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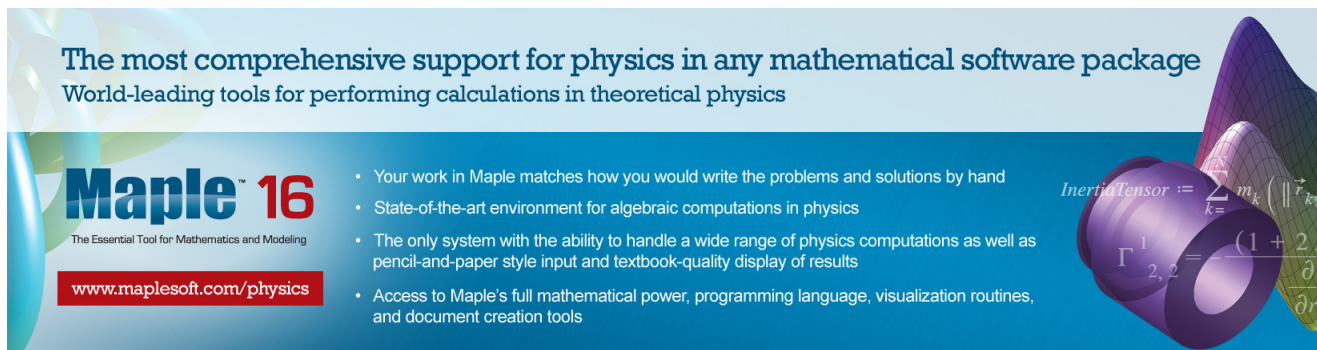
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InertiaTensor := $\sum_{k=1}^n m_k \left(\|\vec{r}_k\|^2 \right)$
 $\Gamma_{2,2}^1 = \frac{(1 + 2)}{\partial r}$



About the ergodic regime in the analogical Hopfield neural networks: Moments of the partition function

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In this paper we introduce and exploit the real replica approach for a minimal generalization of the Hopfield model by assuming the learned patterns to be distributed according to a standard unit Gaussian. We consider the high storage case, when the number of patterns linearly diverges with the number of neurons. We study the infinite volume behavior of the normalized momenta of the partition function. We find a region in the parameter space where the free energy density in the infinite volume limit self-averages around its annealed approximation, as well as the entropy and the internal energy density. Moreover, we evaluate the corrections to their extensive counterparts with respect to their annealed expressions. The fluctuations of properly introduced overlaps, which act as order parameters, are also discussed. © 2008 American Institute of Physics. [DOI: 10.1063/1.3039083]

I. INTRODUCTION

In the past 20 years, from the early work by Hopfield²⁸ and the nowadays historical theory of Amit, Gutfreund, and Sompolinsky (AGS)²⁻⁴ to the modern theory for learning,^{7,15} the neural networks, thought of as spin glasses with a Hebb-like “synaptic matrix,”²⁵ became more and more important in several different contexts, such as artificial intelligence,²⁶ cognitive psychology,²⁷ and problem solving.^{8,17} Despite their fundamental role and due to their very difficult mathematical control, very little is rigorously known about these networks. Along the years several contributions appeared (e.g., Refs. 1, 9–12, and 32–35), often motivated by an increasing understanding of spin glasses (e.g., Refs. 19–21, 31, and 36), and the analysis at low level of stored memories has been achieved.

However, in the high level of stored memories, fundamental inquiries are still rather incomplete. Furthermore, general problems such as the existence of a well defined thermodynamic limit are unsolved, in contrast with the spin glass case.^{22,24}

In this paper, we introduce some techniques (essentially in the real replica framework) developed for the spin glass theory (see, e.g., Refs. 6, 14, 18, and 23) in order to give a description of the Hopfield model in the high temperature region with high level of stored memories (i.e., patterns) without using replica trick.³⁰ We take the freedom of allowing the learned patterns to take all real values, their probability distribution being a standard Gaussian $\mathcal{N}[0, 1]$, and we refer to this minimal generalization as the *analogical* Hopfield model to stress that the memories are no longer discrete as in standard literature.

Within this scenario, we exploit the moment method in order to prove bounds on the critical line for the ergodic phase and give the explicit expression for all the thermodynamical quantities in the infinite volume limit, in complete agreement with the AGS theory. Furthermore we show in

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a simple way self-averaging of free and internal energies and entropy per site and calculate their extensive fluctuations around the annealed expressions, in analogy with what was found by Aizenman *et al.*³⁷ for the Sherrington–Kirkpatrick model in the ergodic region.

We investigate also about the overlap fluctuations in the ergodic region. The paper is organized as follows. In Sec. II we define the analogical Hopfield model, and show that it is equivalent to a bipartite spin glass, where one party is described by Ising spins and the other by Gaussian spins. We introduce also the main thermodynamic quantities and their annealed approximation. In Sec. III we introduce overlaps for replicas of the Ising spins and the Gaussian spins and show how they enter in the expression of thermodynamic quantities, for example, the internal energy. In Sec. IV we state our main results about the validity of the annealed approximation and establish the fluctuations of the extensive thermodynamic variables and the overlaps. In Sec. V we study the momenta of the normalized partition function in the infinite volume limit and prove our results about the annealed approximation. In Sec. VI we prove the log normality of the limiting distribution for the partition function and prove the rest of our results. Finally, Sec. VII is devoted to some conclusion and outlook for future developments.

II. DEFINITION OF THE MODEL

We introduce a large network of N two-state neurons $(1, \dots, N) \ni i \rightarrow \sigma_i = \pm 1$, which are thought of as quiescent (sleeping) when their value is -1 or spiking (emitting a current signal to other neurons) when their value is $+1$. They interact throughout a synaptic matrix J_{ij} defined according to the Hebb rule for learning,

$$J_{ij} = \sum_{\mu=1}^p \xi_i^{\mu} \xi_j^{\mu}. \quad (1)$$

Each random variable $\xi^{\mu} = \{\xi_1^{\mu}, \dots, \xi_N^{\mu}\}$ represents a learned pattern and tries to bring the overall current in the network (or in some part) stable with respect to itself (when this happens, we say we have a retrieval state, see, e.g., Ref. 2). The analysis of the network assumes that the system has already stored p patterns (no learning is investigated here), and we will be interested in the case in which this number increases linearly with respect to the system size (high storage level), so that $p/N \rightarrow \alpha$ as $N \rightarrow \infty$, where $\alpha \geq 0$ is a parameter of the theory denoting the storage level.

In standard literature these patterns are usually chosen at random independently with values ± 1 taken with equal probability of $1/2$. Here, we chose them as taking real values with a unit Gaussian probability distribution, i.e.,

$$d\mu(\xi_i^{\mu}) = \frac{1}{\sqrt{2\pi}} e^{-(\xi_i^{\mu})^2/2}. \quad (2)$$

Of course, avoiding pathological case, in the high storage level and in the high temperature region, the results should show robustness with respect to the particular choice of the probability distribution and we should recover the standard AGS theory. The use of a Gaussian distribution has some technical advantages, as it allows to easily borrow powerful techniques from the spin glass case. The physical interpretation is very simple. While the patterns with ± 1 values describe “images” with white or black pixels, respectively, the patterns with continuous values describe “gray” pixels with a continuously variable luminosity from $-\infty$ (completely black) to $+\infty$ (completely white).

The Hamiltonian of the model involves interactions between any couple of sites according to the definition

$$H_N(\sigma; \xi) = -\frac{1}{N} \sum_{\mu=1}^p \sum_{i<j}^N \xi_i^\mu \xi_j^\mu \sigma_i \sigma_j. \quad (3)$$

By splitting the summations $\sum_{i<j}^N = \frac{1}{2} \sum_{ij}^N - \frac{1}{2} \sum_i^N \delta_{ij}$, we can write the partition function in the following form:

$$Z_{N,p}(\beta; \xi) = \sum_{\sigma} \exp\left(\frac{\beta}{2N} \sum_{\mu=1}^p \sum_{ij}^N \xi_i^\mu \xi_j^\mu \sigma_i \sigma_j - \frac{\beta}{2N} \sum_{\mu=1}^p \sum_i^N (\xi_i^\mu)^2\right) = \tilde{Z}_{N,p}(\beta; \xi) e^{(-\beta/2N) \sum_{\mu=1}^p \sum_{i=1}^N (\xi_i^\mu)^2}, \quad (4)$$

where $\beta \geq 0$ is the inverse temperature and denotes here the level of noise in the network. We have defined

$$\tilde{Z}_{N,p}(\beta; \xi) = \sum_{\sigma} \exp\left(\frac{\beta}{2N} \sum_{\mu=1}^p \sum_{ij}^N \xi_i^\mu \xi_j^\mu \sigma_i \sigma_j\right). \quad (5)$$

Notice that the last term at the right hand side of Eq. (4) does not depend on the particular state of the network.

As a consequence, the control of the last term can be easily obtained. In fact, let us define the random variable \hat{f}_N so that

$$\hat{f}_N = \frac{1}{N} \sum_{\mu}^p \sum_i^N (\xi_i^\mu)^2. \quad (6)$$

Then we have

$$\ln Z_{N,p}(\beta; \xi) = \ln \tilde{Z}_{N,p}(\beta; \xi) - \frac{\beta}{2} \hat{f}_N. \quad (7)$$

Since $\mathbb{E} \hat{f}_N = p$ we have immediately $\lim_{N \rightarrow \infty} (1/N) \mathbb{E} \hat{f}_N = \alpha$. On the other hand, \hat{f}_N is a sum of independent random variables and therefore, by the strong law of large numbers, we have also $\lim_{N \rightarrow \infty} (1/N) \hat{f}_N = \alpha$, ξ almost surely.

Furthermore, by the central limit theorem, we have, in distribution, $\lim_{N \rightarrow \infty} (\hat{f}_N - \mathbb{E} \hat{f}_N) = \sqrt{2\alpha} \chi$, χ being a standard Gaussian $\mathcal{N}(0, 1)$. In fact,

$$\mathbb{E}(\hat{f}_N^2) = \frac{1}{N^2} \sum_{\mu}^p \sum_i^N \sum_{\nu}^p \sum_j^N \mathbb{E}((\xi_i^\mu)^2 (\xi_j^\nu)^2) = p^2 + 2 \frac{p}{N}, \quad (8)$$

so that $\mathbb{E}(\hat{f}_N^2) - \mathbb{E}(\hat{f}_N)^2 = 2p/N$, which in the thermodynamic limit gives the result.

Consequently we focus just on $\tilde{Z}(\beta; \xi)$. Let us apply the Hubbard–Stratonovich lemma¹⁶ to linearize with respect to the bilinear quenched memories carried by the $\xi_i^\mu \xi_j^\mu$. If we define the “ Mattis magnetization ”² m_μ as

$$m_\mu = \frac{1}{N} \sum_i^N \xi_i^\mu \sigma_i, \quad (9)$$

we can write

$$\tilde{Z}_{N,p}(\beta; \xi) = \sum_{\sigma} \exp\left(\frac{\beta N}{2} \sum_{\mu=1}^p m_{\mu}^2\right) = \sum_{\sigma} \int \left(\prod_{\mu=1}^p \frac{dz_{\mu} \exp(-z_{\mu}^2/2)}{\sqrt{2\pi}}\right) \exp\left(\sqrt{\beta N} \sum_{\mu=1}^p m^{\mu} z_{\mu}\right). \tag{10}$$

Using Eq. (10), the expression for the partition function (4) becomes

$$Z_{N,p}(\beta; \xi) = \sum_{\sigma} \int \left(\prod_{\mu=1}^p d\mu(z_{\mu})\right) \exp\left(\sqrt{\frac{\beta}{N}} \sum_{\mu=1}^p \sum_{i=1}^N \xi_i^{\mu} \sigma_i z_{\mu}\right) \exp\left(-\frac{\beta}{2} \hat{f}_N\right), \tag{11}$$

with $d\mu(z_{\mu})$ the standard Gaussian measure for all the z_{μ} .

For a generic function F of the neurons, we define the Boltzmann state $\omega_{\beta}(F)$ at a given level of noise β as the average,

$$\omega_{\beta}(F) = \omega(F) = (Z_{N,p}(\beta; \xi))^{-1} \sum_{\sigma} F(\sigma) e^{-\beta H_N(\sigma; \xi)}, \tag{12}$$

and often we will drop the subscript β for the sake of simplicity. Notice that the Boltzmann state does not involve the function \hat{f}_N , which factors out. The s -replicated Boltzmann state is defined as the product state $\Omega = \omega^1 \times \omega^2 \times \dots \times \omega^s$, in which all the single Boltzmann states are at the same noise level β^{-1} and share an identical sample of quenched memories ξ . For the sake of clearness, given a function F of the neurons of the s replicas and using the symbol $a \in [1, \dots, s]$ to label replicas, such an average can be written as

$$\Omega(F(\sigma^1, \dots, \sigma^s)) = \frac{1}{Z_{N,p}^s} \sum_{\sigma^1} \sum_{\sigma^2} \dots \sum_{\sigma^s} F(\sigma^1, \dots, \sigma^s) \exp\left(-\beta \sum_{a=1}^s H_N(\sigma^a, \xi)\right). \tag{13}$$

The average over the quenched memories will be denoted by \mathbb{E} and for a generic function of these memories $F(\xi)$ can be written as

$$\mathbb{E}[F(\xi)] = \int \left(\prod_{\mu=1}^p \prod_{i=1}^N \frac{d\xi_i^{\mu} e^{-(\xi_i^{\mu})^2/2}}{\sqrt{2\pi}}\right) F(\xi) = \int F(\xi) d\mu(\xi). \tag{14}$$

Of course $\mathbb{E}[\xi_i^{\mu}] = 0$ and $\mathbb{E}[(\xi_i^{\mu})^2] = 1$. We use the symbol $\langle \cdot \rangle$ to mean $\langle \cdot \rangle = \mathbb{E}\Omega(\cdot)$.

Recall that in the thermodynamic limit it is assumed that

$$\lim_{N \rightarrow \infty} \frac{p}{N} = \alpha,$$

α being a given real number, which acts as free parameter of the theory. The main quantity of interest is the intensive pressure defined as

$$A_{N,p}(\beta, \xi) = -\beta f_{N,p}(\beta, \xi) = \frac{1}{N} \ln Z_{N,p}(\beta; \xi), \tag{15}$$

while the quenched intensive pressure is defined as

$$A_{N,p}^*(\beta) = -\beta f_{N,p}^*(\beta) = \frac{1}{N} \mathbb{E} \ln Z_{N,p}(\beta; \xi), \tag{16}$$

and the annealed intensive pressure is defined as

$$\bar{A}_{N,p}(\beta) = -\beta \bar{f}_{N,p}(\beta) = \frac{1}{N} \ln \mathbb{E} Z_{N,p}(\beta; \xi). \quad (17)$$

According to thermodynamics, here $f_{N,p}(\beta, \xi) = u_{N,p}(\beta, \xi) - \beta^{-1} s_{N,p}(\beta, \xi)$ is the free energy density, $u_{N,p}(\beta, \xi)$ is the internal energy density, and $s_{N,p}(\beta, \xi)$ is the intensive entropy (the star and the bar denote the quenched and the annealed evaluations as well). Obviously, by the Jensen inequality, we have $A_{N,p}^*(\beta) \leq \bar{A}_{N,p}(\beta)$.

III. THE ROLE OF THE OVERLAPS AND THE INTERNAL ENERGY

According to the bipartite nature of the Hopfield model expressed by Eq. (11), we introduce two other order parameters beyond the Mattis magnetization [Eq. (9)]: the first is the overlap between the replicated neurons, defined as

$$q_{ab} = \frac{1}{N} \sum_{i=1}^N \sigma_i^a \sigma_i^b \in [-1, +1], \quad (18)$$

and the second the overlap between the replicated Gaussian variables z , defined as

$$p_{ab} = \frac{1}{p} \sum_{\mu=1}^p z_{\mu^a} z_{\mu^b} \in (-\infty, +\infty). \quad (19)$$

These overlaps play a considerable role in the theory as they can express thermodynamical quantities. As an example let us work out the quenched internal energy of the model, $u_{N,p}^*(\beta) = N^{-1} \langle H_N(\sigma; \xi) \rangle$.

Proposition 1: For $\beta \neq 1$, the quenched internal energy $u_N^*(\alpha, \beta)$ of the analogical Hopfield model can be expressed as

$$u_{N,p}^*(\beta) = \frac{\langle H_N(\sigma; \xi) \rangle}{N} = -\frac{\beta}{2} \left(\frac{(p/N) - \sum_{\mu}^p \langle m_{\mu}^1 q_{12} m_{\mu}^2 \rangle}{1 - \beta} \right). \quad (20)$$

Proof: The proof is based on direct calculations. We use Gaussian integration and integration by parts over the Gaussian memories. Let us begin with

$$N^{-1} \mathbb{E} \Omega(H_N(\sigma; \xi)) = -\frac{1}{2} \sum_{\mu}^p \mathbb{E}(\Omega(m_{\mu}^2)) + \frac{1}{2N^2} \sum_{\mu}^p \sum_i^N \mathbb{E}(\xi_i^{\mu 2}) = -\frac{1}{2} \sum_{\mu}^p \mathbb{E}(\Omega(m_{\mu}^2)) + \frac{p}{2N}. \quad (21)$$

Now we write explicitly a Mattis magnetization into the Boltzmann average of (21) so to use integration by parts over the memories ξ_i^{μ} [i.e., $\mathbb{E} \xi F(\xi) = \mathbb{E} \partial_{\xi} F(\xi)$],

$$\begin{aligned} -\frac{1}{2} \sum_{\mu}^p \mathbb{E}(\Omega(m_{\mu}^2)) + \frac{p}{2N} &= -\frac{1}{2N} \sum_{\mu}^p \sum_i^N \mathbb{E}(\xi_i^{\mu} \Omega(\sigma_i m_{\mu})) + \frac{p}{2N} = -\frac{\beta}{2N} \sum_{\mu}^p \sum_i^N \mathbb{E} \left(\Omega((\sigma_i m_{\mu})^2) \right. \\ &\quad \left. - \Omega^2(\sigma_i m_{\mu}) + \frac{1}{\beta N} \right) + \frac{p}{2N} = \beta \frac{\langle H(\sigma; \xi) \rangle}{N} - \frac{\beta p}{2N} + \frac{\beta}{2} \sum_{\mu}^p \langle m_{\mu}^1 q_{12} m_{\mu}^2 \rangle. \end{aligned} \quad (22)$$

□

In particular for $\beta=1$ we get exactly for any N the very remarkable expression

$$\sum_{\mu=1}^p \langle m_{\mu}^1 q_{12} m_{\mu}^2 \rangle = \frac{p}{N}, \tag{23}$$

as it can be understood by looking at the last line of (22) and choosing $\beta=1$. At the end of this section we write down a short formulary in which we consider the streaming of both the partition function and the Boltzmann state with respect to the level of noise β and to a generic stored pattern ξ_i^{μ} , as these calculations will be useful several times along the paper,

$$\begin{aligned} \partial_{\beta} Z_{N,p}(\beta; \xi) &= \sum_{\sigma} \int \left(\prod_{\mu=1}^p d\mu(z^{\mu}) \right) \sum_{i=1}^N \sum_{\mu=1}^p \left(\frac{\xi_i^{\mu} \sigma_i z^{\mu}}{2\sqrt{\beta N}} - \frac{(\xi_i^{\mu})^2}{2N} \right) \exp \left[\sum_{i=1}^N \sum_{\mu=1}^p \left(\sqrt{\frac{\beta}{N}} \xi_i^{\mu} \sigma_i z^{\mu} - \beta \frac{(\xi_i^{\mu})^2}{2N} \right) \right] \\ &= \sum_{i=1}^N \sum_{\mu=1}^p \left(\frac{\omega(\xi_i^{\mu} \sigma_i z^{\mu})}{2\sqrt{\beta N}} - \frac{(\xi_i^{\mu})^2}{2N} \right) Z_{N,p}(\beta; \xi), \end{aligned} \tag{24}$$

$$\partial_{\xi_i^{\mu}} Z_{N,p}^{\lambda}(\beta; \xi) = \lambda \left(\sqrt{\frac{\beta}{N}} \omega(\sigma_i z^{\mu}) - \frac{\beta}{N} \xi_i^{\mu} \right) Z_{N,p}^{\lambda}(\beta; \xi), \tag{25}$$

$$\partial_{\xi_i^{\mu}} \omega(\sigma_i z^{\mu}) = \sqrt{\frac{\beta}{N}} (\omega((z^{\mu})^2) - \omega^2(\sigma_i z^{\mu})), \tag{26}$$

$$\omega((z^{\mu})^2) = Z^{-1}(\beta; \xi) \sum_{\sigma} \int \left(\prod_{\mu=1}^p d\mu(z^{\mu}) \right) \frac{\partial}{\partial z^{\mu}} \left(z^{\mu} \exp \left(\sum_{\mu=1}^p \sum_{i=1}^N \left(\sqrt{\frac{\beta}{N}} \xi_i^{\mu} \sigma_i z^{\mu} - \beta \frac{(\xi_i^{\mu})^2}{2N} \right) \right) \right) \tag{27}$$

$$= 1 + \sqrt{\frac{\beta}{N}} \sum_{i=1}^N \xi_i^{\mu} \omega(\sigma_i z^{\mu}), \tag{28}$$

where, in (25), $\lambda \in \mathbb{R}$ is a generic positive real number.

IV. THE MAIN RESULTS

Now we are ready to state the main results of this paper in the form of the following theorems.

Theorem 1: *There is a $\beta_2(\alpha)$ defined in the following, such that for $\beta < \beta_2(\alpha)$ we have the following limits for the intensive free energy, internal energy, and entropy as $N \rightarrow \infty$ and $p/N \rightarrow \alpha$:*

$$\lim_{N \rightarrow \infty} (-\beta f_{N,p}(\beta; \xi)) = \lim_{N \rightarrow \infty} N^{-1} \ln Z_{N,p}(\beta; \xi) = \ln 2 - \frac{\alpha}{2} \ln(1 - \beta) - \frac{\alpha\beta}{2},$$

$$\lim_{N \rightarrow \infty} (u_{N,p}(\beta; \xi)) = - \lim_{N \rightarrow \infty} N^{-1} \partial_{\beta} \ln Z_{N,p}(\beta; \xi) = - \frac{\alpha\beta}{2(1 - \beta)},$$

$$\lim_{N \rightarrow \infty} (s_{N,p}(\beta; \xi)) = \lim_{N \rightarrow \infty} N^{-1} (\ln Z_{N,p}(\beta; \xi) - \beta \partial_{\beta} \ln Z_{N,p}(\beta; \xi)) = \ln 2 - \frac{\alpha}{2} \ln(1 - \beta) - \frac{\alpha\beta}{2} - \frac{\alpha\beta^2}{2(1 - \beta)},$$

ξ almost surely. The same limits hold for the quenched averages, so that, in particular,

$$\lim_{N \rightarrow \infty} N^{-1} \mathbb{E} \ln Z_{N,p}(\beta; \xi) = \ln 2 - \frac{\alpha}{2} \ln(1 - \beta) - \frac{\alpha\beta}{2}.$$

Theorem 2: *There is a $\beta_4(\alpha)$ defined in the following, such that for $\beta < \beta_4(\alpha)$ we have the convergence in distribution,*

$$\ln \tilde{Z}_{N,p}(\beta; \xi) - \ln \mathbb{E} \tilde{Z}_{N,p}(\beta; \xi) \rightarrow C(\beta) + \chi S(\beta), \quad (29)$$

where χ is a unit Gaussian in $\mathcal{N}[0, 1]$ and

$$C(\beta) = -\frac{1}{2} \ln \sqrt{1/(1 - \sigma^2 \beta^2 \alpha)}, \quad (30)$$

$$S(\beta) = (\ln \sqrt{1/(1 - \sigma^2 \beta^2 \alpha)})^{1/2}, \quad (31)$$

with $\sigma = (1 - \beta)^{-1}$.

If we consider the overlaps among s replicas, then there is a $\beta_{2s}(\alpha)$ defined in the following, such that for $\beta < \beta_{2s}(\alpha)$ the rescaled overlaps converge in distribution under $\langle \cdot \rangle = \mathbb{E} \Omega(\cdot)$ as follows:

$$\sqrt{N} Q_{ab} \rightarrow \frac{\xi_{ab}}{\sqrt{1 - \sigma^2 \beta^2 \alpha}}, \quad (32)$$

$$\sqrt{p} P_{ab} \rightarrow \frac{\sqrt{\alpha} \beta}{1 - \beta^2} \frac{\xi_{ab}}{\sqrt{1 - \sigma^2 \beta^2 \alpha}} + \frac{1}{1 - \beta} \chi_{ab}, \quad (33)$$

where χ_{ab} and ξ_{ab} are unit Gaussian in $\mathcal{N}[0, 1]$ independent for each couple of replicas (a, b) and independent from the χ appearing in the limit for the fluctuation of $\ln Z_{N,p}$.

We remark that the limitation on the parameter regions is strictly related to our technique in the proof. There are good reasons to believe that the theorems can be extended to the whole expected ergodic region $\beta(1 + \sqrt{\alpha}) < 1$.

In order to prove these results, we need some properties about the annealed momenta of the partition function, which will be studied in Sec. V.

V. MOMENTA OF THE NORMALIZED PARTITION FUNCTION

Annealing is a regime in which no retrieval is achievable because the same time scale for both neurons and synapses is implicitly assumed (the synaptic plasticity is thought of as fast as the neuronal current rearrangement). Anyway, annealing is the first step to be investigated in order to have a good control of the statistical mechanics features of the model.

To obtain the annealed intensive pressure, which we call $\bar{A}_{N,p}(\beta)$, we must exchange the logarithm and the average over the memories in the expression of the quenched free energy [i.e., $\mathbb{E} \ln Z_{N,p}(\beta; \xi) \Rightarrow \ln \mathbb{E} Z_{N,p}(\beta; \xi)$]. Therefore at first we need to evaluate $\mathbb{E}[Z_{N,p}(\beta; \xi)]$.

Proposition 2: *For $0 \leq \beta < 1$ we have that*

$$\mathbb{E} \tilde{Z}_{N,p}(\beta; \xi) = 2^N (1 - \beta)^{(-p/2)}. \quad (34)$$

Proof:

$$\begin{aligned} \mathbb{E} \tilde{Z}_{N,p}(\beta; \xi) &= \int \left(\prod_{\mu=1}^p \prod_{i=1}^N d\mu(\xi_i^\mu) \right) \int \prod_{\mu=1}^p d\mu(z_\mu) \sum_{\sigma} \exp \left(\sum_{\mu=1}^p \sum_{i=1}^N \left(\sqrt{\frac{\beta}{N}} \xi_i^\mu \sigma_i z^\mu \right) \right) \\ &= \sum_{\sigma} \int \left(\prod_{\mu=1}^p d\mu(z_\mu) \right) \exp \left((\beta/2) \sum_{\mu=1}^p z_\mu^2 \right) = 2^N (1 - \beta)^{-p/2}. \end{aligned}$$

Then, by recalling the definition (5) of $\tilde{Z}_{N,p}$, we can immediately state the following.

Proposition 3: *In the thermodynamic limit and for every value $0 \leq \beta < 1$, the annealed pres-*

sure per site $\bar{A}(\alpha, \beta) = \lim_{N \rightarrow \infty} \bar{A}_{N,p}(\beta)$ of the analogical Hopfield model is

$$\lim_{N \rightarrow \infty} \frac{1}{N} \ln \mathbb{E}(Z_{N,p}(\beta; \xi)) = \bar{A}(\alpha, \beta) = \ln 2 - \frac{\alpha}{2} \ln(1 - \beta) - \frac{\alpha\beta}{2}. \tag{35}$$

Remark 1: We stress that the annealed free energy (35) turns out to be the same as in the AGS theory with digital patterns. Furthermore,

$$\lim_{\alpha \rightarrow \infty, \beta \rightarrow 0, \sqrt{\alpha}\beta \rightarrow \beta'} \bar{A}(\alpha, \beta) = \ln 2 + (\beta')^2/4, \tag{36}$$

which is the expression for the annealed free energy of the Sherrington–Kirkpatrick model at the temperature β' .³⁷

This is not surprising as, at given N , in the limit $p \rightarrow \infty, \beta \rightarrow 0$ with $\beta\sqrt{p}/N \rightarrow \beta'$ we get in distribution $Z_{N,p}(\beta; \xi) \rightarrow Z_N^{\text{SK}}(\beta'; J)$, where SK stands for the Sherrington–Kirkpatrick model and J is the associated noise (standard unit Gaussian J_{ij} for each couple of sites). At given N , the neural network with infinite stored patterns becomes a Sherrington–Kirkpatrick mean field spin glass.

Remark 2: If we now turn to the energy density it is easy to show that its “annealed expression,” defined as $\bar{u}(\alpha, \beta) = -\partial_\beta \bar{A}(\alpha, \beta)$, is given by

$$\lim_{N \rightarrow \infty} \bar{u}_{N,p}(\beta) = - \lim_{N \rightarrow \infty} \frac{1}{N} \partial_\beta \ln \mathbb{E} Z_{N,p} = - \frac{1}{2} \frac{\alpha\beta}{(1 - \beta)}, \tag{37}$$

where, with respect to (20) thought in the infinite volume limit, the order parameters are missing.

Both the expressions (35) and (37) do diverge in the limit of $\beta \rightarrow 1$, suggesting a point where annealing has to break whatever α . To investigate the region of validity of the annealing, which is defined as the ergodic region, we have to study the momenta of the partition functions and check if and where they are well defined.

It will be easier to deal with the normalized partition function $\bar{Z}_{N,p}(\beta; \xi)$ defined as

$$\bar{Z}_{N,p}(\beta; \xi) \equiv \frac{\tilde{Z}_{N,p}(\beta; \xi)}{\mathbb{E} \tilde{Z}_{N,p}(\beta; \xi)}. \tag{38}$$

As the momenta will turn out to be expressed in terms of overlaps, the following lemma will be of precious help for our purpose.

Lemma 1: At $\beta=0$, in the thermodynamic limit the two replica overlaps $q_{ab}, a \neq b$, become almost surely zero and their rescaled values $\sqrt{N}q_{ab}$ converge in distribution to unit Gaussian ξ_{ab} independent for each couple of replicas.

Proof: At $\beta=0$ the Boltzmann measure Ω_0 becomes flat on all the configurations, so that all σ_i^a are independent and take the values ± 1 with equal probability of $1/2$. By taking into account the expression of the overlap $q_{ab} = N^{-1} \sum_i \sigma_i^a \sigma_i^b$, the lemma follows by the strong law of large numbers and the central limit theorem on sums of independent variables. \square

Theorem 3: For $s \in \mathbb{N}$ and for $\beta < \beta_s(\alpha)$, with $\beta_s(\alpha)$ suitably defined in the following, the following limiting expression for the integer momenta of the normalized partition function holds:

$$\lim_{N \rightarrow \infty} \mathbb{E}[\bar{Z}_{N,p}^s(\beta; \xi)] = \exp\left(\frac{s(s-1)}{4} \left(\ln\left(\frac{1}{1 - \sigma^2 \beta^2 \alpha}\right)\right)\right), \tag{39}$$

where $\sigma = 1/(1 - \beta)$.

Proof: Let us at first notice that there is a constraint $\beta_s(\alpha) \leq s^{-1}$, which becomes effective starting from $s=2$. The key point is that $\mathbb{E}[\bar{Z}_{N,p}^s(\beta; \xi)]$ becomes infinite at fixed N if the constraint is not satisfied. In fact, let us calculate this momentum by introducing the replicated σ_i^a, z_μ^a for $i=1, 2, \dots, N, \mu=1, 2, \dots, p, a=1, 2, \dots, s$,

$$\begin{aligned}
 \mathbb{E} \tilde{Z}_{N,p}^s(\beta; \xi) &= \mathbb{E} \left(\sum_{\sigma^1} \cdots \sum_{\sigma^s} \int \left(\prod_{a=1}^s \prod_{\mu=1}^p d\mu(z_\mu^a) \right) \exp \left\{ \sum_{\mu=1}^p \sum_{i=1}^N \left(\sqrt{\frac{\beta}{N}} \xi_i^\mu \sum_{a=1}^s \sigma_i^a z_\mu^a \right) \right\} \right) \\
 &= \sum_{\sigma^1} \cdots \sum_{\sigma^s} \int \left(\prod_{a=1}^s \prod_{\mu=1}^p d\mu(z_\mu^a) \right) \exp \left(\frac{\beta}{2N} \sum_{\mu=1}^p \sum_{i=1}^N \left(\sum_{a=1}^s \sigma_i^a z_\mu^a \right)^2 \right) = \sum_{\sigma^1} \cdots \sum_{\sigma^s} \int \left(\prod_{\mu=1}^p \right) \\
 &\quad \times \left(\left(\prod_{a=1}^s d\mu(z_\mu^a) \right) \exp \left(\frac{\beta}{2N} \sum_{i=1}^N \left(\sum_{a,b} \sigma_i^a z_\mu^a \sigma_i^b z_\mu^b \right) \right) \right), \tag{40}
 \end{aligned}$$

where we have performed the Gaussian integration over the memories ξ and have expressed the resulting square in the form of the double sum $\sum_{a,b}$. The sum over i gives the overlaps q_{ab} . We notice also that we have complete factorization over μ . Therefore, we can introduce generic variables z^a , $a=1, 2, \dots, s$, and define

$$B_{p,s}(\beta; Q) = \int \left(\prod_{a=1}^s d\mu(z^a) \right) \exp \left(\frac{\beta}{2} \sum_{a,b} q_{ab} z^a z^b \right), \tag{41}$$

where Q is the $s \times s$ matrix with elements q_{ab} , so that

$$\mathbb{E} \tilde{Z}_{N,p}^s(\beta; \xi) = 2^{Ns} \Omega_0(e^{p \ln B_{p,s}(\beta; Q)}). \tag{42}$$

Now we can see the reason for the limitation $\beta < 1/s$. In fact, let us notice that

$$0 \leq \sum_{a,b} q_{ab} z^a z^b \leq \sum_{a,b} |z^a| |z^b| = s^2 \left(s^{-1} \sum_a |z^a| \right)^2 \leq s^2 s^{-1} \sum_a (z^a)^2 = s \sum_a (z^a)^2,$$

where we have introduced the uniform distribution $s^{-1} \sum_a$ on $(1, 2, \dots, s)$ and exploited the Schwartz inequality. Therefore, the integral defining $B_{p,s}(\beta; Q)$ in (41) can be uniformly bound by $(1-s\beta)^{-p/2}$, which is finite in the region $s\beta < 1$. On the other hand, it is easily seen that for some σ configurations the integral in (41) is infinite if $s\beta \geq 1$. The worst case is when all $q_{ab}=1$. Then we have from (41)

$$\begin{aligned}
 B_{p,s} &= \int \left(\prod_{a=1}^s d\mu(z^a) \right) \exp \left(\frac{\beta}{2} \sum_{a,b} z^a z^b \right) = \int \left(\prod_{a=1}^s d\mu(z^a) \right) \exp \left(\frac{\beta}{2} \left(\sum_a z^a \right)^2 \right) \\
 &= \int \left(\prod_{a=1}^s d\mu(z^a) \right) \int d\mu(y) \exp \left(\sqrt{\beta} \sum_a z^a y \right) = \int d\mu(y) \exp \left(\frac{s\beta}{2} y^2 \right),
 \end{aligned}$$

and the integral on the auxiliary variable y is divergent if $s\beta \geq 1$. From now on we remain in the region $s\beta < 1$. Let us go back to the definition (41). We write $\sum_{a,b} q_{ab} z^a z^b = 2 \sum_{(ab)} q_{ab} z^a z^b + \sum_a (z^a)^2$, where (ab) are all couples of different replicas. Then we have

$$\begin{aligned}
 B_{p,s}(\beta; Q) &= \int \left(\prod_{a=1}^s (d\mu(z^a) e^{(1/2)\beta(z^a)^2}) \right) \exp \left(\beta \sum_{(ab)} q_{ab} z^a z^b \right) \\
 &= \left(\int d\mu(z') e^{(1/2)\beta(z')^2} \right)^s \int \left(\prod_{a=1}^s d\bar{\mu}(z^a) \right) \exp \left(\beta \sum_{(ab)} q_{ab} z^a z^b \right) = (1 \\
 &\quad - \beta)^{-s/2} \int \left(\prod_{a=1}^s d\bar{\mu}(z^a) \right) \exp \left(\beta \sum_{(ab)} q_{ab} z^a z^b \right), \tag{43}
 \end{aligned}$$

where we have introduced the normalized deformed measure

$$d\bar{\mu}(z) = \frac{e^{(1/2)\beta z^2} d\mu(z)}{\int e^{(1/2)\beta(z')^2} d\mu(z')}$$

Finally, if we define a modified $\bar{B}_{p,s}(\beta; Q)$ as

$$\bar{B}_{p,s}(\beta; Q) = \int \left(\prod_{a=1}^s d\bar{\mu}(z^a) \right) \exp\left(\beta \sum_{(ab)} q_{ab} z^a z^b \right), \tag{44}$$

we can write

$$\mathbb{E}\tilde{Z}_{N,p}^s(\beta; \xi) = (\mathbb{E}\tilde{Z}_{N,p}(\beta; \xi))^s \Omega_0(e^{p \ln \bar{B}_{p,s}(\beta; Q)}). \tag{45}$$

In order to investigate the $N \rightarrow \infty$ limit, it is convenient to start from the case where $s=2$. Then $\bar{B}_{p,2}$ can be explicitly calculated in the form

$$\bar{B}_{p,2} = -\frac{1}{2} \ln(1 - \beta^2 \sigma^2 q_{12}). \tag{46}$$

In fact, in this case we have

$$\bar{B}_{p,2} = \int d\bar{\mu}(z^1) d\bar{\mu}(z^2) \exp(\beta q_{12} z^1 z^2),$$

where the two Gaussian integrals can be calculated explicitly and lead to (46). Therefore, we are led to consider the (α, β) region where

$$\Omega_0\left(\exp\left(-\frac{1}{2} p \ln(1 - \beta^2 \sigma^2 q_{12}^2)\right)\right)$$

stays finite in the $N \rightarrow \infty$ limit. Through a simple change in variables $\sigma_i^1 = \sigma_i'$, $\sigma_i^2 = \sigma_i' \sigma_i$ we are led to the consideration of a mean field ferromagnetic Ising system $(\sigma_1, \dots, \sigma_N)$ with normalized partition function

$$\Omega_0\left(\exp\left(-\frac{1}{2} p \ln(1 - \beta^2 \sigma^2 m^2)\right)\right), \tag{47}$$

where now $\Omega_0 = 2^{-N \sum_i \sigma_i}$ and $m = N^{-1} \sum_i \sigma_i$, as usual. Now, we can state and prove the following theorem.

Theorem 4: Consider the trial function

$$\phi(\alpha, \beta; M) = -\frac{1}{2} \alpha \ln(1 - \beta^2 \sigma^2 M^2) + \ln \cosh\left(\alpha \frac{\beta^2 \sigma^2 M}{1 - \beta^2 \sigma^2 M^2}\right) - \alpha M \frac{\beta^2 \sigma^2 M}{1 - \beta^2 \sigma^2 M^2},$$

depending on the order parameter M , with $-1 \leq M \leq 1$. Clearly, at $M=0$ we have $\phi(\alpha, \beta; 0)=0$. Define $\beta_2(\alpha)$ as the largest value such that, for any $\beta < \beta_2(\alpha)$, we have $\phi(\alpha, \beta; M) < 0$ for any positive M . It is easily shown that $\beta_2(\alpha) \geq (1 + \sqrt{1 + \alpha})^{-1}$. Then for $\beta < \beta_2(\alpha)$ we have

$$\lim_{N \rightarrow \infty} \Omega_0\left(\exp\left(-\frac{1}{2} p \ln(1 - \beta^2 \sigma^2 m^2)\right)\right) = (1 - \alpha \beta^2 \sigma^2)^{-1/2}. \tag{48}$$

Notice that $\beta_2(\alpha)$ defines the onset of the ferromagnetic phase transition for the model. The proof of the theorem follows standard methods of statistical mechanics. In order to get a lower bound we only notice that

$$-\ln(1 - \beta^2 \sigma^2 m^2) \geq \beta^2 \sigma^2 m^2,$$

and therefore

$$\Omega_0(\exp(-\frac{1}{2}p \ln(1 - \beta^2 \sigma^2 m^2))) \geq \Omega_0(\exp(\frac{1}{2}p \beta^2 \sigma^2 m^2)).$$

The right hand side converges to $(1 - \alpha \beta^2 \sigma^2)^{-1/2}$ provided $\alpha \beta^2 \sigma^2 < 1$. Notice that this last condition can be written also as $\beta < 1/(1 + \sqrt{\alpha})$, which is the critical line according to the AGS theory. For the upper bound, let us introduce the truth functions on the σ configuration space, defined by $\chi_1 = \chi(m^2 \leq \bar{m}^2)$ and $\chi_2 = \chi(m^2 > \bar{m}^2)$, where \bar{m} is some positive number. Then we have that the $\Omega_0(\dots)$ in (47) splits into the sum of two pieces,

$$\Omega_0(\dots) = \Omega_0(\dots \chi_1) + \Omega_0(\dots \chi_2).$$

For $\beta < \beta_2(\alpha)$, the second piece converges to zero as $N \rightarrow \infty$. In fact, let us define for $-1 \leq M \leq 1$

$$\psi(M) = -\frac{p}{2N} \ln(1 - \beta^2 \sigma^2 M^2), \tag{49}$$

with its M derivative

$$\psi'(M) = \frac{p}{N} \frac{\beta^2 \sigma^2 M}{1 - \beta^2 \sigma^2 M^2}. \tag{50}$$

Notice that ψ is convex in M , so that

$$\psi(m) \geq \psi(M) + (m - M)\psi'(M). \tag{51}$$

Now, let us consider M as taking all values of m for which $m^2 > \bar{m}^2$. There are at most $N+1$ such values. For the sake of simplicity, introduce the inequality

$$1 \leq \sum_M \exp(-N(\psi(m) - \psi(M) - (m - M)\psi'(M))), \tag{52}$$

where M is summed over all stated values. The reason of the inequality is clear. In fact, there is one term equal to 1, when $M=m$, while all other terms are positive. By exploiting the inequality, we can write

$$\Omega_0\left(\exp\left(-\frac{1}{2}p \ln(1 - \beta^2 \sigma^2 m^2)\right)\chi_2\right) \leq \sum_M \Omega_0(\exp(-N(-\psi(M) - (m - M)\psi'(M)))\chi_2). \tag{53}$$

If now we remove the χ_2 and perform the average over Ω_0 by taking into account that the exponent is factorized with respect to the σ_i , we get

$$\Omega_0\left(\exp\left(-\frac{1}{2}p \ln(1 - \beta^2 \sigma^2 m^2)\right)\chi_2\right) \leq \sum_M \exp\left(N\left(\psi(M) + \ln \cosh\left(\frac{p}{N} \frac{\beta^2 \sigma^2 M}{1 - \beta^2 \sigma^2 M^2}\right) - M\psi'(M)\right)\right). \tag{54}$$

Clearly, in the region $\beta < \beta_2(\alpha)$ and for large N , each term in the sum is uniformly bounded by an exponential of the type $\exp(-cN)$ for some constant c . Of course there are at most $N+1$ terms. Therefore, as $N \rightarrow \infty$, the right hand side goes to zero, as we were interested to show.

Now we must consider the first term $\Omega_0(\dots \chi_1)$. Let us notice that in the region $m^2 \leq \bar{m}^2$ by convexity we have $-\ln(1 - \beta^2 \sigma^2 m^2) \leq -\ln(1 - \beta^2 \sigma^2 \bar{m}^2)(m^2/\bar{m}^2)$. By inserting the inequality in the first term and neglecting the χ_1 , we have in the infinite volume limit

$$\limsup_{N \rightarrow \infty} \Omega_0 \left(\exp \left(-\frac{1}{2} p \ln(1 - \beta^2 \sigma^2 m^2) \right) \chi_1 \right) \leq \left(1 + \frac{\alpha \ln(1 - \beta^2 \sigma^2 m^2)}{\bar{m}^2} \right)^{-1/2}. \tag{55}$$

Since \bar{m} is arbitrary, we can take $\bar{m} \rightarrow 0$. Collecting all results, we immediately establish the limit in (48). Notice that $\beta_2(\alpha) \leq (1 + \sqrt{\alpha})^{-1}$; otherwise $\phi(\alpha, \beta; M)$ would start with a positive derivative at $M=0$. This ends the proof of Theorem 3 in the case $s=2$.

The general case can be handled in a similar way. Now we encounter ferromagnetic models for the Ising variables σ_i^a , $a=1, 2, \dots, s$, $i=1, 2, \dots, N$, with Boltzmannfaktor $\exp(p \ln \bar{B}_{p,s})$. If $\beta_{s(\alpha)}$ denotes the onset of the associated ferromagnetic transition, then we can immediately prove that

$$\lim_{N \rightarrow \infty} \Omega_0(e^{p \ln \bar{B}_{p,s}}) = \exp \left(\frac{s(s-1)}{4} \left(\ln \left(\frac{1}{1 - \sigma^2 \beta^2 \alpha} \right) \right) \right) \tag{56}$$

for $\beta \leq \beta_{s(\alpha)}$. In fact, as in the proof for the case $s=2$, we see that in the expression of $\bar{B}_{p,s}(\beta; Q)$ only the first two terms in the expansion of the exponent do matter provided the stated condition on β holds. These terms are easily calculated as in the case $s=2$. Then we recall that under Ω_0 , for $a \neq b$, the rescaled overlaps $\sqrt{N}q_{ab}$ converge in distribution to independent unit Gaussian ξ_{ab} . We see that the term $s(s-1)$ in formula (39) comes essentially from the fact that there are $s(s-1)/2$ couples (a, b) for s replicas. Therefore, Theorem 3 is fully established. \square

Now we are ready to prove Theorem 1, at least in the region $\beta < \beta_2(\alpha)$.

First of all, let us recall that if $u_N \geq 0$ is a sequence of random variables normalized to $\mathbb{E}(u_N)=1$, then a simple application of the Markov inequality³⁶ and the Borel–Cantelli lemma gives

$$\limsup_{N \rightarrow \infty} \frac{1}{N} \ln u_N \leq 0,$$

almost surely. Moreover, if $\mathbb{E}(u_N^2) \leq cE^2(u_N)$, uniformly in N , for some finite constant c , then

$$\lim_{N \rightarrow \infty} \frac{1}{N} \ln u_N = 0,$$

almost surely.

If we define

$$u_N = \frac{\mathbb{E}(\tilde{Z}_{N,p}(\beta; \xi))^s}{\mathbb{E}^s(\tilde{Z}_{N,p}(\beta; \xi))},$$

and take into account our previous results, then we immediately find, ξ almost surely,

$$\limsup_{N \rightarrow \infty} \frac{1}{N} \ln \tilde{Z}_{N,p}(\beta; \xi) \leq \ln 2 - \frac{\alpha}{2} \ln(1 - \beta) \quad \text{for any } \beta < 1, \tag{57}$$

$$\lim_{N \rightarrow \infty} \frac{1}{N} \ln \tilde{Z}_{N,p}(\beta; \xi) = \ln 2 - \frac{\alpha}{2} \ln(1 - \beta) \quad \text{for any } \beta < \beta_2(\alpha). \tag{58}$$

In order to get Theorem 1, in the stated region, it is only necessary to recall Eq. (7) connecting $Z_{N,p}$ with $\tilde{Z}_{N,p}$ and the limiting properties of \hat{f}_N . \square

VI. LOG NORMALITY OF THE LIMITING DISTRIBUTION FOR THE PARTITION FUNCTION

In this section we want to show that the limiting distribution of the normalized partition function (38), at least in a given region of the α, β plane, is log normal. This will immediately give us the mean and the fluctuations of the thermodynamical quantities in that region.²⁹

Let us remember that if $C(\beta)$ and $S(\beta)$ are given functions and χ a standard Gaussian $\mathcal{N}[0, 1]$, a variable $\eta(\beta)$ has a log-normal distribution if it is possible to write it down as¹⁶

$$\eta(\beta) = \exp(C(\beta) + \chi S(\beta)). \quad (59)$$

The momenta of $\eta(\beta)$ are

$$\mathbb{E}[\eta^s(\beta)] = \mathbb{E}[\exp(sC(\beta) + sS(\beta)\chi)] = \exp(C(\beta)s + \frac{1}{2}s^2S^2(\beta)). \quad (60)$$

So if we choose

$$C(\beta) = -\frac{1}{2} \ln \left(\sqrt{\frac{1}{1 - \sigma^2 \beta^2 \alpha}} \right), \quad (61)$$

$$S^2(\beta) = \ln \left(\sqrt{\frac{1}{1 - \sigma^2 \beta^2 \alpha}} \right), \quad (62)$$

we see that $\bar{Z}_{N,p}(\beta; \xi)$ and $\eta(\beta)$ have the same integer momenta in the limit provided we restrict the values of β according to the order of the momentum s , as expressed in Theorem 3. This seems to suggest that $\bar{Z}_{N,p}(\beta; \xi)$ is log-normal distributed in the limit. To prove that this is effectively the case, it will be sufficient to prove that the momenta $\bar{Z}_{N,p}(\beta; \xi)^\lambda$ do in fact converge to those of η for all values of λ in some interval of the real line.¹³ In other words, we have to extend the limiting behavior of Theorem 3, from integer values of s to real values λ in some nontrivial interval, at least in some region of the (α, β) plane. To solve this task we have to evaluate the limiting behavior of $\bar{Z}_{N,p}^\lambda(\beta; \xi)$ for λ in some interval of the real line.

We get the result by analyzing

$$\partial_\beta \ln \mathbb{E}[\bar{Z}_{N,p}^\lambda] = \partial_\beta (\ln \mathbb{E}[\bar{Z}_{N,p}^\lambda] - \lambda \ln \mathbb{E}[\bar{Z}_{N,p}]) = \frac{\partial_\beta \mathbb{E}[\bar{Z}_{N,p}^\lambda]}{\mathbb{E}[\bar{Z}_{N,p}^\lambda]} - \lambda \frac{\partial_\beta \mathbb{E}[\bar{Z}_{N,p}]}{\mathbb{E}[\bar{Z}_{N,p}]}, \quad (63)$$

which can be written in terms of overlaps via the following helpful proposition.

Proposition 4: For any real λ , with $\lambda \leq s$, s integer, and $\beta < 1/s$, the β derivative of the annealed real momenta of the partition function can be expressed in terms of overlaps as

$$\frac{\partial_\beta \mathbb{E}[\bar{Z}_{N,p}^\lambda]}{\mathbb{E}[\bar{Z}_{N,p}^\lambda]} = \frac{p\lambda/2}{(1-\beta)} \left((\lambda-1) \mathbb{E} \left(\frac{\bar{Z}_{N,p}^\lambda}{\mathbb{E}[\bar{Z}_{N,p}^\lambda]} \Omega(q_{12} p_{12}) \right) + 1 \right). \quad (64)$$

Notice that $\partial_\beta \ln \mathbb{E}[\bar{Z}_{N,p}^\lambda]$ is convex increasing in λ . Therefore, the limitation on the values of β assures the existence of the relevant averages.

Proof: Using Eq. (24) we can write

$$\partial_\beta \mathbb{E}[\bar{Z}_{N,p}^\lambda] = \mathbb{E}[\lambda \bar{Z}_{N,p}^{\lambda-1} \partial_\beta \bar{Z}_{N,p}] = \sum_{\mu=1}^p \sum_{i=1}^N \left(\frac{\lambda}{2\sqrt{\beta N}} \mathbb{E}[\xi_i^\mu \bar{Z}_{N,p}^\lambda \omega(\sigma_i z^\mu)] \right). \quad (65)$$

Furthermore we can write

$$\begin{aligned} \mathbb{E}[\xi_i^\mu \tilde{Z}_{N,p}^\lambda \omega(\sigma_i z^\mu)] &= \lambda \sqrt{\frac{\beta}{N}} \mathbb{E}[\tilde{Z}_{N,p}^\lambda \omega^2(\sigma_i z^\mu)] - \lambda \frac{\beta}{N} \mathbb{E}[\xi_i^\mu \tilde{Z}_{N,p}^\lambda \omega(\sigma_i z^\mu)] + \sqrt{\frac{\beta}{N}} \mathbb{E}[\tilde{Z}_{N,p}^\lambda] \\ &\quad + \frac{\beta}{N} \sum_{j=1}^N \mathbb{E}[\tilde{Z}_{N,p}^\lambda \xi_j^\mu \omega(\sigma_i z^\mu)] - \sqrt{\frac{\beta}{N}} \mathbb{E}[\tilde{Z}_{N,p}^\lambda \omega^2(\sigma_i z^\mu)], \end{aligned} \tag{66}$$

and, using (25)–(27),

$$\sum_{\mu=1}^p \sum_{i=1}^N \mathbb{E}[\xi_i^\mu \tilde{Z}_{N,p}^\lambda \omega(\sigma_i z^\mu)] = \frac{p\sqrt{\beta N}}{(1-\beta + \beta\lambda/N)} ((\lambda-1)\mathbb{E}[\tilde{Z}_{N,p}^\lambda \Omega(q_{12} p_{12})] + \mathbb{E}[\tilde{Z}_{N,p}^\lambda]). \tag{67}$$

By substituting (67) into (65) and dividing by $\mathbb{E}(Z_{N,p}^\lambda)$ we get the result. □

With the help of (64) we can rewrite (63) as

$$\partial_\beta \ln \mathbb{E}[\tilde{Z}_{N,p}^\lambda] = \frac{\sqrt{\alpha\lambda(\lambda-1)}}{2(1-\beta)} \mathbb{E}\left(\frac{\tilde{Z}_{N,p}^\lambda}{\mathbb{E}(\tilde{Z}_{N,p}^\lambda)} \Omega(\sqrt{N}q_{12}\sqrt{pp_{12}})\right). \tag{68}$$

To proceed further we have now to investigate the distribution of the rescaled overlaps because they do appear into the expression above. Such distribution can be obtained through the evaluation of their momenta generating function. We will see that at least in a given region the distribution of $\tilde{Z}_{N,p}^\lambda$ is not coupled with the one of the overlaps. From this observation, by looking at Eq. (69) the log normality for the normalized partition function is easily achieved. Let us start by proving the following.

Proposition 5: Consider a generic number of replicas s . For each couple (a, b) of replicas, let $(\lambda_{ab}, \eta_{ab})$ be real numbers in the momentum generating functional (which we assume to be very small). Let λ be a real number in the interval $s \leq \lambda \leq 2s$. Then, at least for $\beta < \beta_{2s}(\alpha)$, we have the limit

$$\begin{aligned} \lim_{N \rightarrow \infty} \mathbb{E}\left(\frac{\tilde{Z}_{N,p}^\lambda}{\mathbb{E}[\tilde{Z}_{N,p}^\lambda]} \Omega\left(\exp\left(\sum_{ab} \lambda_{ab} \sqrt{N}q_{ab} + \sum_{ab} \eta_{ab} \sqrt{pp_{ab}}\right)\right)\right) &= \exp\left(\frac{1}{2} \sum_{ab} \lambda_{ab}^2 \frac{1}{1 - \sigma^2 \beta^2 \alpha}\right) \\ &\quad + \frac{1}{2} \sum_{ab} \eta_{ab}^2 \sigma^2 \left(\frac{\alpha \beta^2 \sigma^2}{1 - \sigma^2 \beta^2 \alpha} + 1\right) + \sum_{ab} \frac{\sqrt{\alpha} \beta \sigma^2 \lambda_{ab} \eta_{ab}}{1 - \sigma^2 \beta^2 \alpha}, \end{aligned} \tag{69}$$

where as usual $\sigma = 1/(1-\beta)$ and \sum_{ab} denotes the sum over all couples (ab) .

Proof: We give the proof at first for $\lambda = s$ in the region $\beta < \beta_s(\alpha)$. Then we will enlarge the proof to the interval $s \leq \lambda \leq 2s$ in the region $\beta < \beta_{2s}(\alpha)$. For $\lambda = s$ the left hand side of (69) can be thought of as

$$\begin{aligned} &\left(\prod_{ia} \sum_{\sigma_i^a}\right) \int \left(\prod_{\mu a} d\mu_\sigma(z_\mu^a)\right) e^{((\beta/2)\sum_{\mu=1}^p \sum_{ab} q_{ab} z_\mu^a z_\mu^b)} e^{(\sum_{ab} \lambda_{ab} (\sqrt{N}q_{ab}) + \sum_{ab} \eta_{ab} \sqrt{pp_{ab}})} \\ &= \left(\prod_{ia} \sum_{\sigma_i^a}\right) \int \left(\prod_{\mu a} d\mu_\sigma(z_\mu^a)\right) \exp\left(\sum_{ab} \sqrt{pp_{ab}} (\sqrt{\alpha} \beta \xi_{ab} + \eta_{ab})\right) \exp\left(\sum_{ab} \lambda_{ab} \xi_{ab}\right) \sigma^{ps/2} \\ &= \Omega_0\left(\exp\left(\sum_{ab} \frac{1}{2} \sigma^2 (\alpha \beta^2 \xi_{ab}^2 + \eta_{ab}^2 + 2\sqrt{\alpha} \beta \eta_{ab} \xi_{ab})\right)\right) \exp\left(\sum_{ab} \lambda_{ab} \xi_{ab}\right) \sigma^{ps/2} 2^{Ns} \\ &= \exp\left(\frac{1}{2} \sum_{ab} \frac{1}{1 - \sigma^2 \beta^2 \alpha} (\alpha \beta^2 \sigma^4 \eta_{ab}^2 + \lambda_{ab}^2 + 2\sqrt{\alpha} \beta \sigma^2 \lambda_{ab} \eta_{ab}) + \frac{\sigma^2}{2} \eta_{ab}^2\right) \mathbb{E}[Z_{N,p}^s], \end{aligned}$$

where $d\mu_\sigma$ is the Gaussian with variance $\sigma = (1-\beta)^{-1}$. Therefore, by taking the limit, we prove the

proposition $\lambda=s$. In order to provide the extension to the interval $s \leq \lambda \leq 2s$ we must show that defining

$$\mathcal{A} \equiv \exp\left(\frac{1}{2} \sum_{ab} \lambda_{ab}^2 \frac{1}{1 - \sigma^2 \beta^2 \alpha} + \frac{1}{2} \sum_{ab} \eta_{ab}^2 \sigma^2 \left(\frac{\alpha \beta^2 \sigma^2}{1 - \sigma^2 \beta^2 \alpha} + 1\right) + \sum_{ab} \frac{\sqrt{\alpha} \beta \sigma^2}{1 - \sigma^2 \beta^2 \alpha} \lambda_{ab} \eta_{ab}\right)$$

the following holds:

$$\lim_{N \rightarrow \infty} \mathbb{E} \left(\bar{Z}_{N,p}^\lambda \Omega \left(\exp \left(\sum_{ab} \lambda_{ab} \sqrt{N} q_{ab} + \sum_{ab} \eta_{ab} \sqrt{p} p_{ab} \right) - \mathcal{A} \right) \right) = 0. \tag{70}$$

The proof of (70) can be obtained in the simplest way by using the Cauchy–Schwartz inequality ($\mathbb{E}^2[AB] \leq \mathbb{E}[A^2] \mathbb{E}[B^2]$) choosing $\lambda = \mu + s$, with $0 \leq \mu \leq s$. In fact, we have

$$\mathbb{E}^2 \left(\bar{Z}_{N,p}^\mu \bar{Z}_{N,p}^s \left(\Omega \left(\exp \left(\sum_{ab} \lambda_{ab} \sqrt{N} q_{ab} + \sum_{ab} \eta_{ab} \sqrt{p} p_{ab} \right) \right) - \mathcal{A} \right) \right) \leq \mathbb{E}(\bar{Z}_{N,p}^{2\mu}) \mathbb{E} \left(\bar{Z}_{N,p}^{2s} \left(\Omega \left(\exp \left(\sum_{ab} \lambda_{ab} \sqrt{N} q_{ab} + \sum_{ab} \eta_{ab} \sqrt{p} p_{ab} \right) \right) - \mathcal{A} \right)^2 \right). \tag{71}$$

Here the first factor is bounded in the region $\beta < \beta_{2s}(\alpha)$. In fact, by monotonicity we have $\mathbb{E}(\bar{Z}_{N,p}^{2\mu}) \leq \mathbb{E}(\bar{Z}_{N,p}^{2s})$. The second term is the sum of three terms obtained by calculating the square of $(\Omega(\exp(\sum_{ab} \lambda_{ab} \sqrt{N} q_{ab} + \sum_{ab} \eta_{ab} \sqrt{p} p_{ab})) - \mathcal{A})$, the simplest being $\mathbb{E}(\bar{Z}_{N,p}^{2s}) \mathcal{A}^2$ which is known.

It is easy to check that for the double product we have in the limit

$$\lim_{N \rightarrow \infty} \mathcal{A} \mathbb{E} \left(\bar{Z}_{N,p}^{2s} \Omega \left(\exp \left(\sum_{ab} \lambda_{ab} \sqrt{N} q_{ab} + \sum_{ab} a_{ab} \sqrt{p} p_{ab} \right) \right) \right) = \lim_{N \rightarrow \infty} \mathcal{A}^2 \mathbb{E}(\bar{Z}_{N,p}^{2s}).$$

For the last term we have

$$\lim_{N \rightarrow \infty} \mathbb{E} \left(\bar{Z}_{N,p}^{2s} \Omega \left(\exp \left(\sum_{ab} \lambda_{ab} \sqrt{N} q_{ab} + \sum_{ab} \eta_{ab} \sqrt{p} p_{ab} + \sum_{\bar{a}\bar{b}} \lambda_{\bar{a}\bar{b}} \sqrt{N} q_{\bar{a}\bar{b}} + \sum_{\bar{a}\bar{b}} \eta_{\bar{a}\bar{b}} \sqrt{p} p_{\bar{a}\bar{b}} \right) \right) \right), \tag{72}$$

$$\lim_{N \rightarrow \infty} = \mathcal{A}^2 \mathbb{E}(\bar{Z}_{N,p}^{2s}),$$

where the state Ω is thought of by $2s$ replicas, the sum on the couples of variables (\bar{a}, \bar{b}) taking into account the second set $s+1, \dots, 2s$. The sum of these three terms goes to zero as $N \rightarrow \infty$, proving the proposition. \square

From Proposition 5 we can derive the next corollary, which is a part of Theorem 2.

Corollary 1: For s replicas, at least for $\beta < \beta_{2s}(\alpha)$, in the thermodynamic limit the limiting distributions of the rescaled overlaps are

$$\sqrt{N} Q_{ab} \rightarrow \xi_{ab}, \tag{73}$$

$$\sqrt{p} P_{ab} \rightarrow \frac{\sqrt{\alpha} \beta}{1 - \beta^2} \xi_{ab} + \frac{1}{1 - \beta} \chi_{ab}, \tag{74}$$

where $\chi_{ab} \in \mathcal{N}(0, 1)$ and $\xi_{ab} \in \mathcal{N}(0, 1/(1 - \sigma^2 \beta^2 \alpha))$.

Now we are ready for the proof of the following basic theorem.

Theorem 5: For $\beta < \beta_4(\alpha)$, in the thermodynamic limit, we have that in distribution

$$\bar{Z}_{N,p}(\beta; \xi) \rightarrow \exp(C(\beta) + \chi S(\beta)), \tag{75}$$

where $\chi \in \mathcal{N}[0, 1]$ and

$$C(\beta) = -\frac{1}{2} \ln \sqrt{1/(1 - \sigma^2 \beta^2 \alpha)}, \tag{76}$$

$$S(\beta) = (\ln \sqrt{1/(1 - \sigma^2 \beta^2 \alpha)})^{1/2}. \tag{77}$$

Proof: The limitation $\beta < \beta_4(\alpha)$ is clear. In fact, we will exploit formula (68), which requires two replicas for $2 \leq \lambda \leq 4$, and the results of Proposition 5 that require the stated limitation on β . Therefore, we see immediately that

$$\lim_{N \rightarrow \infty} \mathbb{E}(\bar{Z}_{N,p}^\lambda \Omega(\sqrt{N} q_{ab} \sqrt{p p_{ab}}) / \mathbb{E}(\bar{Z}_{N,p}^\lambda)) \tag{78}$$

$$= \partial_{\lambda_{ab}} \partial_{\eta_{ab}} \lim_{N \rightarrow \infty} \frac{\mathbb{E}(\bar{Z}_{N,p}^\lambda \Omega(\exp(\sum_{ab} \lambda_{ab} \sqrt{N} q_{ab} + \sum_{ab} \eta_{ab} \sqrt{p p_{ab}})))}{\mathbb{E}(\bar{Z}_{N,p}^\lambda)} \tag{79}$$

$$= \frac{\sqrt{\alpha} \beta \sigma^2}{(1 - \sigma^2 \beta^2 \alpha)}. \tag{80}$$

Therefore, we have

$$\lim_{N \rightarrow \infty} \partial_\beta \ln \mathbb{E}[\bar{Z}^\lambda] = \frac{\lambda(\lambda - 1)}{2} \left(\frac{1}{1 - \beta(1 - \beta)^2 - \alpha \beta^2} \right). \tag{81}$$

By exploiting convexity in β , we can integrate this expression and obtain the limit

$$\lim_{N \rightarrow \infty} \mathbb{E}[\bar{Z}_{N,p}^\lambda] \rightarrow \exp\left(\frac{\lambda(\lambda - 1)}{4} \left(\ln\left(\frac{1}{1 - \sigma^2 \beta^2 \alpha}\right)\right)\right)$$

for all values of λ in a nontrivial interval of the real axis. This shows the convergence in distribution of $\bar{Z}_{N,p}$ to a log-normal random variable, as stated in the theorem. □

Finally, we can easily prove Theorem 2 if we recall the definition in (38). □

VII. CONCLUSION

In this work we introduced the framework of the real replicas, successfully applied on spin glasses (see, e.g., Refs. 5, 21, and 23), to neural networks in the ergodic regime. This approach naturally holds for the high storage memory case, which is mathematically challenging. Acting together as a biological generalization to analogical stored memories and as a technical trick to manage easily the mathematical control, we allowed the patterns to live as Gaussian variables on $\mathcal{N}[0, 1]$ instead of ± 1 but, as we checked *a fortiori*, this does not affect (at least in the part of the ergodic region that we can control) any macroscopical distribution once the thermodynamic limit is taken. Thinking of the Hopfield model as a bipartite model in a proper space of variables, beyond the Mattis magnetization, we introduced the other order parameters q_{ab} and p_{ab} , one for each interacting structure, the N dichotomic Ising neurons σ_i , and the p fictitious Gaussians z_μ , which are able to fully describe the high temperature region we investigated.

We showed that the partition function is log-normal distributed in a suitably defined region. Then we evaluated the distribution of the rescaled overlaps, which share centered Gaussian fluctuations with different variances. Finally we proved that all the thermodynamic quantities fluctuate around their annealed approximation and calculated their spread. All the densities (e.g., energy density, free energy density, and entropy density) turn out to be self-averaging on their annealed values.

Further investigation should give us the full control of the whole ergodic phase and bring us exploring the retrieval phase and hopefully the still completely obscure broken replica phase.

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