1}}, \tag{43}$$

with $Z_K \equiv \tilde{Z}_N$ and $Z_0 \equiv \exp(\mathbb{E}_1 \log Z_1)$. Note that this last definition is obtained by the general one (43) in the limit of $m_1 \to 0$. The number K of parameters x_a (dimensions of our fictitious space-time) are then related to the number of steps of broken symmetry. It is useful to define the quantities

$$f_a \equiv \frac{Z_a^{m_a}}{\mathbb{E}_a Z_a^{m_a}}, \tag{44}$$

which are all non-negative and not greater than one, and share with the Z_a the property of depending on the random fields J_i^b only with $b \leq a$.

With these definitions, we are now able to introduce the new states.

Definition 9: The generalized Boltmann-Gibbs states are defined as

$$\omega_a(.) \equiv \mathbb{E}_{a+1} \ldots \mathbb{E}_K(f_{a+1} \ldots f_K \tilde{\omega}(.)), \tag{45}$$

$$\omega_K(.) \equiv \tilde{\omega}(.). \tag{46}$$

Again, it is possible to define Boltmann-Gibbs states Ω_a for replicas of the system and, lastly, introduce the averages

$$\langle . \rangle_a \equiv \mathbb{E}_0 \mathbb{E}_1 \ldots \mathbb{E}_a(f_1 \ldots f_a \Omega_a(.)). \tag{47}$$

We now introduce the generalized free energy $\tilde{\alpha}(t; x_1, \ldots, x_K)$ mirroring Sec. IV.

Definition 10: The generalized free energy associated with the partition function Z_0 is defined as follows:

$$\tilde{\alpha}(t; x_1, \ldots, x_K) \equiv \frac{1}{N} \mathbb{E}_0 \log Z_0 = \frac{1}{N} \mathbb{E}_0 \mathbb{E}_1 \log Z_1. \tag{48}$$

We want to use this expression to write down a proper Hamilton-Jacobi equation, generalizing Eq. (25) and find the physical free energy in this enlarged space. To this aim, we need the derivatives of the generalized free energy with respect to the interpolating parameters, whose cumbersome computation is reported in the Appendix.

Lemma 3: The derivatives of the generalized free energy with respect to the interpolating parameters are given by

$$\partial_t \tilde{\alpha}_N(t; x_1, \ldots, x_K) = \frac{1}{4} \left[1 - \sum_{a=1}^{K} (m_{a+1} - m_a) \langle q_{\sigma\sigma'}^p \rangle_a \right], \tag{49}$$

$$\frac{\partial}{\partial x_a} \tilde{\alpha}_N(t; x_1, \ldots, x_K) \equiv \partial_a \tilde{\alpha}_N(t; x_1, \ldots, x_K) = \frac{1}{2} \left(\frac{p}{2} Q_a^{p-2}(\beta) \right)^{1/2} \left[1 - \sum_{b=a}^{K} (m_{b+1} - m_b) \langle q_{\sigma\sigma'} \rangle_b \right], \tag{50}$$

where we recall that

$$\langle q_{\sigma\sigma'}^p \rangle_a = \mathbb{E}_0 \mathbb{E}_1 \ldots \mathbb{E}_a(f_1 \ldots f_a \Omega_a(q_{\sigma\sigma'}^p)) = \mathbb{E}_0 \mathbb{E}_1 \ldots \mathbb{E}_a \left(f_1 \ldots f_a \frac{1}{N^p} \sum_{i_1, \ldots, i_p} \omega_a^2(\sigma_{i_1} \ldots \sigma_{i_p}) \right). \tag{51}$$

We are now ready to introduce the proper Hamilton principal function in this generalized framework.

Definition 11: The Hamilton principal function is defined as follows:

$$S(t; x_1, \ldots, x_K) \equiv 2\tilde{\alpha}(t; x_1, \ldots, x_K) - \sum_{a=1}^{K} x_a \left(\frac{p}{2} Q_a^{p-2}(\beta) \right)^{1/2}$$

$$- \frac{t}{2} \left[1 + \left(\frac{p}{2} - 1 \right) \sum_{a=1}^{K} (m_{a+1} - m_a) Q_a^p(\beta) \right]. \tag{52}$$

Using (49) and (50), we may easily compute its derivatives

$$\partial_t S(t; x_1, \ldots, x_K) = -\frac{1}{2} \sum_{a=1}^{K} (m_{a+1} - m_a) \langle q_{\sigma\sigma'}^p \rangle_a - \left(\frac{p}{4} - \frac{1}{2} \right) \sum_{a=1}^{K} (m_{a+1} - m_a) Q_a^p(\beta),$$

$$\partial_a S(t; x_1, \ldots, x_K) = -\left(\frac{p}{2} Q_a^{p-2}(\beta) \right)^{1/2} \sum_{b=a}^{K} (m_{b+1} - m_b) \langle q_{\sigma\sigma'} \rangle_b, \tag{53}$$

and write down the Hamilton-Jacobi equation which implicitly defines the potential $V(t; x_1, \ldots, x_K)$ to whom our auxiliary mechanical system is subject

$$\partial_t S(t; x_1, \ldots, x_K) + \frac{1}{2} \sum_{a,b=1}^{K} \partial_a S(t; x_1, \ldots, x_K) \times M_{ab}^{-1} \times \partial_b S(t; x_1, \ldots, x_K) + V(t; x_1, \ldots, x_K) = 0. \tag{54}$$

Here M^{-1} is the inverse of the mass matrix, which we are going to define in a convenient way through the kinetic energy $T(t; x_1, \ldots, x_K)$.

Definition 12: The kinetic energy is defined as

$$T(t; x_1, \ldots, x_K) \equiv \frac{1}{2} \sum_{a,b=1}^{K} \partial_a S(t; x_1, \ldots, x_K) \times M_{ab}^{-1} \times \partial_b S(t; x_1, \ldots, x_K). \tag{55}$$

Using (53), $T(t; x_1, \ldots, x_K)$ may be written as

$$T(t; x_1, \ldots, x_K) = \frac{p}{4} \sum_{a,b=1}^{K} (M^{-1})_{ab} \left[Q_a(\beta) Q_b(\beta) \right]^{\frac{p-2}{2}} \sum_{c \geq a}^{K} \sum_{d \geq b}^{K} (m_{c+1} - m_c) \langle q_{\sigma\sigma'} \rangle_c (m_{d+1} - m_d) \langle q_{\sigma\sigma'} \rangle_d$$

$$= \frac{p}{4} \sum_{c,d=1}^{K} D_{cd} (m_{c+1} - m_c) \langle q_{\sigma\sigma'} \rangle_c (m_{d+1} - m_d) \langle q_{\sigma\sigma'} \rangle_d, \tag{56}$$

where we introduced the matrix D, whose generic entry is defined as

$$D_{cd} \equiv \sum_{a=1}^{c} \sum_{b=1}^{d} (M^{-1})_{ab} Q_a^{(p-2)/2}(\beta) Q_b^{(p-2)/2}(\beta). \tag{57}$$

To decouple the overlaps $\langle q_{\sigma\sigma'} \rangle_c$ and $\langle q_{\sigma\sigma'} \rangle_d$, we now pose

$$D_{cd}(m_{c+1} - m_c) = \delta_{cd} Q_c^{(p-2)/2}(\beta) Q_d^{(p-2)/2}(\beta), \tag{58}$$

where δ_{cd} is the Kronecker delta, and then

$$T(t; x_1, \ldots, x_K) = \frac{p}{4} \sum_{a=1}^{K} (m_{a+1} - m_a) \langle q_{\sigma\sigma'} \rangle_a^2 Q_a^{p-2}(\beta). \tag{59}$$

Definition 13: Within this mechanical analogy, the potential $V(t; x_1, \ldots, x_K)$ is, again, directly related to the fluctuations of the overlaps and can be introduced as follows:

$$V(t; x_1, \ldots, x_K) = \frac{1}{2} \sum_{a=1}^{K} (m_{a+1} - m_a)\{\langle q_{\sigma\sigma'}^p \rangle_a - Q_a^p(\beta) + \frac{p}{2}\left[Q_a^p(\beta) - \langle q_{\sigma\sigma'}\rangle_a^2 Q_a^{p-2}(\beta)\right]\}. \quad (60)$$

The condition (58) completely determines the elements of M^{-1}. These are all vanishing except on the diagonal and the terms whose indexes differ only by one, which are symmetric

$$(M^{-1})_{aa} = \frac{1}{m_{a+1} - m_a} + \frac{1}{m_a - m_{a-1}}\left(\frac{Q_a(\beta)}{Q_{a+1}(\beta)}\right)^{p-2} \quad a \geq 2,$$

$$(M^{-1})_{a,a+1} = (M^{-1})_{a,a+1} = -\frac{1}{m_{a+1} - m_a}\left(\frac{Q_a(\beta)}{Q_{a+1}(\beta)}\right)^{(p-2)/2} \quad a \geq 2, \quad (61)$$

and with

$$(M^{-1})_{11} = \frac{1}{m_2}. \quad (62)$$

The matrix M^{-1} clearly admits an inverse, its determinant being non-null

$$\det M^{-1} = \prod_{a=2}^{K+1} (m_a - m_{a-1}) \neq 0. \quad (63)$$

Notice that the elements of M^{-1} and consequently M depend on the overlaps q_a, differently from the case $p = 2$.[10] In this case, in fact, the system energy is no longer a quadratic form in the overlap averages, and this has deep physical consequences; in particular, the phase transition is first order for $p > 2$, meaning that the order parameter changes discontinuously at the critical temperature.

B. The first step of broken replica symmetry

Using results from Sec. IV A, here we find out the expression of the free-energy corresponding to the first step of broken replica symmetry (1-RSB).

Definition 14: The generalized partition function and free-energy are defined as

$$\tilde{Z}_N(t; x_1, x_2) = \sum_{\sigma} \exp\left[\sqrt{\frac{tp!}{2N^{p-1}}} \sum_{1 \leq i_1 < \cdots < i_p \leq N} J_{i_1, \ldots i_p} \sigma_{i_1} \ldots \sigma_{i_p} + \sum_{a=1}^{2} \sqrt{x_a}\left(\frac{p}{2}Q_a^{p-2}\right)^{1/4} \sum_{i=1}^{N} J_i^a \sigma_i\right],$$

$$\tilde{\alpha}_N(t; x_1, x_2) = \frac{1}{Nm}\mathbb{E}_0\mathbb{E}_1 \log \mathbb{E}_2 Z_2^m, \quad (64)$$

where we took $m_2 \equiv m$ and we remind that in this case

$$Z_2 \equiv \tilde{Z}_N,$$
$$0 = Q_0(\beta) < Q_1(\beta) < Q_2(\beta) < 1, \quad (65)$$
$$0 = m_1 < m_2 < 1 = m_3.$$

The principal Hamilton function $S(t, x_1, x_2)$ can be introduced as follows:

Definition 15: The principal Hamilton function for the associated 1-RSB mechanical problem is

$$S(t, x_1, x_2) = 2\tilde{\alpha}(t, x_1, x_2) - \left(\frac{p}{2}Q_1^{p-2}(\beta)\right)^{1/2} x_1 - \left(\frac{p}{2}Q_2^{p-2}(\beta)\right)^{1/2} x_2$$

$$- \frac{t}{2}\left[1 + \left(\frac{p}{2} - 1\right)(mQ_1^p(\beta) + (1 - m)Q_2^p(\beta))\right]. \quad (66)$$

As shown in the general case in Sec. IV A we must now evaluate its derivatives

$$\partial_t S(t, x_1, x_2) = -\frac{1}{2} m \langle q_{\sigma\sigma'}^p \rangle_1 - \frac{1}{2}(1-m)\langle q_{\sigma\sigma'}^p \rangle_2 - \left(\frac{p}{4} - \frac{1}{2}\right)(m Q_1^p(\beta) + (1-m)Q_2^p(\beta)),$$

$$\partial_1 S(t, x_1, x_2) = -m \left(\frac{p}{2} Q_1^{p-2}(\beta)\right)^{1/2} \langle q_{\sigma\sigma'}\rangle_1 - (1-m)\left(\frac{p}{2} Q_2^{p-2}(\beta)\right)^{1/2} \langle q_{\sigma\sigma'}\rangle_2, \qquad (67)$$

$$\partial_2 S(t, x_1, x_2) = -(1-m)\left(\frac{p}{2} Q_2^{p-2}(\beta)\right)^{1/2} \langle q_{\sigma\sigma'}\rangle_2.$$

Proposition 4: Choosing the inverse of the mass matrix (and so the mass matrix itself with the condition (58))

$$M^{-1} = \begin{bmatrix} \frac{1}{m} & -\frac{1}{m}(\frac{Q_1}{Q_2})^{(p-2)/2} \\ -\frac{1}{m}(\frac{Q_1}{Q_2})^{(p-2)/2} & \frac{1}{1-m} + \frac{1}{m}(\frac{Q_1}{Q_2})^{p-2} \end{bmatrix} \Rightarrow M = \begin{bmatrix} m + (1-m)(\frac{Q_1}{Q_2})^{p-2} & (1-m)(\frac{Q_1}{Q_2})^{(p-2)/2} \\ (1-m)(\frac{Q_1}{Q_2})^{(p-2)/2} & 1-m \end{bmatrix}.$$

we can write down explicitly the kinetic term $T(t; x_1, x_2)$ and the potential $V(t; x_1, x_2)$ of the equivalent mechanical system

$$T(t; x_1, x_2) = \frac{p}{4} m \langle q_{\sigma\sigma'}\rangle_1^2 Q_1^{p-2}(\beta) + \frac{p}{4}(1-m)\langle q_{\sigma\sigma'}\rangle_2^2 Q_2^{p-2}(\beta),$$

$$V(t; x_1, x_2) = \frac{1}{2} m \left[\langle q_{\sigma\sigma'}^p \rangle_1 - Q_1^p(\beta) + \frac{p}{2}(Q_1^p(\beta) - \langle q_{\sigma\sigma'}\rangle_1^2 Q_1^{p-2}(\beta)) \right] \qquad (68)$$
$$+ \frac{1}{2}(1-m)\left[\langle q_{\sigma\sigma'}^p \rangle_2 - Q_2^p(\beta) + \frac{p}{2}(Q_2^p(\beta) - \langle q_{\sigma\sigma'}\rangle_2^2 Q_2^{p-2}(\beta)) \right].$$

We can consequently state the following.

Proposition 5: There is a mechanical analogy between the 1-RSB statistical mechanics of the p-spin-glass and an equivalent mechanical system that moves in a two-dimensional space-time with equations of motion given by

$$x_1(t) = x_1^0 + v_1(t; x_1, x_2) t,$$
$$x_2(t) = x_2^0 + v_2(t; x_1, x_2) t. \qquad (69)$$

The corresponding velocities are defined as

$$v_1(t; x_1, x_2) \equiv \sum_{a=1}^{2} (M^{-1})_{1a} \partial_a S(t; x_1, x_2) = -\left(\frac{p}{2} Q_1^{p-2}(\beta)\right)^{1/2} \langle q_{\sigma\sigma'}\rangle_1,$$

$$v_2(t; x_1, x_2) \equiv \sum_{a=1}^{2} (M^{-1})_{2a} \partial_a S(t; x_1, x_2) = \left(\frac{p}{2}\right)^{1/2} \frac{Q_1^{p-2}(\beta)}{Q_2^{(p-2)/2}(\beta)} \langle q_{\sigma\sigma'}\rangle_1 - \left(\frac{p}{2}\right)^{1/2} Q_2^{(p-2)/2}(\beta)\langle q_{\sigma\sigma'}\rangle_2.$$
$$(70)$$

As discussed before, we are interested in studying the free motion, i.e., the motion in absence of potential, and deduce the physical free-energy from the solution of the Hamilton-Jacobi equation

$$\partial_t S(t; x_1, x_2) + \frac{1}{2} \sum_{a,b=1}^{K} \partial_a S(t; x_1, x_2) \times M_{ab}^{-1} \times \partial_b S(t; x_1, x_2).$$

We stress that here the potential is related to a more complex kind of fluctuations of the overlap as we are requiring much more than the simple self-averaging: Physically, we can think at each step of RSB as a refinement, a zoom, in the analysis of the free energy landscape, that allows to see rugged valleys otherwise averaged out and we are asking for adiabatic thermalization within each of these (sub)-valleys ("sub" with respect to the macro-ones already encoded in the RS-approximation). Coherently, a sufficient condition for a vanishing 1-RSB potential is an overlap variance inside the

bracket denoted with $\langle . \rangle_a$ equal to zero and the identification of the averages of the overlap with the functions $Q_a(\beta)$,

$$\langle q_{\sigma\sigma'}^2 \rangle_a = \langle q_{\sigma\sigma'} \rangle_a^2 = Q_a^2(\beta), \ a = 1, 2. \tag{71}$$

In the absence of a potential, the velocities (and so the kinetic energy) are conserved quantities and we can then consider their values at the initial instant $t = 0$, in perfect analogy with the RS case

$$\bar{q}_1 \equiv \langle q_{\sigma\sigma'} \rangle_1(0; x_1^0, x_2^0) = \int d\mu(z_1) \left[\frac{\int d\mu(z_2) \cosh^m \theta(z_1, z_2) \tanh \theta(z_1, z_2)}{\int d\mu(z_2) \cosh^m \theta(z_1, z_2)} \right]^2,$$

$$\bar{q}_2 \equiv \langle q_{\sigma\sigma'} \rangle_2(0; x_1^0, x_2^0) = \int d\mu(z_1) \frac{\int d\mu(z_2) \cosh^m \theta(z_1, z_2) \tanh^2 \theta(z_1, z_2)}{\int d\mu(z_2) \cosh^m \theta(z_1, z_2)}, \tag{72}$$

$$\theta(z_1, z_2) \equiv \sqrt{x_1^0} \left(\frac{p}{2} Q_1^{p-2} \right)^{1/4} z_1 + \sqrt{x_2^0} \left(\frac{p}{2} Q_2^{p-2} \right)^{1/4} z_2,$$

where θ will be defined in Eq. (75) and

$$d\mu(z) = \exp(-z^2/2)dz \tag{73}$$

is the Gaussian measure. This computation essentially leads us to the 1-RSB self-consistence equations for overlaps when considering the statistical-physics point $t = \beta^2$, $x_1 = x_2 = 0$. In this point, and with the condition (71), the equations of motion (69) give

$$x_1^0 = \beta^2 \left(\frac{p}{2} Q_1^p(\beta) \right)^{1/2},$$

$$x_2^0 = \beta^2 \left(\frac{p}{2} \right)^{1/2} Q_2^{\frac{p}{2}}(\beta) - \beta^2 \left(\frac{p}{2} \right)^{1/2} \frac{Q_1^{p-1}(\beta)}{Q_2^{\frac{p-2}{2}}(\beta)}, \tag{74}$$

so that the explicit self-consistence equations contain

$$\theta(z_1, z_2) \equiv \beta \left(\frac{p}{2} \right)^{1/2} z_1 Q_1^{\frac{p-1}{2}}(\beta) + \beta \left(\frac{p}{2} \right)^{1/2} z_2 \sqrt{Q_2^{p-1}(\beta) - Q_1^{p-1}(\beta)}. \tag{75}$$

Remark 2: In the second term of the rhs of Eq. (75) the two overlaps are decoupled, and in the limit $p \to 2$ we get the correct 1-RSB self-consistence equation for the SK model too.

To compute the free-energy we use the usual recipe: As we assume that the mechanical potential is zero, we write (easily) the solution for the Hamilton-Jacobi problem and then we evaluate it in the point $t = \beta^2$, $x_a = 0$. First of all, we need the free-energy at the initial instant, which is straightforward to obtain, since it contains no spin interactions

$$\tilde{\alpha}(0; x_1^0, x_2^0) = \log 2 + \frac{1}{m} \int d\mu(z_1) \log \int d\mu(z_2) \cosh^m \theta(z_1, z_2), \tag{76}$$

with $\theta(z_1, z_2)$ given by (75). The Hamilton function which is solution of (54) for a vanishing potential $V \equiv 0$ is simply given by the function at the initial instant plus the integral of the Lagrangian, (which corresponds to the kinetic energy only), over time

$$S(t; x_1, x_2) = S(0; x_1^0, x_2^0) + \int_0^t ds \, T(s; x_1, x_2) = S(0; x_1^0, x_2^0) + T(0; x_1^0, x_2^0)t, \tag{77}$$

where we used the fact that the kinetic energy is a conserved quantity. We obtain in this way

$$S(t; x_1, x_2) = 2\tilde{\alpha}(0; x_1^0, x_2^0) - \left(\frac{p}{2} Q_1^{p-2}(\beta) \right)^{1/2} x_1^0 - \left(\frac{p}{2} Q_2^{p-2}(\beta) \right)^{1/4} x_2^0$$

$$+ \frac{tp}{4} Q_1^p(\beta) + \frac{tp}{4}(1-m)Q_2^p(\beta), \tag{78}$$

and from this, the generalized free-energy $\tilde{\alpha}(t; x_1, x_2)$,

$$\tilde{\alpha}(t; x_1, x_2) = \frac{1}{2} S(t; x_1, x_2) + \frac{1}{2} \left(\frac{p}{2} Q_1^{p-2}(\beta)\right)^{1/2} x_1 + \frac{1}{2} \left(\frac{p}{2} Q_2^{p-2}(\beta)\right)^{1/2} x_2)$$
$$+ \frac{t}{4} \left[1 + \left(\frac{p}{2} - 1\right)(m Q_1^p(\beta) + (1-m) Q_2^p(\beta))\right]. \tag{79}$$

Then the physical free-energy is easily computed by taking $t = \beta^2$, $x_1 = x_2 = 0$, and we can state the next theorem.

Theorem 4: *Making the assumption of vanishing potential $V(t, x_1, x_2)$ in the mechanical analogy, the corresponding free energy for the p-spin glass model corresponds to the so-called "1-RSB"* and is given by

$$\alpha(\beta) = \log 2 + \frac{1}{m} \int d\mu(z_1) \log \int d\mu(z_2) \cosh^m \left(\beta z_1 Q_1^{(p-1)/2}(\beta) + \beta z_2 \sqrt{Q_2^{p-1}(\beta) - Q_1^{p-1}(\beta)}\right)$$
$$+ \frac{\beta^2}{4} \left[1 + (p-1)m Q_1^p(\beta) + (p-1)(1-m) Q_2^p(\beta) - p Q_1^{p-1}(\beta) - p Q_2^{p-1}(\beta)\right]. \tag{80}$$

We skip here any digression on the physics behind these formulas as these are in perfect agreement with the original investigation by Gardner[14] and by Gross and Mezard,[17] so to highlight only the mathematical methods, to which this paper is dedicated.

C. Conservation laws: Polynomial identities

We conclude this section with an analysis of the conserved quantities deriving from the internal symmetries of the theory. We will approach them as Nöther integrals within the Hamilton-Jacobi formalism, while at the end of next section we will re-obtain (and discuss more deeply) the same constraints within a more familiar thermodynamic approach.

Let us restate the Hamilton-Jacobi equation

$$\partial_t S(t, x) + H(\partial_x S(t, x), t, x) = 0,$$

where the Hamiltonian function reads off as[25]

$$H(\partial_x S(t, x), t, x) = T(t, x) + V(t, x). \tag{81}$$

Hamilton equations are nothing but characteristics given by

$$\begin{cases} \dot{x} = v(t, x), \\ \dot{t} = 1, \\ \dot{P} = -v(t, x)\partial_x v(t, x) - \partial_x V(t, x), \\ \dot{E} = -v(t, x)\partial_x (\partial_t S(t, x)) - \partial_t V(t, x), \end{cases} \tag{82}$$

the latter two equations display space-time translational invariance and express the conservation laws for momentum and energy for our system, further, these can be written in form of streaming equations as

$$\begin{cases} D P(t, x) = -\partial_x V(t, x), \\ D \partial_t S(t, x) = -\partial_t V(t, x). \end{cases}$$

Since we are interested in evaluating the free motion, bearing in mind that $v(x, t) = -\langle q_{12}^{p/2}\rangle$ and $\partial_t S(x, t) = -\frac{1}{2}\langle q_{12}^p\rangle$, so $D = \partial_t - \langle q^{p/2}\rangle\partial_x$, we conclude

$$\begin{cases} D\langle q^{p/2}\rangle = 0, \\ D\langle q^p\rangle = 0, \end{cases} \tag{83}$$

i.e.,

$$\begin{cases} \langle q_{12}^p \rangle - 4\langle q_{12}^{p/2} q_{13}^{p/2} \rangle + 3\langle q_{12}^{p/2} q_{34}^{p/2} \rangle = 0, \\ \langle q_{12}^{2p} \rangle - 4\langle q_{12}^p q_{23}^p \rangle + 3\langle q_{12}^p q_{34}^p \rangle = 0. \end{cases} \tag{84}$$

Remark 3: The orbits of the Nöther groups of the theory coincide with the streaming lines of the Hamilton-Jacobi Hamiltonian, and conservation laws along these lines give well-known identities in the statistical mechanics of the model often known as Ghirlanda-Guerra relations and Aizenman-Contucci identities.[4, 16]

We will deserve Secs. V C and V D of the paper to deepen our understanding of these identities within the smooth cavity field approach, hence we do not investigate them further here.

V. SECOND APPROACH: THE SMOOTH CAVITY FIELD

A. Smooth cavity field and stochastic stability

The main heuristic idea of the *cavity field* method is to look for an explicit expression of $\alpha(\beta) = -\beta f(\beta)$ upon increasing the size of the system from N particles to $N + 1$ (originally the technique was developed by removing a spin instead of adding, hence "cavity," but we will follow the approach recently developed in Ref. 6). As a consequence, within this framework attention will be payed at the system size and all the N dependencies will be explicitly introduced.

On the other hand, in order to formulate stochastic stability, we have to consider the statistical properties of the system with a Hamiltonian given by the original Hamiltonian H plus a random perturbation $\tilde{H}$ so to write $H' = H + \epsilon \tilde{H}$. Stochastic stability states that all the properties of the system are smooth functions of ϵ around $\epsilon = 0$, after the appropriate averages over the original Hamiltonian and the random Hamiltonian have been taken. We stress that, even though initially it was only postulated,[4] stochastic stability has recently[11] been rigorously proven for a wide class of disordered Hamiltonians.

Our idea, to be explored in detail later on, is that for a system with a gauge-invariant Hamiltonian (like the even p-spin model at zero external field) we can choose, as generic random perturbation $\tilde{H}$ in the stochastic stability approach, a term proportional to $\sum_{i_1 < \cdots < i_{p-1}} J_{i_1,\ldots,i_{p-1}} \sigma_{i_1} \ldots \sigma_{i_{p-1}}$. Here the $J_{i_1,\ldots,i_{p-1}}$ are random fields, taken from the same Gaussian i.i.d. distribution as the original $J_{i_1,\ldots,i_p}$. The key insight is that this is a "hidden" cavity field: by applying the transformation $\sigma_i \to \sigma_i \sigma_{N+1} \,\forall i$ (which leaves the Hamiltonian H invariant), it is possible to switch the stochastic stability approach into the standard cavity field approach. As we are going to see, this technique offers more freedom than the two single, non-interacting, approaches as we can turn one into the other as desired.

To explain the method, we need some preliminary definitions. First, let us introduce an extended partition function that includes an interaction with an added hidden spin σ_{N+1} through a control parameter $t \in [0, \beta^2]$ such that for $t = 0$ we have the classical partition function of N spins, while for $t = \beta^2$ we get the partition function (times one half) for the larger system, with a little temperature shift which vanishes in the thermodynamic limit

$$Z_N(\beta, t) = \sum_\sigma e^{-\beta H_N(\sigma; J) + \sqrt{\frac{t p!}{2N^{p-1}}} \sum_{1 \le i_1 < \cdots < i_{p-1} \le N} J_{i_1,\ldots,i_{p-1}} \sigma_{i_1} \ldots \sigma_{i_{p-1}}}. \tag{85}$$

Indeed, when $t = \beta^2$, by redefining $J_i \to J_{i,N+1}$ and making the transformation $\sigma_i \to \sigma_i \sigma_{N+1} \,\forall i$, we obtain the partition function for a system of $N + 1$ spins at a shifted temperature β^* such that

$$\beta^* = \beta \left((N+1)/N \right)^{\frac{p-1}{2}} \to \beta \text{ for } N \to \infty. \tag{86}$$

The only other, trivial, difference is that of course the sum over σ_{N+1} in the partition function for $N + 1$ spins gives an additional factor 2.

Next, we state the two key symmetries whose breaking we will be concerned with. These apply to the unperturbed ($t = 0$) system; recall that $\langle . \rangle \equiv \mathbb{E}\omega(.)$.

Proposition 6: The averages $\langle \cdot \rangle$ are replica-symmetric, i.e., invariant under permutation of replicas. In other words, for any function $F_s(\{q_{ab}\})$ of the overlaps among s replicas and any permutation g of s elements, $\langle F_s(\{q_{ab}\}) \rangle = \langle F_s(\{q_{g(a)g(b)}\}) \rangle$.

Note that there is no issue with replica symmetry breaking here, as we are concerned with *real* replicas.

Proposition 7: The averages $\langle \cdot \rangle$ are invariant under gauge transformation, i.e., for any assignment of the $\epsilon^a = \pm 1$ we have

$$\langle F_s(\{q_{ab}\}) \rangle = \langle F_s(\{\epsilon^a \epsilon^b q_{ab}\}) \rangle. \tag{87}$$

This second symmetry is a consequence of the fact that the Hamiltonian (in zero field) is even in the spins, i.e., it remains unchanged when we transform $\sigma_i^a \to \epsilon^a \sigma_i^a$. Next we will formalize some terminology and concepts which will be useful for developing our *smooth* version of the cavity method.

Definition 16: We define as "filled" a monomial of the overlaps in which every replica appears an even number of times.

Definition 17: We define as "fillable" a monomial of the overlaps in which the above property is obtainable by multiplying with exactly one two-replica overlap.

Definition 18: We define as "unfillable" a monomial which is neither filled nor fillable.

Polynomials that are sums of filled monomials will themselves be called filled, etc. We give a few examples:

- The monomials q_{12}^p and $q_{12}^p q_{34}^p$ are filled (as p is even by definition).
- The monomial q_{12}^{p-1} is fillable: multiplication by q_{12} gives the filled monomial q_{12}^p. Similarly, $q_{12}^p q_{34}^{p-1}$ is fillable: it is filled by multiplication with q_{34}.
- The following monomials are unfillable: $q_{12}^{p-1} q_{34}^{p-1}$, $q_{12} q_{23} q_{45}$.

Now the plan to gain information on the p-spin-glass free energy as follows: First, we define the cavity function and we prove some properties (related stochastic stability) of the classes of overlap monomials defined above. Then, we show that the free energy can be written as the internal energy plus the cavity function, and lastly, we expand the cavity function through the overlap monomials. Merging all together we have an irreducible expression of the free energy in terms of overlap monomials (which physically correspond to overlap correlation functions).

Definition 19: We define the cavity function $\Psi_N(\beta, t)$ *as*

$$\Psi_N(\beta, t) = \mathbb{E}[\ln \omega(e^{\sqrt{\frac{tp!}{2N^{p-1}}} \sum_{1 \le i_1 < \cdots < i_{p-1} \le N} J_{i_1,\ldots,i_{p-1}} \sigma_{i_1} \ldots \sigma_{i_{p-1}}})] = \mathbb{E}\left[\ln \frac{Z_N(\beta, t)}{Z_N(\beta)}\right]. \tag{88}$$

Definition 20: We define the generalized Boltzmann state that corresponds to the partition function (85) as

$$\omega_t(F) = \frac{\omega(F e^{\sqrt{\frac{tp!}{2N^{p-1}}} \sum_{1 \le i_1 < \cdots < i_{p-1} \le N} J_{i_1,\ldots,i_{p-1}} \sigma_{i_1} \ldots \sigma_{i_{p-1}}})}{\omega(e^{\sqrt{\frac{tp!}{2N^{p-1}}} \sum_{1 \le i_1 < \cdots < i_{p-1} \le N} J_{i_1,\ldots,i_{p-1}} \sigma_{i_1} \ldots \sigma_{i_{p-1}}})}, \tag{89}$$

where F is a generic function of the N-spin configuration σ.

The next step is to motivate why we have introduced these definitions. We will first state two Theorems (5 and 6) that show that the filled and the fillable monomials have peculiar properties. Monomials in the first class do not depend on the perturbation (i.e., they are stochastically stable), while those in the second class become filled (via the $\sigma_i \to \sigma_i \sigma_{N+1}$ gauge transformation) in the thermodynamic limit.

Theorem 5: *In the $N \to \infty$ limit, the averages $\langle Q \rangle$ of the filled monomial Q are t-independent for almost all values of β, such that*

$$\lim_{N \to \infty} \partial_t \langle Q \rangle_t = 0.$$

Proof: We will prove the theorem in a key case, namely, for $Q = q_{12}^p$, and refer to Ref. 6 for further generalizations. Let us write the cavity function as

$$\Psi_N(\beta, t) = \mathbb{E}[\ln Z_N(\beta, t)] - \mathbb{E}[\ln Z_N(\beta)] \tag{90}$$

and take its derivative with respect to β (writing again $\langle . \rangle_t \equiv \mathbb{E}\omega_t(.)$), we have

$$\partial_\beta \Psi_N(\beta, t) = \frac{\beta N}{2}(\langle q_{12}^p \rangle - \langle q_{12}^p \rangle_t). \tag{91}$$

We want to show now that the function $\Upsilon_N(\beta, t) = \langle q_{12}^p \rangle - \langle q_{12}^p \rangle_t$ vanishes for $N \to \infty$. From Eq. (91), we have

$$\Upsilon_N(\beta, t) = \frac{4}{N} \partial_{\beta^2} \Psi_N(\beta, t) \tag{92}$$

and integrating this in a generic interval $[\beta_1^2, \beta_2^2]$ gives

$$\int_{\beta_1^2}^{\beta_2^2} \Upsilon_N(\beta, t) d\beta^2 = \frac{4}{N}[\Psi_N(\beta_2, t) - \Psi_N(\beta_1, t)]. \tag{93}$$

To finish the proof, we show that $\Psi_N(\beta, t)$ is of order unity. The simplest way to do this is by looking at its "streaming," i.e., its variation with t. By a direct calculation, one finds

$$\frac{d\Psi_N(\beta, t)}{dt} = \frac{1}{2}\mathbb{E}\left[1 - \frac{1}{N} \sum_{1 \leq i_1 < \cdots < i_{p-1} \leq N} \omega_t^2(\sigma_{i_1} \ldots \sigma_{i_{p-1}}) \right] = \frac{1}{2}(1 - \langle q_{12}^{p-1} \rangle_t). \tag{94}$$

Hence, since $\langle q_{12}^{p-1} \rangle_t \in [-1, 1]$, and with $\Psi_N(\beta, 0) = 0$ (due to $Z_N(\beta, t = 0) = Z_N(\beta)$), we have $0 \leq \Psi_N(\beta, t) \leq t$. Therefore, the rhs of (93) goes to zero for $N \to \infty$, and the same holds for the average of $\Upsilon_N(\beta, t)$ over any small temperature interval (with the exception of singularities). Consequently, $\Upsilon_N(\beta, t)$ itself goes to zero, implying the claimed t-independence of the filled overlap monomials $\langle q_{12}^p \rangle_t \to \langle q_{12}^p \rangle$. $\qquad\square$

The next theorem is crucial for this section, so we first prove a lemma which contains the core idea. We temporarily introduce subscripts on the Boltzmann states to clearly distinguish the different quantities considered.

Lemma 4: Let $\omega_{N, \beta}(.)$ and $\omega_{N, \beta, t}(.)$ be the Boltzmann states defined, on a system of N spins, respectively, by the canonical partition function and by the extended one (85). Consider a set of r distinct spin sites $\{i_1, .., i_r\}$ with $1 \leq r \leq N$. Then for $t = \beta^2$, the extended state becomes comparable to the canonical state of an $N + 1$ spin system, in that the following relation holds:

$$\omega_{N, \beta, t=\beta^2}(\sigma_{i_1} \cdots \sigma_{i_r}) = \omega_{N+1, \beta^*}(\sigma_{i_1} \cdots \sigma_{i_r} \sigma_{N+1}^r). \tag{95}$$

Note that the r in the last factor is an exponent, not a replica index, so that $\sigma_{N+1}^r = 1$ if r is even and $\sigma_{N+1}^r = \sigma_{N+1}$ if r is odd.

Proof: The proof is based on an application of the gauge symmetry, i.e., the substitution $\sigma_i \rightarrow \sigma_i \sigma_{N+1}$. Let us write out explicitly the lhs of Eq. (95), abbreviating $\pi \equiv \sigma_{i_1} \cdots \sigma_{i_r}$,

$$\omega_{N,\beta,t=\beta^2}(\pi) = \mathbb{E} \frac{\sum_\sigma \pi e^{-\beta H_N(\sigma,J) + \beta \sqrt{\frac{p!}{2N^{p-1}}} \sum_{1 \le i_1 < \cdots < i_{p-1} \le N} J_{i_1,\ldots,i_{p-1}} \sigma_{i_1} \ldots \sigma_{i_{p-1}}}}{\sum_\sigma e^{-\beta H_N(\sigma,J) + \beta \sqrt{\frac{p!}{2N^{p-1}}} \sum_{1 \le i_1 < \cdots < i_{p-1} \le N} J_{i_1,\ldots,i_{p-1}} \sigma_{i_1} \ldots \sigma_{i_{p-1}}}}. \tag{96}$$

Introducing a sum over σ_{N+1} into the numerator and the denominator (which is the same as multiplying and dividing by 2 because there is no dependence on σ_{N+1}) and making the transformation $\sigma_i \rightarrow \sigma_i \sigma_{N+1}$, the factor π in the numerator is transformed into $\pi \sigma_{N+1}^r$. The exponential becomes the extended Boltzmann factor of an $(N + 1)$-spin system at the modified temperature (86), so that

$$\omega_{N,\beta,t=\beta^2}(\pi) = \omega_{N+1,\beta^*}(\pi \sigma_{N+1}^r) \tag{97}$$

as claimed.   □

Using this lemma, it is straightforward to prove the following theorem, whose proof we omit as it is identical to the one shown in Ref. 6.

Theorem 6: *Let Q be a fillable overlap monomial, such that $q_{ab}Q$ is filled. Then for $N \rightarrow \infty$,*

$$\langle Q \rangle_{t=\beta^2} = \langle q_{ab} Q \rangle, \tag{98}$$

where the average on the right is evaluated in the canonical Boltzmann state ($t = 0$). We will refer to this property as saturability.

To motivate physically why Theorem 5 should indeed be true for all filled monomials, let us make a clear example: Suppose that such a monomial Q is a function of overlaps among *s* replicas. Consider as before the Boltzmann measure perturbed by a smooth cavity field and call σ^a the *N*-spin configuration of the replica *a*. We apply the gauge transformation $\sigma_i^a \rightarrow \sigma_i^a \sigma_{N+1}^a$, calling $\sigma_+^a = (\sigma_1^a, \ldots, \sigma_{N+1}^a)$ the enlarged spin vector obtained. The key feature of a filled monomial Q is that it is left invariant by this transformation, so that (all sums run over $a = 1 \ldots s$ and $i = 1 \ldots N$)

$$
\begin{aligned}
\langle Q \rangle_t &= \mathbb{E} \frac{\sum_{\{\sigma_{N+1}^a\}} \sum_{\{\sigma^a\}} Q(\{q_{ab}^{(N)}\}) e^{-\sum_a \beta H(\sigma^a) + \sqrt{\frac{tp!}{2N^{p-1}}} \sum_{i_1 < \cdots < i_{p-1}} J_{i_1 \cdots i_{p-1}} \sigma_{i_1}^a \ldots \sigma_{i_{p-1}}^a}}{\sum_{\{\sigma_{N+1}^a\}} \sum_{\{\sigma^a\}} e^{-\sum_a \beta H(\sigma^a) + \sqrt{\frac{tp!}{2N^{p-1}}} \sum_{i_1 < \cdots < i_{p-1}} J_{i_1 \cdots i_{p-1}} \sigma_{i_1}^a \ldots \sigma_{i_{p-1}}^a}} \\[2mm]
&= \mathbb{E} \frac{\sum_{\{\sigma_+^a\}} Q(\{q_{ab}^{(N)}\}) e^{-\sum_a \beta H(\sigma^a) + \sqrt{\frac{tp!}{2N^{p-1}}} \sum_{i_1 < \cdots < i_{p-1}} J_{i_1 \cdots i_{p-1}} \sigma_{i_1}^a \ldots \sigma_{i_{p-1}}^a \sigma_{N+1}^a}}{\sum_{\{\sigma_+^a\}} e^{-\sum_a \beta H(\sigma^a) + \sqrt{\frac{tp!}{2N^{p-1}}} \sum_{i_1 < \cdots < i_{p-1}} J_{i_1 \cdots i_{p-1}} \sigma_{i_1}^a \ldots \sigma_{i_{p-1}}^a \sigma_{N+1}^a}} \\[2mm]
&= \mathbb{E} \frac{\sum_{\{\sigma_+^a\}} Q(\{q_{ab}^{(N+1)} + O(N^{-1})\}) e^{-\sum_a \beta H(\sigma^a) + \sqrt{t/N} \sum_{i,a} J_i \sigma_i^a \sigma_{N+1}^a}}{\sum_{\{\sigma_+^a\}} e^{-\sum_a \beta H(\sigma^a) + \sqrt{t/N} \sum_{i,a} J_i \sigma_i^a \sigma_{N+1}^a}} \\[2mm]
&= \mathbb{E} \frac{\sum_{\{\sigma_+^a\}} Q(\{q_{ab}^{(N+1)}\}) e^{-\sum_a \beta^* H_+(\sigma_+^a)}}{\sum_{\{\sigma_+^a\}} e^{-\sum_a \beta^* H_+(\sigma_+^a)}} + O\left(\frac{1}{N}\right),
\end{aligned} \tag{99}
$$

with β^* defined as before and

$$H_+(\sigma_+^a) = -\sqrt{\frac{tp!}{2(N+1)^{p-1}}} \sum_{1 \leq i_1 < \cdots < i_p \leq N} J_{i_1,\ldots,i_p} \sigma_{i_1} \ldots \sigma_{i_p}$$

$$- \sqrt{\frac{tp!}{2(N+1)^{p-1}}} \sum_{1 \leq i_1 < \cdots < i_{p-1} \leq N} \sqrt{\frac{t}{\beta^2}} J_{i_1 \ldots i_{p-1}} \sigma_{i_1} \ldots \sigma_{i_{p-1}} \sigma_{N+1}. \tag{100}$$

For $t = \beta^2$, we have an $N + 1$ spin system at the slightly shifted temperature β^*, and for $N \to \infty$ this will give the same result as for an N spin system at the original temperature up to vanishingly small corrections: $\langle Q \rangle_{t=\beta^2} = \langle Q \rangle + O(1/N)$. For generic nonzero t one has in addition a modified strength of the interaction of one spin (σ_{N+1}) with all others. Also, this should only give $O(1/N)$ corrections because to produce a non-vanishing perturbation, one expects that a finite fraction of spins should have non-standard interaction strengths.

As a final ingredient for later developments, let us show the streaming of a generic observable, i.e., its variation with respect to the parameter t ruling the strength of the smooth cavity perturbation, that we state without the proof as it is a long but straightforward generalization of the once given for instance in Refs. 2 and 6.

Proposition 8: Let F_s be a monomial of overlaps among s replicas; then for any N the following streaming equation for F_s holds:

$$\frac{\langle F_s \rangle_t}{dt} = \langle F_s (\sum_{1 \leq a < b \leq s} q_{ab}^{p-1} - s \sum_{1 \leq a \leq s} q_{a,s+1}^{p-1} + \frac{s(s+1)}{2} q_{s+1,s+2}^{p-1}) \rangle_t. \tag{101}$$

B. Stochastically stable expansions

For the sake of clearness, let us outline briefly the plan for this section: first, we link the free energy, the internal energy, and the cavity function (which carries the information about the entropy). As we are interested in the free energy and an explicit expression for the internal energy is obtained through a direct calculation as

$$\langle H_N(\sigma, J) \rangle = -\frac{\beta}{2} \left(1 - \langle q_{12}^p \rangle \right),$$

our attention is focused on the cavity function: We show that it is possible to represent it in terms of filled overlap monomials. These are evaluated initially in the perturbed Boltzmann state ω_t but because they are stochastically stable according to Theorem 5, we can also evaluate them in the unperturbed state. Adding the internal energy part then gives us the desired expansion of the free energy in terms of overlap correlation functions as stated by the following theorem.

Theorem 7: *Assuming that the infinite volume limit of the cavity function*

$$\Psi(\beta, t = \beta^2) = \lim_{N \to \infty} \Psi_N(\beta, t = \beta^2)$$

is well behaved, the following relation holds in the thermodynamic limit:

$$\alpha(\beta) + \frac{\beta}{2}(p-1)\partial_\beta \alpha(\beta) = \ln 2 + \Psi(\beta, t = \beta^2). \tag{102}$$

Proof: Let us consider the partition function of a system of $N + 1$ spins and at an inverse temperature β^*, which is slightly larger than the "true" inverse temperature β according to (86).

Then, using the gauge transformation $\sigma_i \to \sigma_i \sigma_{N+1}$ in reverse, we get

$$Z_{N+1}(\beta^*) = \sum_{\{\sigma\}, \sigma_{N+1}} e^{\frac{\beta^* \sqrt{p!}}{\sqrt{2(N+1)^{p-1}}} \sum_{1 \le i_1 < \cdots < i_p \le N+1} J_{i_1, \dots, i_p} \sigma_{i_1} \dots \sigma_{i_p}}$$

$$\sim 2 \sum_{\sigma} e^{\frac{\beta \sqrt{p!}}{\sqrt{2N^{p-1}}} \sum_{1 \le i_1 < \cdots < i_p \le N} J_{i_1, \dots, i_p} \sigma_{i_1} \dots \sigma_{i_p}} e^{\frac{\beta \sqrt{p!}}{\sqrt{2N^{p-1}}} \sum_{1 \le i_1 < \cdots < i_{p-1} \le N} J_{i_1, \dots, i_{p-1}} \sigma_{i_1} \dots \sigma_{i_{p-1}}}$$

$$= 2 Z_N(\beta) \omega_{N,\beta}(e^{\frac{\beta \sqrt{p!}}{\sqrt{2N^{p-1}}} \sum_{1 \le i_1 < \cdots < i_{p-1} \le N} J_{i_1, \dots, i_{p-1}} \sigma_{i_1} \dots \sigma_{i_{p-1}}}). \tag{103}$$

Taking logarithms and averaging over the disorder, the last term just becomes the cavity function (as the $J_{i, N+1}$ have the same distribution as the J_i in the original definition)

$$[\mathbb{E} \ln Z_{N+1}(\beta^*) - \mathbb{E} \ln Z_{N+1}(\beta)] + [\mathbb{E} \ln Z_{N+1}(\beta) - \mathbb{E} \ln Z_N(\beta)]$$

$$= \ln 2 + \Psi_N(\beta, t = \beta^2).$$

The first combination in square brackets on the lhs can now be expanded in the small difference

$$\beta^* - \beta = \beta \left[\left(\frac{N+1}{N} \right)^{\frac{(p-1)}{2}} - 1 \right] = (p-1) \frac{\beta}{2N} + O(\frac{1}{N^2}), \tag{104}$$

according to

$$\mathbb{E} \ln Z_{N+1}(\beta^*) - \mathbb{E} \ln Z_{N+1}(\beta) = (p-1) \frac{\beta}{2N} \partial_\beta \mathbb{E} \ln Z_{N+1}(\beta) + O(1/N)$$

$$= (p-1) \frac{\beta}{2} \partial_\beta \alpha_{N+1}(\beta) + O(1/N).$$

The difference in the second set of square brackets will give the pressure $\alpha(\beta)$ for large N, and taking $N \to \infty$ therefore directly gives the statement of the theorem. □

Strictly speaking, the existence of the thermodynamic limit is not sufficient to guarantee that the free energy increments converge, as assumed above. This technical difficulty can be avoided by taking a Cesàro limit (see, for instance, Ref. 9) rather than a standard limit $N \to \infty$, and the large-N value of the cavity function then should be understood in this sense. This theorem states that we need to study the cavity function to extrapolate properties of the free energy. To do this, let us recall its streaming with respect to t, as given in (94),

$$\frac{d \Psi_N(\beta, t)}{dt} = \frac{p}{4} (1 - \langle q_{12}^{p-1} \rangle_t). \tag{105}$$

Since the cavity function vanishes for $t = 0$, it can then be written as

$$\Psi_N(\beta, t) = \frac{p}{4} \int_0^t dt' (1 - \langle q_{12}^{p-1} \rangle_{t'}). \tag{106}$$

The plan now is to expand $\langle q_{12}^{p-1} \rangle_t$ in t, by evaluating successive t-derivatives via the streaming equation (Proposition 8). A key insight that makes this expansion possible is that at $t = 0$ all averages of monomials that are not filled must vanish because they would otherwise acquire a minus sign under a gauge transformation (Proposition 7).

Applying the streaming equation first to $\langle q_{12}^{p-1} \rangle_t$ gives

$$\frac{d \langle q_{12}^{p-1} \rangle_t}{dt} = \langle q_{12}^{p-1} \left(q_{12}^{p-1} - 4 q_{13}^{p-1} + 3 q_{34}^{p-1} \right) \rangle_t, \tag{107}$$

where we have also exploited the permutation symmetry among replicas. As a consequence, because filled monomials do not depend on t in the thermodynamic limit and in β-average, we can write

$\langle q_{12}^{p-1} \rangle_t \sim \langle q_{12}^p \rangle t + O(q^{2p})$, such that the first terms of the cavity function read off as

$$\Psi(t = \beta^2) = \frac{\beta^2}{4} p - \frac{\beta^4}{8} p \langle q_{12}^{2(p-1)} \rangle + O(q_{12}^{2(p-1)}). \tag{108}$$

Hence, we can write the representation of the free energy in terms of irreducible overlap correlation functions as stated in the next proposition.

Proposition 9: The leading terms of the free energy of the p-spin glass model are given by the following expression in terms of overlap correlation functions:

$$\alpha(\beta) = \ln 2 + \frac{\beta^2}{4} \left(1 + (p-1)\langle q^p \rangle - \frac{\beta^2}{2} p \langle q^{2(p-1)} \rangle + O(\langle q_{12}^{2(p-1)} \rangle) \right). \tag{109}$$

Note that this expression coincides with the corresponding expression for the SK model when $p = 2$, in fact in this case $\langle q^p \rangle = \langle q^{2(p-1)} \rangle$ and the coefficient for the second moment, i.e., $\langle q_{12}^2 \rangle$, is given by $(1 - \beta^2)$, which when equal to zero, i.e., at $\beta = 1$, reverses the concavity of the term, implying a second-order phase transition, so that criticality is restored, as expected. Note further that this coincides with the expansion (40) of the free energy previously obtained with the Hamilton-Jacobi technique.

1. Locking of the order parameters

The free energy expression above has an interesting interpretation if we regard the pressure as a function of temperature *and* of all the averages of filled overlap monomials. To emphasize this we write in the following discussion $\alpha(\beta, \langle \cdot \rangle)$ instead of $\alpha(\beta)$; here $\langle \cdot \rangle$ refers to the collection of all (averages of) filled monomials and we associate with any combination of monomials a graph where each node represents a different replica and each link corresponds to an overlap between the connected nodes/replicas.[6] We will show that the total temperature derivative of α equals its partial derivative; in the latter, the graphs are taken as constant, i.e., their temperature dependence is not accounted for. This is reminiscent of the situation where a free energy is expressed as an extremum over some order parameters, and the first order variation with temperature can be found while keeping the order parameters constant. The result we prove shows that the filled graphs in our framework behave similar to such order parameters, even though of course their values are not determined via an extremization.

In order to prove our statement, it is convenient to work with derivatives with respect to β^2; of course β-derivatives can be recovered trivially by multiplying by 2β. From Theorem 7, we have for the pressure the expression

$$\alpha(\beta, \langle . \rangle) = \ln 2 + \Psi(\beta, t = \beta^2) - \frac{\beta}{2}(p-1)\partial_\beta \alpha(\beta). \tag{110}$$

Its total derivative with respect to β^2 is

$$\frac{d}{d\beta^2} \alpha(\beta, \langle . \rangle) = \partial_{\beta^2} \alpha(\beta, \langle . \rangle) + \sum_{\langle . \rangle} \frac{\partial \alpha(\beta, \langle . \rangle)}{\partial \langle . \rangle} \frac{\partial \langle . \rangle}{\partial \beta^2}, \tag{111}$$

where the sum $\sum_{\langle . \rangle}$ runs over all filled graphs. Of course, we already know the value of this total derivative as it is proportional to the internal energy

$$\frac{d}{d\beta^2} \alpha(\beta, \langle . \rangle) = \frac{1}{2\beta} \frac{d\alpha(\beta)}{d\beta} = \frac{1}{4}(1 - \langle q_{12}^p \rangle). \tag{112}$$

But we can also calculate the partial β^2 derivative: from (110),

$$\partial_{\beta^2} \alpha(\beta, \langle . \rangle) = \partial_{\beta^2} \Psi(\beta, t = \beta^2) - \frac{(p-1)}{4}(1 - \langle q_{12}^p \rangle). \tag{113}$$

To understand how to calculate the partial derivative of the cavity function, where all filled monomials are held constant, recall the expression (109). We need to substitute $t = \beta^2$ there as we are concerned

with $\Psi(\beta, t = \beta^2)$. The explicit dependence on β^2 of the result then comes only from the prefactors of the filled graphs, i.e., from the original t-dependence of the cavity function. The latter is already known (see Eq. (105)), and so we get

$$\partial_{\beta^2}\Psi(\beta, t = \beta^2) = \partial_t\Psi(\beta, t) \mid_{t=\beta^2} = \frac{p}{4}(1 - \langle q_{12}^{p-1}\rangle_{t=\beta^2}) = \frac{p}{4}(1 - \langle q_{12}^p\rangle), \qquad (114)$$

where in the last step we have exploited Theorem 6. Inserting the previous expression into (113) shows that the total and partial derivatives of α are indeed the same, as claimed

$$\frac{d}{d\beta^2}\alpha(\beta, \langle.\rangle) = \partial_{\beta^2}\alpha(\beta, \langle.\rangle). \qquad (115)$$

As a consequence, the second term in the rhs of Eq. (111) has to be identically zero

$$\sum_{\langle.\rangle}\frac{\partial\alpha(\beta, \langle.\rangle)}{\partial\langle.\rangle}\frac{\partial\langle.\rangle}{\partial\beta^2} = 0. \qquad (116)$$

We will see in Sec. V D how this relates to the well-known polynomial identities that we revise in Sec. V C.

C. A digression on Ghirlanda-Guerra and Aizenman-Contucci identities

In the $p = 2$ case (namely, the paradigmatic SK model[26]), Parisi went beyond the solution for the free energy and gave an ansatz about the pure states of the model as well, prescribing the so-called ultrametric or hierarchical organization of the phases (see Ref. 21 and references therein). From a rigorous point of view, the closest the community has so far got to ultrametricity is in the proof of identities constraining the probability distribution of the overlaps, namely, the Aizenman-Contucci (AC) and the Ghirlanda-Guerra identities (GG) (see Refs. 4 and 16, respectively). These are consistent with, but weaker than, Parisi's ultrametric structure, despite recent fundamental step forward have been achieved.[22]

In a nutshell, here, we summarize what the GG or AC identities state for the $p = 2$ case. Consider the overlaps among s replicas. Add one replica $s + 1$; then the overlap $q_{a, s+1}$ between one of the first s replicas (say a) and the added replica $s + 1$ is either independent of all other overlaps, or it is identical to one of the overlaps q_{ab}, with b ranging across the first s replicas except a. Each of these cases has equal probability s^{-1}.

This property is very close to the relation obtained within the Parisi picture: Integrating over q_{23} in this equation, the joint probability distribution for the overlaps q_{12} and q_{13} corresponding to the case $s = 2$, $a = 1$ above becomes

$$P(q_{12}, q_{13}) = P(q_{12})\left[\frac{1}{2}\delta(q_{12} - q_{13}) + \frac{1}{2}P(q_{13})\right], \qquad (117)$$

where $P(.)$ is the probability distribution of the overlap between any two replicas. Dividing by $P(q_{12})$ gives the conditional probability $P(q_{13}|q_{12})$, and the formula above then says precisely that the two overlaps are independent with probability one half and identical with the same probability. Even when we consider two overlaps between two distinct pairs of replicas the correlation remains strong; in fact, still following Parisi

$$P(q_{12}, q_{34}) = \frac{2}{3}P(q_{12})P(q_{34}) + \frac{1}{3}P(q_{12})\delta(q_{12} - q_{34}). \qquad (118)$$

D. Zero average polynomials at even *p*

Let us now see how to prove these properties in p-spin glasses (or at least the equality of the second moments of the relevant distributions) following Ghirlanda and Guerra argument.[16] Denote by $e(\sigma) = H_N(\sigma)/N$ the energy density; the dependence on N will be left implicit below. This quantity

is self-averaging

$$\lim_{N\to\infty} (\langle e(\sigma)^2\rangle - \langle e(\sigma)\rangle^2) = 0. \tag{119}$$

Let us sketch an euristic proof of (119),

$$\langle e(\sigma)^2\rangle - \langle e(\sigma)\rangle^2 = \mathbb{E}\omega(e(\sigma)^2) - [\mathbb{E}\omega(e(\sigma))]^2 = \mathbb{E}[\omega(e(\sigma)^2) - \omega^2(e(\sigma))]$$
$$+ [\mathbb{E}\omega^2(e(\sigma)) - (\mathbb{E}\omega(e(\sigma)))^2]. \tag{120}$$

The second term is the variance with the disorder of the Boltzmann average of the energy density and, as $N\to\infty$, it goes to zero. The first term is equal to $-N^{-1}\partial_\beta\mathbb{E}\omega(e(\sigma))$ and, since $\mathbb{E}\omega(e(\sigma))$ is finite, the prefactor N^{-1} forces also this contribution to go to zero as $N\to\infty$. A rigorous proof for the $p=2$ case can be found in Refs. 12 and 24 for a generic even p.

The property (119) is fundamental because it implies, for any function F_s of overlaps among s replicas,

$$\lim_{N\to\infty} (\langle e(\sigma^a)F_s\rangle - \langle e(\sigma)\rangle\langle F_s\rangle) = 0, \tag{121}$$

where by $e(\sigma^a)$ we mean $e(\sigma)$ calculated on replica a, taken to be one of the replicas that appear in F_s. Equation (121) can be obtained easily from the Schwartz inequality

$$\lim_{N\to\infty} (\langle e(\sigma^a)F_s\rangle - \langle e(\sigma)\rangle\langle F_s\rangle)^2 \tag{122}$$

$$= \lim_{N\to\infty} \langle (e(\sigma^a) - \langle e(\sigma)\rangle)F_s\rangle^2 \tag{123}$$

$$\le \lim_{N\to\infty} \langle (e(\sigma^a) - \langle e(\sigma)\rangle)^2\rangle\langle F_s^2\rangle = 0. \tag{124}$$

The first term in (121) can be evaluated again using Gaussian integration by parts

$$\langle e(\sigma^a)F_s\rangle = -\sqrt{\frac{p!}{2N^{p-1}}} \sum_{i_1<\cdots<i_p} \mathbb{E}J_{i_1,\ldots,i_p}\Omega(F_s\sigma_{i_1}^a\ldots\sigma_{i_p}^a) = -\frac{\beta}{2}\langle F_s(\sum_{1\le b\le s} q_{ab}^p - sq_{a,s+1}^p)\rangle, \tag{125}$$

while the second term is simply

$$\langle e(\sigma)\rangle\langle F_s\rangle = -\frac{\beta}{2}(1 - \langle q_{12}^p\rangle)\langle F_s\rangle. \tag{126}$$

Combining Eqs. (125) and (126), we obtain the first type of GG relation

$$\lim_{N\to\infty} \langle F_s\Big(\sum_{1\le b\le s} q_{ab}^p - sq_{a,s+1}^p - (1 - \langle q_{12}^p\rangle)\Big)\rangle = 0. \tag{127}$$

Since F_s is a generic function, this result implies[16] that, conditionally on all the overlaps q_{cd} with $1\le c<d\le s$,

$$\langle q_{a,s+1}^p\rangle = \frac{1}{s}\langle q_{12}^p\rangle + \frac{1}{s}\sum_{1\le b\le s, b\ne a} q_{ab}^p. \tag{128}$$

This is consistent with our description above of the physical content of the GG relations; the particular example $s=2$, $a=1$ corresponds to the second moment of (117).

In the same way it is possible to derive a constraint for averages involving $s+2$ replicas by using

$$\mathbb{E}\Omega(e(\sigma))\Omega(F_s) - \mathbb{E}\Omega(e(\sigma))\mathbb{E}\Omega(F_s) = 0, \tag{129}$$

which is based on the vanishing of the second term of Eq. (120). One obtains the second type of GG identity,

$$\langle F_s\Big(\sum_{1\le b\le s} q_{b,s+1}^p + \langle q_{12}^p\rangle - (s+1)q_{s+1,s+2}^p\Big)\rangle = 0. \tag{130}$$

Again, invoking the arbitrariness of F_s, this tells us that conditional on the overlaps among the first s replicas

$$\langle q^p_{s+1,s+2} \rangle = \frac{1}{s+1} \sum_{1 \leq b \leq s} \langle q^p_{b,s+1} \rangle + \frac{1}{s+1} \langle q^p_{12} \rangle$$

$$= \frac{2}{s+1} \langle q^p_{12} \rangle + \frac{2}{s(s+1)} \sum_{1 \leq a < b \leq s} q^p_{ab}, \tag{131}$$

where the second equation follows by inserting (128). The specific case $s = 2$ corresponds to the second moment of (118) as expected.

Finally, subtracting (129) from (121), which is equivalent to exploiting the vanishing of the first term in Eq. (120), leads to the self-averaging relation

$$\mathbb{E}[\Omega(e(\sigma^a)F_s) - \Omega(e(\sigma^a))\Omega(F_s)] = 0, \tag{132}$$

from which it is possible to obtain, again for some fixed $1 \leq a \leq s$,

$$\langle F_s \Big(\sum_{1 \leq b \leq s, b \neq a} q^p_{ab} - s q^p_{a,s+1} - \sum_{1 \leq b \leq s} q^p_{b,s+1} + (s+1) q^2_{s+1,s+2} \Big) \rangle. \tag{133}$$

Summing over a and dividing by two, this last relation becomes

$$\langle F_s \Big(\sum_{1 \leq a < b \leq s} q^p_{ab} - s \sum_{1 \leq a \leq s} q^p_{a,s+1} + \frac{s(s+1)}{2} q^p_{s+1,s+2} \Big) \rangle = 0, \tag{134}$$

which is the general form of the AC relations. It is interesting to note that the lhs of Eq. (134) equals $2N\beta\partial_\beta \langle F_s \rangle$, as one verifies by direct calculation: As the β-derivative must be $O(1)$ we can then directly argue that (134) vanishes for large N and does so generically as $1/N$.

Moving on to concrete examples, the most famous GG relations are those obtained from $F_s = q^p_{12}$, where the exponent p makes us focus on the energy term of the p-spin model. They are typically written in the form

$$\langle q^p_{12} q^p_{13} \rangle = \frac{1}{2} \langle q^{2p}_{12} \rangle + \frac{1}{2} \langle q^p_{12} \rangle^2, \tag{135}$$

$$\langle q^p_{12} q^p_{34} \rangle = \frac{1}{3} \langle q^{2p}_{12} \rangle + \frac{2}{3} \langle q^p_{12} \rangle^2. \tag{136}$$

Eliminating $\langle q^p_{12} \rangle^2$, we get, as expected, the AC relation for $F_s = q^p_{12}$,

$$\langle q^{2p}_{12} \rangle - 4 \langle q^p_{12} q^p_{13} \rangle + 3 \langle q^p_{12} q^p_{34} \rangle = 0. \tag{137}$$

1. Overlap constraint generators

We now show that within our smooth cavity field framework these relations can be obtained very simply from the stochastic stability of filled monomials (Theorem 5). Specifically, we claim that the AC identities follow from the t-independence that obtains for averages of such monomials when $N \to \infty$, and specifically from the vanishing of the t-derivative at $t = \beta^2$: if F_s is a filled monomial, then

$$\lim_{N \to \infty} \partial_t \langle F_s \rangle|_{t=\beta^2} = 0. \tag{138}$$

This property, for generic t, has already been used in our smooth cavity expression, where we did not evaluate the streaming of filled graphs like q^p_{12} because they are independent of t.

To see that we can also generate constraints for the overlaps, we combine t-independence with the fact that for $t = \beta^2$, by Theorem 6, the perturbed Boltzmann state effectively reverts to the

unperturbed state of an enlarged system. Explicitly, we have by evaluating the t-derivative in (138) using the streaming equation (Theorem 8),

$$\lim_{N\to\infty} \langle F_s \Big(\sum_{1\le a<b\le s} q_{ab}^{p-1} - s \sum_{1\le a\le s} q_{a,s+1}^{p-1} + \frac{s(s+1)}{2} q_{s+1,s+2}^{p-1} \Big) \rangle_{t=\beta^2} = 0. \tag{139}$$

Now, given that F_s is filled, all the terms here are fillable. Because we are evaluating at $t = \beta^2$, then from Theorem 6 their averages reduce to unperturbed averages of the corresponding filled expressions. Filling in this case just means squaring all the overlaps inside the brackets, and so we get directly

$$\lim_{N\to\infty} \langle F_s \Big(\sum_{1\le a<b\le s} q_{ab}^{p} - s \sum_{1\le a\le s} q_{a,s+1}^{p} + \frac{s(s+1)}{2} q_{s+1,s+2}^{p} \Big) \rangle = 0. \tag{140}$$

This is nothing but the general AC relation (134), as claimed.

From the streaming of the simplest filled monomial (i.e., $\langle q_{12}^p \rangle$) we find the first AC relation

$$\lim_{N\to\infty} \partial_t \langle q_{12}^p \rangle_{t=\beta^2} = \lim_{N\to\infty} \langle q_{12}^{2p} - 4q_{12}^p q_{23}^p + 3q_{12}^p q_{34}^p \rangle = 0, \tag{141}$$

which denotes overall a perfect agreement among results from our approach and previous knowledge on p-spin models.

VI. CONCLUSIONS

In recent years, spin-glasses have attracted a growing interest raised as, day after day, these systems are becoming the bricks for building models to describe behavior of complex systems, ranging from biology to economics. As a consequence, there is a need for stronger and stronger analytical methods possibly related with the numerical and experimental findings. This paper was written with the intention of offering a detailed analysis of a well-known model, namely, the p-spin-glass, through two recent methods: the Hamilton-Jacobi technique and the smooth cavity expression. We first provide a picture of the behavior of the p-spin-glass, from a perspective which is intermediate between that of the pure theoretical physicist and that of the rigorous mathematician, hoping to help in bridging the gap between these two approaches. Then, we explain the methods and use them with abundance of details, so to allow the reader to learn them. Indeed, our focus is more on techniques and on their versatility rather than on results themselves, which are mostly already known.[14,17,24]

Summarizing, after a streamlined introduction to the basic properties of the model (expression for the internal energy and convergence of the infinite volume limit), within the Hamilton-Jacobi technique, we obtained analytical expressions for both the RS and the 1-RSB free energies and we gain a clear mathematical control of the underlying physical assumptions. Within the smooth cavity method, we showed how to build the expression of the free energy through overlap correlation functions and we analyzed the polynomial identities, that always develop in frustrated systems, derived as consequences of the stability of the measure $\lim_{N\to\infty} \sum_\sigma$ with respect to small (negligible) random stochastic perturbations.

It is interesting to note that in the mean field techniques we developed, there is a certain degree of complementary between the two approaches as within the former we use a trial overlap coupled with a single particle free spin, while in the latter we use $p - 1$ spins for the cavity: both methods essentially work by reducing the problem to a single-body one, the Hamilton-Jacobi in a direct way, the smooth cavity in a complementary way.

Further investigations should be addressed to the study of the diluted frustrated p-spin model and its relation with K-satisfiability problems and *P/NP* completeness.

ACKNOWLEDGMENTS

APPENDIX: DERIVATIVES OF THE GENERALIZED PARTITION FUNCTION RESPECT TO THE INTERPOLATING PARAMETERS

We show here the derivation of expressions (49) and (50). Let us start with the first one

$$\partial_t \tilde{\alpha}_N = \frac{1}{N} \mathbb{E}_0 Z_0^{-1} \partial_t Z_0. \tag{A1}$$

It is easy to see that

$$Z_a^{-1} \partial_t Z_a = \mathbb{E}_{a+1} f_{a+1} Z_{a+1}^{-1} \partial_t Z_{a+1}, \tag{A2}$$

so that

$$\mathbb{E}_0 Z_0^{-1} \partial_t Z_0 = \mathbb{E}_0 \mathbb{E}_1 \dots \mathbb{E}_K f_1 \dots f_K Z_K^{-1} \partial_t Z_K \equiv \mathbb{E} f_1 \dots f_K Z_K^{-1} \partial_t Z_K \tag{A3}$$

and, remembering that $Z_K \equiv \tilde{Z}_N$,

$$\mathbb{E}_0 Z_0^{-1} \partial_t Z_0 = \frac{1}{2\sqrt{t}} \sqrt{\frac{tp!}{2N}} \sum_{1 \le i_1 < \dots < i_p \le N} \mathbb{E}(f_1 \dots f_K J_{i_1 \dots i_p} \tilde{\omega}(\sigma_{i_1} \dots \sigma_{i_p})). \tag{A4}$$

Integrating by parts the $J_{i_1 \dots i_p}$ inside the expectation becomes a derivative

$$\frac{1}{2\sqrt{t}} \sqrt{\frac{tp!}{2N}} \sum_{1 \le i_1 < \dots < i_p \le N} \left[\sum_{a=2}^K \mathbb{E}(f_1 \dots \partial_{J_{i_1 \dots i_p}} f_a \dots f_K \tilde{\omega}(\sigma_{i_1} \dots \sigma_{i_p})) + \mathbb{E}(f_1 \dots f_K \partial_{J_{i_1 \dots i_p}} \tilde{\omega}(\sigma_{i_1} \dots \sigma_{i_p})) \right]. \tag{A5}$$

We now proceed by computing separately the two addends in the square brackets. for the second term we easily find

$$\partial_{J_{i_1 \dots i_p}} \tilde{\omega}(\sigma_{i_1} \dots \sigma_{i_p})) = \sqrt{\frac{tp!}{2N^{p-1}}} (1 - \tilde{\omega}^2(\sigma_{i_1} \dots \sigma_{i_p})), \tag{A6}$$

while for the first term we have

$$\partial J_{i_1 \dots i_p} f_a = m_a f_a Z_a^{-1} \partial J_{i_1 \dots i_p} Z_a - m_a f_a E_a(f_a Z_a^{-1} \partial J_{i_1 \dots i_p} Z_a). \tag{A7}$$

Now, using the analogous of (A2), one has

$$Z_a^{-1} \partial J_{i_1 \dots i_p} Z_a = \mathbb{E}_{a+1} \dots \mathbb{E}_K (f_{a+1} \dots f_K Z_K^{-1} \partial J_{i_1 \dots i_p} Z_K) \tag{A8}$$

$$= \sqrt{\frac{tp!}{2N^{p-1}}} \omega_a(\sigma_{i_1} \dots \sigma_{i_p}) \tag{A9}$$

and then for $a \ge 2$ (remind that $f_1 = 1$ so its derivative is zero)

$$\partial J_{i_1 \dots i_p} f_a = \sqrt{\frac{tp!}{2N^{p-1}}} m_a f_a (\omega_a(\sigma_{i_1} \dots \sigma_{i_p}) - \omega_{a-1}(\sigma_{i_1} \dots \sigma_{i_p})). \tag{A10}$$

Putting together the terms computed, we find

$$\mathbb{E}_0 Z_0^{-1} \partial_t Z_0 = \frac{1}{4} \frac{p!}{N^{p-1}} \sum_{1 \le i_1 < \dots < i_p \le N} \left[\sum_{a=2}^K \mathbb{E}_0 \dots \mathbb{E}_a (f_1 \dots f_a \omega_a(\sigma_{i_1} \dots \sigma_{i_p}) f_{a+1} \dots f_K \tilde{\omega}(\sigma_{i_1} \dots \sigma_{i_p}) \right.$$

$$- \sum_{a=2}^K \mathbb{E}_0 \dots \mathbb{E}_a (f_1 \dots f_a \omega_a(\sigma_{i_1} \dots \sigma_{i_p}) f_{a+1} \dots f_K \tilde{\omega}(\sigma_{i_1} \dots \sigma_{i_p})$$

$$\left. + 1 - \mathbb{E}_0 \dots \mathbb{E}_K (f_1 \dots f_K \tilde{\omega}^2(\sigma_{i_1} \dots \sigma_{i_p})) \right] \tag{A11}$$

and noting that in the thermodynamic limit $p! \sum_{i_1 < \cdots < i_p} \sim \sum_{i_1,\ldots,i_p}$, we have

$$\partial_t \tilde{\alpha}_N = \frac{1}{4} \left[\sum_{a=1}^{K} m_a (\langle q_{\sigma\sigma'}^p \rangle_a - \langle q_{\sigma\sigma'}^p \rangle_{a-1}) + 1 - \langle q_{\sigma\sigma'}^p \rangle_K \right] \tag{A12}$$

from which the derivation of (49) is straightforward.

Let us now compute the derivatives of the free energy respect to the "spatial" parameters

$$\partial_a \tilde{\alpha}_N = \frac{1}{N} \mathbb{E}_0 Z_0^{-1} \partial_a Z_0$$

$$= \frac{1}{N} \mathbb{E}(f_1 \ldots f_K Z_K^{-1} \partial_a Z_K) \tag{A13}$$

$$= \frac{1}{N} \frac{1}{2\sqrt{x_a}} q^{\frac{p-2}{4}} \mathbb{E}(f_1 \ldots f_K \sum_{i=1}^{N} J_i^a \tilde{\omega}(\sigma_i)),$$

where we used the analogous of (A3). Integrating by parts this becomes

$$\partial_a \tilde{\alpha}_N = \frac{1}{N} \mathbb{E}_0 \sum_i \left[\mathbb{E}_1 \ldots \mathbb{E}_K (\sum_{b=2}^{K} f_1 \ldots \partial_{J_i^a} f_b \ldots f_K \tilde{\omega}(\sigma_i)) \right. \tag{A14}$$

$$\left. + \mathbb{E}_1 \ldots \mathbb{E}_K (f_1 \ldots f_K \partial_{J_i^a} \tilde{\omega}(\sigma_i)) \right].$$

The derivatives of the state and of f_b respect to the random fields are, respectively, given by

$$\partial_{J_i^a} \tilde{\omega}(\sigma_i) = \sqrt{x_a} q_a^{\frac{p-2}{4}} (1 - \tilde{\omega}^2(\sigma_i))$$

$$\partial_{J_i^a} f_b = \begin{cases} m_b f_b (Z_b^{-1} \partial_{J_i^a} Z_b - E_b f_b Z_b^{-1} \partial_{J_i^a} Z_b) & \text{if } a < b, \\ m_b f_b Z_b^{-1} \partial_{J_i^a} Z_b & \text{if } a = b, \\ 0 & \text{if } a > b. \end{cases} \tag{A15}$$

Using again the iterative derivation formula we have

$$Z_b^{-1} \partial_{J_i^a} Z_b = E_{b+1} (f_{b+1} Z_{b+1}^{-1} \partial_{J_i^a} Z_{b+1})$$

$$= E_{b+1} \ldots E_K (f_{b+1} \ldots f_K Z_K^{-1} \partial_{J_i^a} Z_K)$$

$$= \sqrt{x_a} q_a^{\frac{p-2}{4}} E_{b+1} \ldots E_K (f_{b+1} \ldots f_K \tilde{\omega}(\sigma_i)) \tag{A16}$$

$$= \sqrt{x_a} q_a^{\frac{p-2}{4}} \omega_b(\sigma_i),$$

so that

$$\partial_{J_i^a} f_b = \begin{cases} \sqrt{x_a} q_a^{\frac{p-2}{4}} m_b f_b (\omega_b(\sigma_i) - \omega_{b-1}(\sigma_i)) & \text{if } a < b, \\ \sqrt{x_a} q_a^{\frac{p-2}{4}} m_b f_b \omega_b(\sigma_i) & \text{if } a = b, \\ 0 & \text{if } a > b. \end{cases} \tag{A17}$$

Substituting these in the (A14), the first term inside the square brackets becomes

$$\mathbb{E}_1 \ldots \mathbb{E}_K (\sum_{b=2}^{K} f_1 \ldots \partial_{J_i^a} f_b \ldots f_K \tilde{\omega}(\sigma_i)) = \sqrt{x_a} q_a^{\frac{p-2}{4}} \big[m_a \mathbb{E}_1 \ldots \mathbb{E}_a (f_1 \ldots f_a \omega_a^2(\sigma_i))$$

$$+ \sum_{b=a+1}^{K} m_b \mathbb{E}_1 \ldots \mathbb{E}_b (f_1 \ldots f_b \omega_b^2(\sigma_i)) \qquad (A18)$$

$$- \sum_{b=a+1}^{K} m_b \mathbb{E}_1 \ldots \mathbb{E}_{b-1} (f_1 \ldots f_{b-1} \omega_b^2(\sigma_i)) \big]$$

and we find

$$\partial_a \tilde{\alpha}_N = \frac{1}{2} q_a^{\frac{p-2}{2}} \big[m_a \langle q_{\sigma\sigma'} \rangle_a + \sum_{b>a} m_b (\langle q_{\sigma\sigma'} \rangle_b - \langle q_{\sigma\sigma'} \rangle_{b-1}) + 1 - \langle q_{\sigma\sigma'} \rangle_K \big] \qquad (A19)$$

from which, after some manipulations, one can easily obtain the (50).

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[25] Here, we name P our velocity, i.e., the velocity field coincides with the generalized time dependent momentum.

[26] Among the several many body theories developed in statistical mechanics along the years, the mean field two body ones are the most welcome as the Hamiltonian – even though no longer in classical space-time sense – are still quadratic forms such that linear response for the forces is still kept. However, especially in disordered system, many real features of glass forming dynamics seem better reproduced by violation of the linear response and in this sense by p-spin models.