



A statistical mechanics approach to Granovetter theory

Adriano Barra^{a,b,*}, Elena Agliari^{c,d}

^a Dipartimento di Fisica, Sapienza Università di Roma, Italy

^b Gruppo Nazionale per la Fisica Matematica, Sezione di Roma I, Italy

^c Dipartimento di Fisica, Università di Parma, Italy

^d Istituto Nazionale di Fisica Nucleare, Gruppo Collegato di Parma, Italy

ARTICLE INFO

Article history:

Received 18 March 2011

Received in revised form 26 June 2011

Available online 15 January 2012

Keywords:

Quantitative sociology

Small world

Neural network

Stochastic stability

ABSTRACT

In this paper we try to bridge breakthroughs in quantitative sociology/econometrics, pioneered during the last decades by Mac Fadden, Brock–Durlauf, Granovetter and Watts–Strogatz, by introducing a minimal model able to reproduce essentially all the features of social behavior highlighted by these authors.

Our model relies on a pairwise Hamiltonian for decision-maker interactions which naturally extends the multi-populations approaches by shifting and biasing the pattern definitions of a Hopfield model of neural networks. Once introduced, the model is investigated through graph theory (to recover Granovetter and Watts–Strogatz results) and statistical mechanics (to recover Mac-Fadden and Brock–Durlauf results). Due to the internal symmetries of our model, the latter is obtained as the relaxation of a proper Markov process, allowing even to study its out-of-equilibrium properties.

The method used to solve its equilibrium is an adaptation of the Hamilton–Jacobi technique recently introduced by Guerra in the spin-glass scenario and the picture obtained is the following: shifting the patterns from $[-1, +1] \rightarrow [0, +1]$ implies that the larger the amount of similarities among decision makers, the stronger their relative influence, and this is enough to explain both the different role of strong and weak ties in the social network as well as its small-world properties. As a result, imitative interaction strengths seem essentially a robust request (enough to break the gauge symmetry in the couplings), furthermore, this naturally leads to a discrete choice modelization when dealing with the external influences and to imitative behavior *à la* Curie–Weiss as the one introduced by Brock and Durlauf.

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1. Summarizing some main results of quantitative sociology

In recent years there has been an increasing awareness towards the problem of finding a quantitative way to study the role played by human interactions in shaping behavior observed at a population level, ranging from the context of pure sociology to the one belonging to economic sciences. The conclusion reached by all these studies is that mathematical models have the potential of describing several features of social behavior (e.g., the sudden shifts often observed in society's aggregate behavior [1–3]), and that these are unavoidably linked to the way individual people influence each other when deciding how to behave (the phase transitions in the language of thermodynamics [4]), the whole suggesting a promising potential application of disordered statistical mechanics to this field of research [5–7].

* Corresponding author at: Dipartimento di Fisica, Sapienza Università di Roma, Rome, Italy.

E-mail addresses: adriano.barra@roma1.infn.it (A. Barra), elena.agliari@fis.unipr.it (E. Agliari).

Here we summarize what we understood as real breakthroughs in these analysis, highlighting two main aspects dealing with topological investigations on the structure of the graph built by social interactions and the kind of interactions themselves. Namely, the discovery of the fundamental role of weak ties in bridging different communities (due to Granovetter [6,8–10]) and the “small world” feature of the social structure (obtained by Watts and Strogatz [11–13]) for the first analysis and the discrete choice of decision makers in econometrics (due to McFadden [14–16]) and the essentially imitative behavior among these agents (due to Brock and Durlauf [4,7,17]).

Even though fundamental experiments dealing with social networks may constellate modern society analysis (i.e., the paradigmatic Milgram experiment in the sixties [18]), a real breakthrough in our understanding of network structure inside modern societies was achieved when Granovetter reversed the Chicago school of social-psychology showing how a person who built weak ties (which were previously seen, at individual level, as ancestors of depressive states) was much more able to adapt his behavior to the social fitness due to the much broader amount of available information: in particular he noticed that these weak ties may often carry information of little significance (but not redundant as in highly clustered community of similar agents linked by strong ties), however they allow a primarily transmission of new information across otherwise disconnected clusters of the social network (with a consequent great potential benefit by these bridges).

Two decades after this achievement, Watts and Strogatz, through a mathematical technique (rewiring) have been able to recover Milgram results (sometimes known as “six degrees of separation”) by which they understood that social (as well as others, i.e., biological [19,20]) networks cannot be described by purely ordered or purely random graphs (e.g., Erdős–Rényi (ER) ones [20,21]), due to correlations among nodes which allow for a much faster transmission of information (real graphs show high degree of cliqueness [22,23]), pioneering a quantitative approach to these new networks, nowadays often called “small worlds”.

In a different but related context, McFadden has shown how to infer a model for econometric estimation of binary decision makers (reflecting accurately several real cases in social structures [15]) by introducing fundamental dichotomic degrees of freedom inside each agent mirroring its personal attitudes (e.g., a bit string ξ of K entries $\mu = 1, \dots, K$ where each entry represents an attribute, i.e., $\mu = 1$ accounting for smoking such that $\xi_i^{\mu=1} = +1$ means that the i -th agent smokes, while $\xi_i^{\mu=1} = 0$ means that he does not, and so on for $\mu = 2, \dots, K$). Once indexed individuals by i , $i = 1, \dots, N$, and assigned an Ising spin to each individual's choice $\sigma_i = +1$ for the agreement (or $\sigma_i = -1$ for disagreement), he chooses to exploit data by assuming a single particle model into a suitable external field h_i (the “field” influencing the choice of i) which is a function of the vector of attributes ξ_i . Since, for the sake of simplicity, attributes are taken as binary variables, the whole theory can be described in terms of an effective one-body Hamiltonian H as

$$H_N(\sigma; \xi) = \sum_i^N h_i \sigma_i = \frac{h}{K} \sum_i^N \sum_{\mu}^K \xi_i^{\mu} \xi_i^{\mu} \sigma_i,$$

where h is a scalar parameter ruling the overall intensity of the external stimulus (whose capabilities of influencing a generic agent i are encoded into its bit-string ξ_h). This parametrization of h_i corresponds to what economists call a *discrete choice* model [15], and shows a remarkable link between econometrics and statistical mechanics ($H_N(\sigma; \xi)$ can be seen as a (suitably adapted) random field Ising model): in fact discrete choice theory has the same variational flavor of thermodynamics and states that, when making a choice, each agent weights out various factors such as his own gender, age, income, etc, in order to maximize in probability the benefit arising from his/her decision.

Beyond this result, there exist many examples from economics and sociology where it has been observed how the global behavior of large groups of people can change in an abrupt manner as a consequence of slight variations in the social structure (such as, for instance, a change in the pronunciation of a language due to a small immigration rate, or as a substantial decrease in crime rates due to seemingly minor action taken by the authorities [3,24]). From a statistical mechanical point of view, these abrupt transitions should be considered as phase transitions caused by the interaction between individuals that cannot be accounted for by a pure one-body theory. Indeed, Brock and Durlauf have shown [17] how discrete choice can be extended to the case where a global mean-field interaction is present (providing an interesting mapping to the Curie–Weiss theory (CW) [4,25,26]), thus further highlighting the close relation existing between the econometric and the statistical mechanical approaches to these problems.

Instead of introducing the Brock–Durlauf approach (which is a deep and systematic translation of the CW scenario in social sciences) we go one step forward following the subsequent generalization obtained by dividing the ensemble of the N decision makers into clusters, due to Contucci and coworkers [14,27]: introducing a general two-body Hamiltonian $H_N(\sigma; J)$ as

$$H_N(\sigma; J) = - \sum_{i,l=1}^N J_{il} \sigma_i \sigma_l - \sum_{i=1}^N h_i \sigma_i, \quad (1.1)$$

they went over by defining a suitable parametrization for the interaction coefficients J_{il} . Since each agent is characterized by k binary socio-economic attributes, the population can be naturally partitioned into 2^k subgroups, which for convenience are taken of equal size: this leads to consider a mean field kind of interaction, where coefficients J_{il} depend explicitly on such a partition as follows

$$J_{il} = \frac{J}{2^k N} \delta_{gg'}, \quad \text{if } i \in g \text{ and } l \in g',$$

which in turn allows us to rewrite (1.1) as

$$H_N(\sigma; J) = -\frac{NJ}{2^k} \left(\sum_{g, g'=1}^{2^k} \delta_{gg'} m_g m_{g'} + \sum_{g=1}^{2^k} h_g m_g \right)$$

where m_g is the average opinion of group g , namely $m_g = \frac{1}{2^{kN}} \sum_{i=(g-1)N/2^k+1}^{gN/2^k} \sigma_i$.

If one focuses now on a finite amount of attributes, say $K = 2$ (e.g., male/female and young/old), the interaction matrix is block-like, something far from Hebbian style. However, this idea of partitioning the interaction matrix into clusters of similar agents can be extended to a natural limit (that we work out here) such that the size of these clusters approaches zero in the thermodynamic limit as $K \rightarrow \infty$ ¹: interestingly this leads to an interaction matrix so that

$$J_{ij} = \frac{1}{K} \sum_{\mu}^K \xi_i^{\mu} \xi_j^{\mu}, \tag{1.2}$$

and naturally collapses the concept of magnetization in spin glasses [28] to the one of retrieval in neural networks [29], ultimately switching frustration into dilution (as $\xi \in [0, +1]$ instead of $\xi \in [-1, +1]$).

2. The model and the emergent network

In this section we represent the population and the mutual interactions among nodes by means of a graph and we study its topological properties: each agent i is represented by a node; couples of agents (i, j) displaying a positive coupling $J_{ij} > 0$ are said to be in contact or to interact with each other and this is envisaged by means of a link between i and j , whose weight is just J_{ij} (cf. Eq. (1.2)). This picture mirrors the idea that socio-economic relations between individuals or firms are embedded and organized in actual social networks, which follows from the seminal work by Granovetter.

As anticipated, each agent i is characterized by a binary string ξ_i , which might be thought of as the codification of the attitude of the agent himself, either positive (1) or negative (0), towards a given issue. All strings are taken of length K and each entry is extracted randomly according to

$$P(\xi_i^{\mu} = +1) = \frac{1+a}{2}, \quad P(\xi_i^{\mu} = 0) = \frac{1-a}{2}, \tag{2.1}$$

in such a way that, by tuning the parameter $a \in [-1, +1]$, the concentration of non null-entries for the i -th string $\rho_i = \sum_{\mu} \xi_i^{\mu}$ can be varied. The set of strings ξ generates a weighted graph $G(N, K, a)$ whose topology is ruled by Eq. (1.2) and tuned via a . In particular, when $a \rightarrow -1$ the system is completely disconnected (and only discrete choice survives), while when $a \rightarrow +1$ each link is present, being $J_{ij} = 1$ for any couple (so to retain the fully Brock-Durlauf approach). As we will see, small values of a give rise to highly correlated, diluted networks, while, as a gets larger the network gets more and more connected and correlation among links vanishes. In agreement with our modelization intent, repetitions among strings are allowed.

The main topological features of the emergent network have been investigated in Ref. [19,30], where it was shown that the average link probability among two generic nodes is

$$p = 1 - \left[1 - \left(\frac{1+a}{2} \right)^2 \right]^K, \tag{2.2}$$

and that, for large enough N and K , with K growing slower than linearly with N , the degree distribution is multimodal. Therefore, the average degree for a generic node reads as $z = pN$.

Apart from these global, long-scale features, the model also displays interesting properties concerning correlation among links, as we are going to deepen.

Small-world (SW) networks are characterized by two basic properties, that are a large clustering coefficient, i.e., they display subnetworks where almost any two nodes within them are connected, and a small diameter, i.e., the mean-shortest path length among two nodes grows logarithmical with N . While the latter requirement is a common property of random graphs [20,21], and is therefore implicitly satisfied also by the graph under study (provided that a is properly chosen so to keep the graph connected under scaling), the clustering coefficient deserves more attention. Several attempts in the past have been made in order to define network models able to display such a feature [12,20,21]. For instance, in their seminal work, Watts and Strogatz [12] introduced a rewiring procedure on links, which can yield the desired degree of correlation.

¹ Essentially, two scalings, deeply different for both physical meaning and mathematical techniques involved, should be considered: a polynomial growth and a transcendental one, namely $K \sim N$, $K \sim \log N$. In Ref. [19] we considered the former, here, as it is unreasonable to think at a society with say 10^9 people each of which having $O(10^9)$ different attributes, we focus on the logarithmic case, such that the amount of attributes is reasonably small with respect to the size of the population.

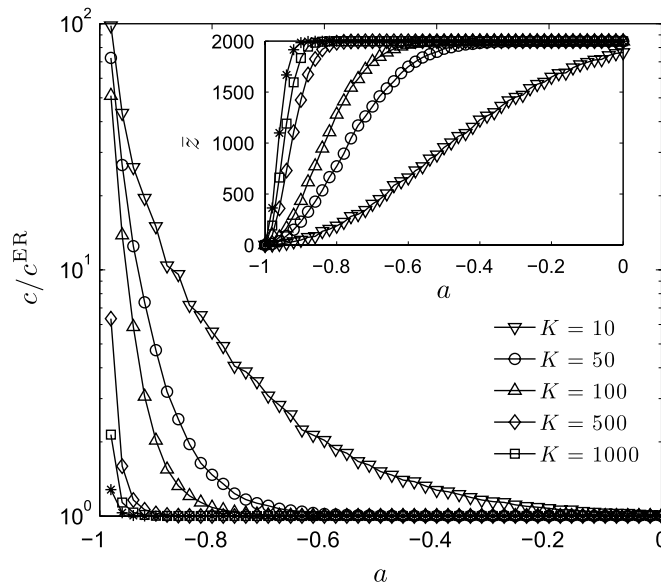


Fig. 2.1. Main figure: ratio between the clustering coefficient c measured for the correlated network under consideration and the coefficient c^{ER} corresponding to a purely random graph displaying an analogous density of links, as a function of the parameter a and for different values of K ; the volume N is kept fixed and equal to 2000. Inset: behavior of the average degree \bar{z} with respect to a for the same set of realizations corresponding to the main figure.

As we are going to show, in our approach SW effects emerge naturally from the definition of patterns and from the rule in Eq. (1.2); so to say that interactions based on sharing of interests (i.e., non-null entries) intrinsically generate a clustered society.

Before proceeding, we notice that the same property can be addressed in different ways: we can say that the graph exhibits a large transitivity, meaning that if i is connected to both j and k , then j and k are likely to be connected; in modern network theory we say that the graph displays a large “cliquishness” and we measure it by means of the so-called clustering coefficient [21,23]: the local clustering coefficient for node i is defined as

$$c_i = \frac{2E_i}{z_i(z_i - 1)}, \quad (2.3)$$

where E_i is the number of actual links present within the neighborhood of i , whose upper bound is just $z_i(z_i - 1)/2$, namely is the number of connections for a fully-connected group of z_i neighbors nodes. Then, the (global) clustering coefficient for the whole graph simply follows as the average of c_i over all nodes. Usually, for a graph with average degree equal to z , having a large clustering means that c is larger than $z/N = p$, which represents the clustering coefficient c^{ER} for an ER random graph with an analogous degree of dilution.

As for the graph under study, we found that [30] $c \approx p + 1/\rho - 1/(z - 1) > p$, where in the last equality we used $\rho < z - 1$, which holds when N is large enough and the graph topology non-trivial [30]. More generally, one can notice that for this graph, the z_i neighbors of i are all nodes displaying at least one non-null entry corresponding to any non-null entries of ξ_j ; this condition biases the distribution of strings relevant to neighboring nodes, so that they are more likely to be connected with each other. Indeed, when $\rho_i = 1$, it is easy to see that $c_i = 1$, to be compared with c_i^{ER} , namely the average link probability for node i , which turns out to be $(1 + a)/2 \leq 1$; analogous arguments apply also for larger values of ρ [30]. A numerical corroboration can be found in Fig. 2.1, which shows that $c > c^{ER}$ in a wide region of values of a corresponding to non-trivial networks, i.e., for a larger than the percolation threshold and smaller than the fully-connected threshold. The clustering effect is especially manifest in the region of high dilution, where, for graphs analyzed here, c is even two orders of magnitude larger than c^{ER} . Of course, when a approaches 1, the graph gets fully connected and $c \rightarrow c^{ER} \rightarrow 1$. These results are robust as N is varied.

Finally, we mention another quantity used in ecology and epidemiology to quantify the existence of correlations among links, that is the so-called *assortativity coefficient* [21,22]: a network is said to show “assortative mixing” (“dissortative mixing”) on their degrees whenever high-degree vertices prefer to attach to other high-degree (low-degree) vertices. While assortativity is typical of social networks, dissortativity is often found in technological and biological networks.

The assortativity coefficient r can be defined as a Pearson coefficient to measure the correlation between the coordination numbers at either ends of a link; the ER graph corresponds to $r = 0$ [21]. The measures performed on the graph under study suggest a dissortative behavior ($r < 0$), which is corroborated by the quantity $\langle z \rangle_z$, representing the average degree over

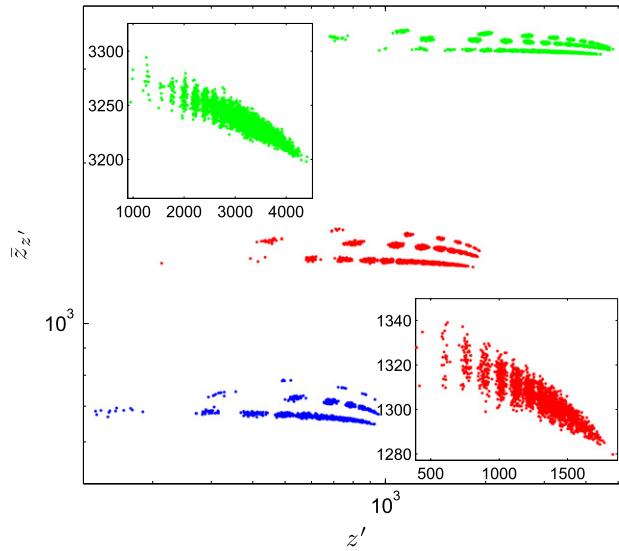


Fig. 2.2. Neighbor assortativity for different system sizes (from bottom to top $N = 2000$, $N = 5000$, $N = 8000$, depicted in different colors) and different values of N/K (for each choice of N , from bottom to top of each series, $N/K = 25$, $N/K = 100$, $N/K = 250$). Notice that for K small different modes can be distinguished. The two insets show in detail the case $N = 8000$, $K = 320$ (upper panel) and $N = 5000$, $K = 200$ (lower panel). The value of a is fixed and equal to -0.6 .

the nearest-neighbors of a node with degree z' , namely:

$$\bar{z}_{z'} = \sum_{z=0}^{N-1} zP(z; z'), \tag{2.4}$$

where $P(z; z')$ is the conditional probability that a link stemming from a node with degree z' points to a node with degree z . As shown in Fig. 2.2, a slow decreasing behavior of $\bar{z}_{z'}$ is consistent with a slight assortative mixing. The reason of this behavior is clear to see: while nodes corresponding to strings with large ρ can connect to most other nodes, nodes with small ρ have poor chances to connect to other similar strings. This gets more evident when N/K is small, vice versa, one can distinguish several classes of nodes (denoted to as “modes” in Refs. [19,30], each corresponding to strings displaying a different value of ρ) with a non-monotonic behavior of neighbor assortativity. Interestingly, as highlighted in Ref. [22], assortativity has significant effects on the resilience (see next section) of the structure itself: assortatively mixed networks are less robust to the deletion of their vertices than assortatively mixed or neutral networks.

As first remarked in Ref. [8], real social networks are not only characterized by a small-world topology, which basically means large clustering and small diameter, but they also feature a peculiar coupling pattern. In fact, not only the neighbors of a given node i are likely to be connected, but they form communities such that intra-group links are expected to be stronger than inter-group links. In this way weaker ties work as *bridges* connecting communities strongly linked up. Interestingly, analogous properties are found also for $\mathcal{G}(N, K, a)$, in fact, it is intuitive to see that nodes displaying very similar strings are likely to be intensively connected with each other, hence forming a group, while, each of them, separately and according to the pertaining string, can be weakly connected with other nodes/groups.

In order to deepen the role of weak ties, we perform two percolation processes where links are deleted either deterministically or randomly: given the pattern of couplings, in the former case we delete those with magnitude lower than a given threshold β meant as a tunable parameter. This has a physical meaning: in the forthcoming thermodynamics of the model, β will rule the level of noise in the network. For any particular choice of this parameter, all the links whose strength will be smaller than β , despite topologically counting, will have essentially zero weight in the composition of statistical mechanics averages, so that we can think at these as effectively being zero; in the latter case we progressively delete nodes in a random way. We call $1 - f$ the fraction of links erased (in the former case f is a function of β) and we measure the size of the largest connected cluster (see also Ref. [31] for more details). Results are shown and compared in Fig. 2.3. Interestingly, when weak links are deleted first, the graph starts to be disconnected at a value of f rather small, on the other hand, strong ties are highly redundant [31]. Indeed, starting from a connected graph, the first nodes to get disconnected are those with small ρ , to fix ideas, those with $\rho = 1$; among these, the ones with the non-null entry in the same position were completely clustered in the original network. As the threshold β is increased, more and more nodes get disconnected; most of them remain isolated (typically those with $\rho < \beta$), however, some non-trivial components survive. Such clusters are made up of very similar strings with a relatively large number of non-null entries ($\rho > \beta$) and are therefore all closely connected.

It is worth noting that such strongly clustered components emerge just in the “critical region”, namely where it is possible to detect nodes bridging two clusters and which play as “brokerage” between distinct groups; this is a strategic position since

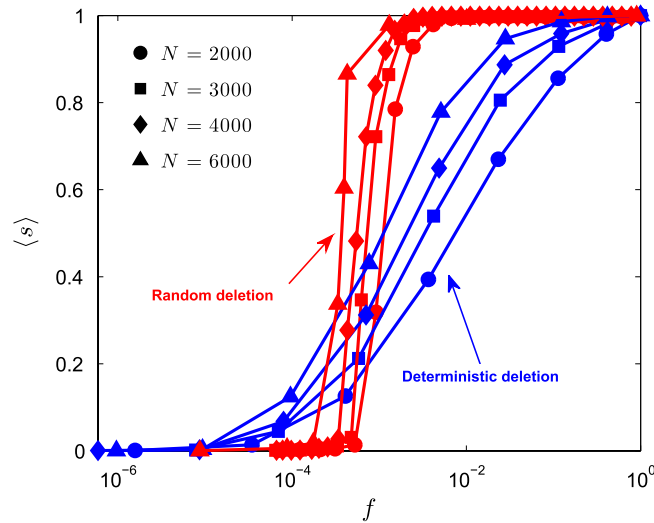


Fig. 2.3. Size of the largest component versus the fraction of links in the system. Several sizes are depicted $N = 2000, N = 3000, N = 4000, N = 6000$, all corresponding to the same value $\alpha = 100$ and $\gamma = 1$. Notice that when weak ties are deleted first, the way the size of the largest component $\langle s \rangle$ increases with f is smoother with respect to the random deletion and also that $\langle s \rangle$ is smaller than 1 at relatively large values of f .

it allows access to a more diverse set of ideas and information. The notions of homogeneity within groups and intermediacy between groups form the basis for the theory of “structural holes” introduced by Burt [32].

We finally comment on the resilience properties of the network under study, which can be as well inferred from the analysis on percolation processes. According to the situation, the stability of the network can be defined as its ability to remain connected or to still exhibit a giant component, under edge removal. In the former case, if weak links are the most prone to failure, our correlated network performs rather badly. Conversely, if we are interested in the maintenance of a macroscopic connected component, given that weak links are the first to be deleted, our correlated network performs definitely better, as the percolation threshold grows slowly with N (see also Ref. [31]).

3. The model and its relaxation: stochastic dynamics

We saw that the general structure of the Hamiltonian obeys

$$H_N(\sigma; \xi) = \frac{1}{NK} \sum_{i < j}^{N,N} \sum_{\mu=1}^K \xi_i^\mu \xi_j^\mu \sigma_i \sigma_j + \sum_i^N h_i \sigma_i. \tag{3.1}$$

It is worth recalling that the attributes ξ are drawn randomly once for all, and so are treated as quenched variables: this does not mean that a particular agent does not evolve in time changing his attribute arrangement, but that, overall, one agent may switch to another and viceversa as far as the global attribute distribution is kept constant.

Following standard disordered statistical mechanics approach [33] we introduce the latter according to

$$\sigma_i(t + 1) = \text{sign}(\tanh(\beta \varphi_i(t)) + \eta_i(t)), \tag{3.2}$$

where $\varphi_i(t)$ is the overall stimulus felt by the i -th agent, given by

$$\varphi_i(t) = N^{-1} \sum_j^N J_{ij} \sigma_j(t) + h_i(t), \tag{3.3}$$

and the randomness is in the noise implemented via the random numbers η_i , uniformly drawn over the set $[-1, +1]$. β rules the impact of this noise on the state $\sigma_i(t + 1)$, such that for $\beta = \infty$ the process is completely deterministic while for $\beta = 0$ completely random.

In the sequential dynamics we are introducing, at each time step t , a single agent l_t -randomly chosen among the N - is updated, such that its evolution becomes

$$P[\sigma_{l_t}(t + 1)] = \frac{1}{2} (1 + \sigma_{l_t}(t) \tanh(\beta \varphi_{l_t}(t))), \tag{3.4}$$

whose deterministic zero-noise limit is immediately recoverable by sending $\beta \rightarrow \infty$.

If we now look at the probability of the state at a given time $t + 1$, $P_{t+1}(\sigma)$, we get

$$P_{t+1}(\sigma) = \frac{1}{N} \sum_i^N \frac{1}{2} (1 + \sigma_i \tanh(\beta \varphi_i(\sigma))) P_t(\sigma) + \frac{1}{N} \sum_i^N \frac{1}{2} (1 + \sigma_i \tanh(\beta \varphi_i(F_i \sigma))) P_t(F_i \sigma), \tag{3.5}$$

where we introduced the N flip-operators F_i , $i \in (1, \dots, N)$, acting on a generic observable $\phi(\sigma)$, as

$$F_i \Phi(\sigma_1, \dots, +\sigma_i, \dots, \sigma_N) = \Phi(\sigma_1, \dots, -\sigma_i, \dots, \sigma_N) \tag{3.6}$$

such that we can write the evolution of the network as a Markov process

$$p_{t+1}(\{\sigma\}) = \sum_{\{\sigma'\}} W[\{\sigma\}; \{\sigma'\}] p_t(\{\sigma'\}), \tag{3.7}$$

$$W[\{\sigma\}; \{\sigma'\}] = \delta_{\{\sigma\}, \{\sigma'\}} + \frac{1}{N} \sum_{i=1}^N \left(w_i(F_i \{\sigma\}) \delta_{\{\sigma\}, F_i \{\sigma'\}} - w_i(\{\sigma\}) \delta_{\{\sigma\}, \{\sigma'\}} \right),$$

with the transition rates $w_i(\{\sigma\}) = \frac{1}{2} [1 - \sigma_i \tanh(\beta \varphi_i)]$.

As the affinity matrix is symmetric, detailed balance ensures that there exists a stationary solution $P_\infty(\{\sigma\})$ such that (restricting $\tilde{h}_i(t) \rightarrow \tilde{h}_i \in \mathbb{R} \forall i \in (1, \dots, N)$)

$$W[\{\sigma\}, \{\sigma'\}] P_\infty(\{\sigma'\}) = W[\{\sigma'\}, \{\sigma\}] P_\infty(\{\sigma\}).$$

This key feature ensures equilibrium, which implies

$$p_\infty(\sigma; J, h) \propto \exp\left(\frac{\beta}{2NK} \sum_{ij} \sum_{\mu} \xi_i^\mu \xi_j^\mu \sigma_i \sigma_j - \beta \sum_i h_i \sigma_i\right) = \exp(-\beta H_N(\sigma; \xi)), \tag{3.8}$$

namely the Maxwell–Boltzmann distribution [4,25] for the Hamiltonian (3.1).

In absence of external stimuli, and skipping here the question about the needed timescales for “thermalization”, the system reaches an equilibrium that is possible to work out explicitly and that reproduces all the features stressed in the first section (as we are going to show).

For this detailed balanced system furthermore, the sequential stochastic process (3.2) reduces to Glauber dynamics such that the following simple expression for the transition rates W_i can be implemented

$$W_i(\{\sigma\}) = \left(1 + \exp(\beta \Delta H(\sigma_i; \xi))\right)^{-1}, \quad \Delta H(\sigma_i; \xi) = H(F_i \{\sigma\}; \xi) - H(\{\sigma\}; \xi). \tag{3.9}$$

4. The model and its equilibrium: statistical mechanics

In the previous section we showed that, if the affinity matrix is symmetric (i.e., $J_{ij} = J_{ji}$), so that detailed balance holds, the stochastic evolution of our social model approaches the Maxwell–Boltzmann distribution (see Eq. (3.8)), which determines the thermodynamic equilibria.

The latter are obtained by extremizing the free energy $A(\beta, a, h) = -\beta f(\beta, a, h) = u(\beta, a, h) - \beta^{-1} s(\beta, a, h)$ (u being the internal energy and s being the intensive entropy) that, as it is straightforward to check, corresponds to both maximizing entropy and minimizing energy (at the given level of noise β , attribute’s bias a and external influences h). Furthermore, and this is the key bridge with stochastic processes, there is a deep relation among statistical mechanics and their equilibrium measure P_∞ , in fact

$$P_\infty(\sigma; \xi, h) \propto \exp(-\beta H(\sigma; \xi, h)), \quad A(\beta, a, h) = -\beta f(\beta, a, h) \equiv \frac{-1}{N} \mathbb{E} \log \sum_{\sigma} \exp(-\beta H_N(\sigma; \xi, h)).$$

The operator \mathbb{E} , that averages over the quenched distribution of attributes ξ , makes the theory not “sample-dependent”: for sure each realization of the network will be different with respect to some other in its details, but we expect that, after sufficient long sampling, the averages and variances of observable become unaffected by the details of the quenched variables.

Hence, once the microscopic interaction laws are encoded into the Hamiltonian, we can achieve a specific expression for the free energy, from which we can derive both the internal energy $u(\beta, a, h)$ as well as its related entropy $s(\beta, a, h)$:

$$u(\beta, a, h) = -\partial_\beta (\beta f(\beta, a, h)) = N^{-1} \langle H(\sigma; \xi, h) \rangle, \tag{4.1}$$

$$s(\beta, a, h) = f(\beta, a, h) + \beta^{-1} \partial_\beta (\beta f(\beta, a, h)). \tag{4.2}$$

The Boltzmann state is given by

$$\omega(\Phi(\sigma, \xi)) = \frac{1}{Z_N(\beta, a, h)} \sum_{\{\sigma_N\}} \Phi(\sigma; \xi) e^{-\beta H_N(\sigma, \xi)}, \quad (4.3)$$

where the normalization Z is called “partition function” and the total average $\langle \Phi \rangle$ is defined as

$$\langle \Phi \rangle = \mathbf{E}[\omega(\Phi(\sigma, \xi))]. \quad (4.4)$$

We want to tackle the problem of solving the thermodynamics of the model through the Hamilton–Jacobi technique [25,34–36].

Before outlining the strategy, some further definitions are in order here to lighten the notation (see also Ref. [30] for more details): taken g as a generic function of the quenched variables we have

$$\mathbb{E}_\xi g(\xi) = \sum_{l_b=0}^N \sum_{l_c=0}^K \binom{N}{l_b} \binom{K}{l_c} \left(\frac{1+a}{2}\right)^{l_b+l_c} \left(\frac{1-a}{2}\right)^{N+K-l_b-l_c} \delta_{l_b l_c = l} g(\xi), \quad (4.5)$$

where we summed over the probability $P(l)$ that in the graph a number l of weighted links out of the possible $N \times K$ display a non-null coupling, i.e., $\xi \neq 0$; this problem has been rewritten in terms of $P(l_b)$ and $P(l_c)$, where $P(l_b)$ is the probability that l_b (out of N random links) are active and analogously, mutatis mutandis, for $P(l_c)$ (on K random attributes): in fact, ξ_i^μ can be looked at as an $N \times K$ matrix generated by the product of two given vectors like η and χ , namely $\xi_i^\mu = \eta_i \chi_\mu$, in such a way that the number of non-null entries in the overall matrix ξ is just given by the number of non-null entries displayed by η times the number of non-null entries displayed by χ . Hence, $P(l)$ is the product of $P(l_b)$ and $P(l_c)$ conditional to $l_b l_c = l$.

We can introduce now the following order parameters

$$M_l = \frac{1}{N} \sum_i \omega_{l+1}(\sigma_i), \quad (4.6)$$

and the Boltzmann states ω_l are defined by taking into account only l terms among the elements of the whole involved.

Namely, ω_{l+1} has only $l+1$ terms of the type $\sigma\sigma$ in the Maxwell–Boltzmann exponential, all the others being zero: by these “partial Boltzmann states” we can define the average of the order parameters as

$$\langle M \rangle = \sum_l^{N-1} P(l) M_l. \quad (4.7)$$

We are now ready to show our strategy by defining the following interpolating free energy, depending on two interpolants, t, x , which can be thought of as time and space in a mechanical analogy [25,34–36]

$$A(t, x) = \frac{1}{N} \mathbb{E} \log \sum_\sigma \exp \left(\frac{-t}{2NK} \sum_{ij} \sum_{\mu} \xi_i^\mu \xi_j^\mu \sigma_i \sigma_j + \sum_i h_i \sigma_i + x \sum_i \tilde{\xi}_i \sigma_i \right), \quad (4.8)$$

where the random fields $\tilde{\xi}_i$ have the same distributions of the standard ξ_i^μ as in any standard stochastic stability approach. Of course statistical mechanics is obtained when evaluating this trial free energy at $t = -\beta, x = 0$. Let us work out the derivatives now:

$$\frac{\partial A(t, x)}{\partial t} = -\frac{1}{2} \left(\frac{1+a}{2} \right)^2 \langle M \rangle, \quad \frac{\partial A(t, x)}{\partial x} = \left(\frac{1+a}{2} \right) \langle M \rangle. \quad (4.9)$$

If we now introduce the following potential $V(t, x)$:

$$V(t, x) = \frac{1}{2} \left(\frac{1+a}{2} \right)^2 \left(\langle M^2 \rangle - \langle M \rangle^2 \right), \quad (4.10)$$

we can write the following Hamilton–Jacobi equation for the trial free energy

$$\partial_t A(t, x) + \frac{1}{2} \left(\partial_x A(t, x) \right)^2 + V(t, x) = 0. \quad (4.11)$$

When interested at the replica symmetric regime (in a nutshell, this constitutes an approximation, widely believed to be correct in diluted ferromagnets—even though not yet rigorously proved- in which we do not consider fluctuations of the order parameters in the large size of the population limit) we simply have to solve the free motion because replica symmetry means $\lim_{N \rightarrow \infty} V(t, x) = 0$. The free field solution is given by the action in a generic point of the space–time plus the time-integral of the Lagrangian $\mathcal{L} = \left(\frac{1+a}{2} \right)^2 \langle M^2 \rangle / 2$. Namely we can write

$$A(x, t) = A(x_0, 0) + \int_0^t \mathcal{L}(t') dt'. \quad (4.12)$$

So we have

$$A(x_0, 0) = \log 2 + \langle \log \cosh(x_0 \xi + \beta h) \rangle = \frac{1+a}{2} \log \cosh(x_0 + \beta h) + \frac{1-a}{2} \log \cosh(\beta h). \tag{4.13}$$

The time integral of the Lagrangian (as there is no potential) is simply $\frac{1}{2}(\frac{1+a}{2})^2 \langle M^2 \rangle t$ and the equation of motion is a straight line $x(t) = x_0 + \frac{1+a}{2} \langle M \rangle t$, such that overall we can write at $t = -\beta$ and $x = 0 \rightarrow x_0 = \frac{1+a}{2} \beta$

$$A(\beta) = \log 2 + \left(\frac{1+a}{2} \right) \log \cosh \left(\beta \left[\left(\frac{1+a}{2} \langle M \rangle + h \right) \right] \right) + \left(\frac{1-a}{2} \right) \log \cosh(\beta h) - \frac{\beta}{2} \left(\frac{1+a}{2} \right)^2 \langle M \rangle^2. \tag{4.14}$$

Now we want to deepen the information encoded into Eq. (4.14); namely we want to recover by this solution all the theories of interaction introduced in section one in a quantitative way.

Let us start forgetting the network, so with the two limits of McFadden independent particle model ($a \rightarrow -1$) and the pure Brock and Durlauf theory ($a \rightarrow +1$):

$$\lim_{a \rightarrow -1} A(\beta, a) = \log 2 + \langle \log \cosh(\beta h) \rangle, \tag{4.15}$$

$$\lim_{a \rightarrow +1} A(\beta, a) = \log 2 + \langle \log \cosh(\beta M_{CW} + h) \rangle - \frac{\beta}{2} \langle M_{CW}^2 \rangle, \tag{4.16}$$

in perfect agreement with thermodynamics [4,25].

Note that when extremizing the free energy with respect to the order parameter (which is just one in both cases because in the former, as there is no network, the only decomposition through Eq. (4.7) is the independent sum of all the disconnected agents, while in the latter only one graph survives -the unweighted fully connected- and $P(\langle M \rangle) = \delta(M - M_{CW})$, where with M_{CW} we meant the standard CW magnetization), the response of the system is described by the hyperbolic tangent (nothing but the logit fit function used in econometrics [14]):

$$\partial_M A(\beta, a = -1, h) = 0 \Rightarrow M = \langle \tanh[\beta h] \rangle, \tag{4.17}$$

$$\partial_{(M_{CW})} A(\beta, a = +1, h) = 0 \Rightarrow M_{CW} = \langle \tanh[\beta(M_{CW} + h)] \rangle. \tag{4.18}$$

In all the other cases of interest (so for $a \neq \pm 1$) a distribution for weights on links is always present and weights are stronger for links among nodes that share higher attribute similarity as shown in Section 2.

Last step now should be achieving the critical line, e.g., by the control of the fluctuations of $\mathcal{M} = \sqrt{N}(M - \bar{M})$. To fulfill this task it is possible to follow the approach of Ref. [30, section four], with the streaming now given by the transport derivative $D = \partial_t + (\frac{1+a}{2}) \langle M \rangle \partial_x$ [34]. Instead of performing these calculations which mirrors the ones thoroughly exposed in Ref. [30] and depict a phase transition at $\beta_c = (\frac{1+a}{2})^{-2}$ (as naively expected), we find instructive to bridge the two solutions (at $h = 0$), namely the one obtained in section tree of Ref. [30, Eq. (4.25) in that paper] and (4.14).

Starting from the former, let us at first use the self consistency relation (Eq. 4.24 of the above-cited paper) to transform $\tanh^{-1}(\sum_l P(l)M_l) = (\frac{1+a}{2})\beta \langle M \rangle$. This gives

$$A(\beta, a) = \log 2 + \left(\frac{1+a}{2} \right) \log \cosh \left(\beta \left(\frac{1+a}{2} \langle M \rangle \right) \right) + \frac{\beta \left(\frac{1+a}{2} \right)^2}{2} \langle M^2 \rangle - \left(\frac{1+a}{2} \right)^2 \beta \langle M \rangle^2.$$

The latter can be written exactly as

$$A(\beta, a) = \log 2 + \left(\frac{1+a}{2} \right) \log \cosh \left(\beta \left(\frac{1+a}{2} \langle M \rangle \right) \right) - \frac{\beta \left(\frac{1+a}{2} \right)^2}{2} \langle M^2 \rangle$$

by assuming $\langle M \rangle^2 = \langle M^2 \rangle$, which, in a nutshell, is sharply the request $S = 0$ (zero source limit) in the double stochastic stability approach and $V = 0$ in the Hamilton–Jacobi approach.

5. Summary and outlooks

In this paper we tried to bridge over different aspects of modern quantitative sociology in a unifying perspective, ultimately offered by a simple shift of the patterns in a Hopfield model of neural networks. The fundamental prescriptions of Granovetter and Watts–Strogatz theories, from a topological viewpoint, and of McFadden and Brock–Durlauf theories, from social influences, are recovered as different limits of this larger model, where, in proper (wide) regions of the parameters (β, a, h), all these features can be retained contemporarily, offering a systemic view of social interaction.

The idea that a model for the associative memory of the brain (thought of as an ensemble of many interacting elementary agents) may work even for quantifying social behavior is in general agreement with the “universality” found in all these complex systems; however, while the neural networks share both positive and negative links (so to preserve a low synaptic activity, i.e., $\sum_{\mu} \xi^{\mu} \rightarrow \mathcal{N}(0, 1)$), this property is avoided in our context (for otherwise the role of weak ties could be played

by highly conflicting peoples). As a consequence, despite the role of anti-imitative coupling is fundamental (as discussed for instance in Ref. [37]), it turns out that the greatest part of social interactions should be essentially imitative (as sociologists have known for a long time).

Moreover, by focusing on couplings generated from the sharing of common attributes, a small-world structure naturally emerges: Indeed, small-world features can be looked at as an indication of transitivity and in our model the coupling among nodes is based on the similarity between the related attributes, which is a transitive property. Also, consistently with real networks, it is possible to detect strongly clustered sub-communities.

So, our main goal when dealing with these techniques, is not discovering other hidden breakthroughs among pioneering speculations, but offering a quantitative, predictive model (and related methods for its solution) by which recovering agreement with data and existing theories.

In this sense, we think that a great deal of effort should now be put forward in dealing with the inverse problem and its related data analysis, on which we plan to investigate soon.

Acknowledgments

The authors are pleased to thank Raffaella Burioni, Mario Casartelli, Claudia Cioli, Pierluigi Contucci, Mark Granovetter, Enore Guadagnini and Francesco Guerra for useful discussions.

This work is supported by the FIRB grant: RBFRO8EKEV.

GNFM and INFN are also acknowledged.

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