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On quantum and relativistic mechanical analogues in mean-field spin models

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Conceptual analogies among statistical mechanics and classical or quantum mechanics have often appeared in the literature. For classical two-body mean-field models, such an analogy is based on the identification between the free energy of Curie–Weiss-type magnetic models and the Hamilton–Jacobi action for a one-dimensional mechanical system. Similarly, the partition function plays the role of the wave function in quantum mechanics and satisfies the heat equation that plays, in this context, the role of the Schrödinger equation. We show that this identification can be remarkably extended to include a wider family of magnetic models that are classified by normal forms of suitable real algebraic *dispersion curves*. In all these cases, the model turns out to be completely solvable as the free energy as well as the order parameter are obtained as solutions of an integrable nonlinear PDE of Hamilton–Jacobi type. We observe that the mechanical analogue of these models can be viewed as the *relativistic* analogue of the Curie–Weiss model and this helps to clarify the connection between generalized self-averaging in statistical thermodynamics and the semiclassical dynamics of viscous conservation laws.

This work is dedicated to Sandro Graffi in honour of his seventieth birthday.

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1. Introduction

A powerful approach for mean-field spin glass models is based on the formal analogy between mean-field statistical mechanics and the Hamilton–Jacobi formulation of classical mechanics.

Such an analogy has been pointed out and investigated over the past few decades, and tracing back in time the genesis of such an approach, because of the vast popularity of these magnetic mean-field models, is not a simple task. Newman pointed out the analogy in 1981, as did Bogolyubov and co-workers in the early 1980s [1,2]; more recently, Choquard & Wagner [3] as well as the present authors and colleagues (see [4–8] and also [9–12]) have done the same.

However, the discovery of such an analogy turns out to be nothing but the tip of an iceberg requiring further exploration. This correspondence is indeed very profound and shows a hidden (and at first glance even counterintuitive) relation between the minimum action principle in mechanics (which is often used to describe determinism) and the second principle of thermodynamics (which is often used to justify randomness and stochasticity). Indeed, one can show that the free energy of a statistical mechanical model can be interpreted as the Hamilton–Jacobi function of a suitable one-dimensional mechanical system. For the Curie–Weiss model, the Hamilton–Jacobi equations imply that the magnetization satisfies the celebrated Burgers equation, perhaps the simplest scalar model for the propagation of nonlinear waves in a viscosity regime. The thermodynamic limit for the magnetic model is equivalent to the inviscid limit of the Burgers equation and leads to the so-called inviscid Burgers equation that is also known as the Riemann–Hopf equation. This limit is interpreted as a second principle definition because it turns out to be equivalent to a minimal action principle for the free energy function. The Riemann–Hopf equation is the simplest example of the nonlinear conservation law introduced to describe the propagation of nonlinear hyperbolic waves in the zero dispersion regime. Despite its simplicity, this equation already possesses several interesting features that make it suitable for the description of thermodynamic phase transitions. For instance, solutions to the Rieman–Hopf equation generically fail as they develop a gradient catastrophe in finite time. The gradient catastrophe point is associated with the caustics of the characteristic lines and it is usually interpreted as the critical point for a magnetic phase transition. The critical point develops into a classical shock wave that explains the mechanism responsible for discontinuities of the order parameter or its derivatives.

A model based on the Riemann–Hopf equation is completely integrable via the characteristics method and its general solution provides the equation of state, that is, the consistency equation, of the model. This description seems to be very general, as it has also been observed in the context of van der Waals models and their virial extensions [9] and in pure glassy scenarios [5] and it leads to the construction of a one-to-one correspondence table between some standard concepts in classical thermodynamics and the theory of classical shocks and conservation laws [11]. Although the Riemann–Hopf equation turns out to provide an accurate description of the model away from the critical region, in the vicinity of the critical point a suitable multi-scale asymptotic analysis of the Burgers equation is required. It was shown in [13] that the asymptotic behaviour in the vicinity of the critical point is universally expressed in terms of the Pearcey integral and it is argued in [14] (see also [15]) that such a description extends to more general Burgers-type equations.

In this paper, we determine the formal analogy between mean-field models and one-dimensional mechanical systems at the level of the partition function that in this context plays the role of a (real-valued) quantum-mechanical wave function and satisfies a linear PDE. Consistent with the description outlined above, the associated Hamilton–Jacobi function is interpreted as the free energy of the model. In particular, we focus on a class of solvable generalized models of N interacting spins where the Hamiltonian function is given, as in the cases mentioned above, by the linear combination of the potential associated with the internal spin interaction and the one associated with the external field

$$H_N = H_{\text{int}}(m_N) + hH_{\text{ext}}(m_N), \tag{1.1}$$

where

$$m_N = \frac{\sum_i \sigma_i}{N}$$

is the mean magnetization per spin particle. We argue that a natural generalization of the Curie–Weiss model can be obtained from the assumption that the internal and external potentials satisfy a certain polynomial relation referred to as the *dispersion curve*. This implies, as for the Curie–Weiss model, that the partition function solves a linear PDE, where temperature and external magnetic field coupling are the independent variables and the number of particles N plays the role of a scale parameter. The solution in the large N limit is obtained via the standard WKB approach leading to a Hamilton–Jacobi-type equation for the free energy function. Similarly to the semiclassical approximation of quantum mechanical models and the geometric optics approximation of the Maxwell equations, the Hamilton–Jacobi-type equation so obtained provides an accurate description of the magnetic system in the thermodynamic limit away from the caustic lines associated with the boundary of the critical region. We analyse in detail models associated with a second-order dispersion curve whose normal form reduces to a conic. We note that the parabolic case, referred to as the F-type scenario, gives the Curie–Weiss model. The elliptic and the hyperbolic case, the P-type and K-type scenario, respectively (i.e. Poisson-like and Klein–Gordon-like), can be viewed as a deformation of the Curie–Weiss model involving infinitely many p-spin contributions. We observe that in all cases the Hamilton–Jacobi-type equation for the free energy reduces to a Riemann–Hopf-type equation for the expected value of the magnetization. The model is then completely integrable via the characteristics method (e.g. [16]) and the critical point of the gradient catastrophe is signified by the occurrence of a magnetic phase transition.

The paper is structured as follows: in §2, we illustrate the methodology in general terms. Section 3 is dedicated to examples, one for each case. Section 4 contains our conclusion and outlooks.

2. Generalized models and techniques for mean-field many-body problems

Given N Ising spins $\sigma_i = \pm 1$, $i \in \{1, \ldots, N\}$, let us consider a general *ferromagnetic* model of a Hamiltonian of form

$$\frac{H_N}{N} = -F(m_N) - hG(m_N), \tag{2.1}$$

where

$$m_N = \frac{1}{N}\sum_{i=1}^{N} \sigma_i$$

is the magnetization, $F(m_N)$ models the generic p-spin mean-field interaction and $G(m_N)$ accounts for the interaction with an external magnetic field h (that, in many cases, is one body, i.e. $G(m_N) = m_N$).

Note that, generally, with the adjective *ferromagnetic*, we mean models whose interaction matrix has only positive entries, e.g. $H_N = -(1/N)\sum_{i<j}^{N} J_{ij}\sigma_i\sigma_j$, with $J_{ij} = J > 0$ for all the $N(N-1)/2$ couples. However, as the effect of J on the model's thermodynamics is only to shift the critical temperature, in the following we simply set $J \equiv 1$.

The Boltzmann average of the magnetization is standardly denoted as follows:

$$\langle m \rangle = \lim_{N\to\infty} \frac{\sum_{\{\sigma\}}^{2^N} \sigma_i \exp(-\beta H_N)}{\sum_{\{\sigma\}}^{2^N} \exp(-\beta H_N)}, \tag{2.2}$$

where the sum is evaluated over all spin configurations $\{\sigma\}$, and $\beta = 1/k_B T$, where T is the temperature and k_B is the Boltzmann constant (that we set to one in proper units). The main object of interest is the free energy function $f(\beta, h) = -\alpha(\beta, h)/\beta$, where

$$\alpha(\beta, h) = \lim_{N\to\infty} \frac{1}{N} \ln \sum_{\{\sigma\}}^{2^N} \exp(-\beta H_N) \tag{2.3}$$

is called the *mathematical pressure*.

The free energy is related to the thermodynamic average of the intensive entropy S and the internal energy E via the standard formula $f = E - \beta^{-1}S$ (or, alternatively, in terms of the mathematical pressure, $\alpha(\beta, h) = S - \beta E$) that allows us to deduce all thermodynamic properties of the system induced by the Hamiltonian H_N. However, as the mathematical pressure $\alpha(\beta, h)$ is more convenient for computational purposes w.r.t. $f(\beta, h)$, and its usage largely prevailed in the community of disordered statistical mechanics (where most of the applications—of the theory we are going to develop—lie), in the following we will use the former with a little language abuse.

(a) Generalized thermodynamic limit and its variational formulation

We introduce two scalar variables $t \in \mathscr{R}^+$ and $x \in \mathscr{R}$ (which can be thought of as time and space in the *mechanical analogy* that we are going to develop), and we consider, first, F such that $F(m) = F(-m)$, $\partial^2_{xx}F(m) > 0$ and $F(0) = 0$, and then we set $G(m) \equiv m$; next, we consider the class of models associated with a Hamiltonian $-N[F(m) + hm] \equiv H\colon (0, 1) \ni m \mapsto H(m)$.

We now prove that, under the above assumptions, the thermodynamic limit for the system defined via H_N is well defined. We have the following theorem.

Theorem 2.1. *The thermodynamic limit for the free energy $\alpha_N(t, x)$ exists and is*

$$\lim_{N\to\infty} \frac{1}{N} \ln Z_N(t, x) = \inf_N \frac{1}{N} \ln Z_N(t, x) = \alpha(t, x), \tag{2.4}$$

where $Z(x, t)$ is the partition function

$$Z_N(t, x) = \sum_{\{\sigma\}}^{2^N} \exp(N(tF(m_N) + xm_N)), \tag{2.5}$$

with $\forall t > 0$ and $\forall x \in \mathscr{R}$ (which, in order to bridge with thermodynamics, should be related to temperature and magnetic field via $t = 1/T$ and $x = h/T$).

The proof of this statement works within the classical Guerra–Toninelli scheme [17]. It is sufficient to prove the model sub-additivity as stated in the following lemma.

Lemma 2.2. *The extensive free energy related to the generalized models defined by $-H(m)/N = F(m) + hG(m)$ is sub-additive in the volume N, namely*

$$\ln Z_N(t, x) \le \ln Z_{N_1}(t, x) + \ln Z_{N_2}(t, x). \tag{2.6}$$

Proof. Let us split the system into two subsystems of size N_1 and N_2 such that $N = N_1 + N_2$. Let m_1 and m_2 be the partial magnetizations associated with the two subsystems such that $m = (N_1/N)m_1 + (N_2/N)m_2$. Hence, because of the convexity of F, we have

$$F(m) = F\left(\frac{N_1}{N}m_1 + \frac{N_2}{N}m_2\right) \le \frac{N_1}{N}F(m_1) + \frac{N_2}{N}F(m_2). \tag{2.7}$$

By virtue of the above inequality, the partition function (2.5) satisfies the following:

$$Z_N(t, x) \le Z_{N_1}(t, x) \cdot Z_{N_2}(t, x), \tag{2.8}$$

which proves the lemma. ■

The route from lemma 2.2 to theorem 2.1 is the classical one set out by Ruelle [18].

Now we proceed showing that the variational formulation of statistical mechanics is preserved even in this extended scenario. Let us prove the following theorem.

Theorem 2.3. *Given the variational parameter $-1 \leq M \leq +1$ and the trial free energy*

$$\tilde{\alpha}(t,x|M) = \ln 2 + \ln\cosh(x + t\partial_x F(M)) + t(F(M) - M\partial_x F(M)), \tag{2.9}$$

and its optimized value (w.r.t. M)

$$\hat{\alpha}(t,x) = \max_M \tilde{\alpha}(t,x|M),$$

then we can write $\alpha(t,x) = \hat{\alpha}(t,x)$.

Proof. Let us introduce the auxiliary function $g(m,M)$ as

$$g(m,M) = \exp(-tN(F(m) - F(M) - \partial_x F(M)(m-M))). \tag{2.10}$$

Clearly, because of the convexity we have $g(m,M) \leq 1$. Let us consider only those values of M that can also be assumed by m and let us restrict only those values of the sum over M, which will be denoted with a star, i.e. $\sum_M \to \sum_M^*$. Then

$$\sum_M^* g(m,M) \geq 1, \tag{2.11}$$

because, with probability one, a term in the sum will have $m = M$ and its corresponding $g(M,M) \equiv 1$; as all the others are non-negative, equation (2.11) holds. Then, we have

$$Z_N(t,x) = \sum_\sigma \mathrm{e}^{tNF(m)}\,\mathrm{e}^{xNm} \geq \sum_\sigma \mathrm{e}^{tNF(m)}\,\mathrm{e}^{xNm} g(m,M) = \mathrm{e}^{N\tilde{\alpha}_N(t,x|M)}, \tag{2.12}$$

as $1 \geq g(m,M)$; thus, the sum factorizes, the $F(m)$ terms cancel and we can conclude the first bound—namely, taking the thermodynamic limit and optimizing w.r.t. M

$$\alpha(t,x) \geq \hat{\alpha}(t,x). \tag{2.13}$$

To prove the reverse bound, we can write

$$Z_N(t,x) \leq \sum_\sigma \mathrm{e}^{tNF(m)}\,\mathrm{e}^{xNm} \sum_M^* g(m,M) = \sum_M^* \mathrm{e}^{N\tilde{\alpha}(t,x|M)} \leq \sum_M^* \mathrm{e}^{N\hat{\alpha}(t,x)}. \tag{2.14}$$

Thus, $\alpha_N(t,x) \leq \hat{\alpha} + \ln(1+N)/N$ because $\sum_M^*$ now gives $N+1$ identical terms (because, as stated above, there is no longer dependence on M during the summation procedure), hence $Z_N(t,x) \leq (N+1)\exp N\hat{\alpha}(t,x)$: taking the logarithm of $Z_N(t,x)$ and dividing by N, we obtain the expression above, which in the thermodynamic limit returns the expected bound and ends the proof. ∎

In the following subsections, we will investigate those values of $M(t,x)$ that optimize the evolution through the mechanical approach.

(b) Dispersion curve and generalized models

Let us assume that the potentials $F(m_N)$ and $G(m_N)$ that define the Hamiltonian (2.1) belong to the *dispersion curve* given by the equation

$$P_d(F,G) = 0, \tag{2.15}$$

where

$$P_d(\eta,\xi) = \sum_{k,l} c_{k,l}\eta^k \xi^l$$

is a polynomial of degree $d = \max\{k + l \,|\, c_{k,l} \neq 0\}$. Introducing the linear differential operator of order d

$$L_d = \sum_{k,l} c_{k,l}\partial_t^k(-\partial_x)^l,$$

one can readily verify that, given the condition (2.15), the partition function (2.5) can be obtained as a solution to the following linear differential equation:

$$L_d[Z_N] = 0. \tag{2.16}$$

Equation (2.16) can be viewed as the statistical analogue of a quantum mechanical wave equation, where Z_N plays the role of the wave function. More explicitly, setting $\nu = 1/N$, equation (2.16) becomes

$$\sum_{k,l} \nu^{k+l} c_{k,l} \partial_t^k (-\partial_x)^l Z_N = 0. \tag{2.17}$$

From the definition of the free energy α_N in (2.3), we get $\alpha_N = \nu \log Z_N$ and then $Z_N = e^{\alpha_N/\nu}$.

Substituting the above change of variable into equation (2.17), we obtain at leading order as $\nu \to 0$ (according to the standard WKB approximation) the following Hamilton–Jacobi-type equation:

$$P_d(\alpha_t, \alpha_x) = 0,$$

where $\alpha = \lim_{N\to\infty} \alpha_N$.

Let us now analyse the particular class of models associated with a polynomial relation of the form (2.15) of degree $d = 2$, that is,

$$c_1 F^2 + c_2 FG + c_3 G^2 + c_4 F + c_5 G + c_6 = 0. \tag{2.18}$$

The quadratic equation (2.18) can be reduced via a suitable linear change of variables to one of the following canonical forms:

$$F^2 + G^2 - 1 = 0, \tag{2.19}$$

$$F^2 - G^2 - 1 = 0 \tag{2.20}$$

and

$$F - G^2 = 0. \tag{2.21}$$

The corresponding partition function satisfies one the following normal forms:

$$\nu^2 (Z_{tt} + Z_{xx}) = Z, \tag{2.22a}$$

$$\nu^2 (Z_{tt} - Z_{xx}) = Z \tag{2.22b}$$

and

$$Z_t - \nu Z_{xx} = 0. \tag{2.22c}$$

Many-body problems associated with a quadratic dispersive curve will be referred to as P-type, K-type and F-type according to whether their canonical form is the Poisson equation (2.22*a*), the Klein–Gordon equation (2.22*b*) or the Fourier (or heat) equation (2.22*c*), respectively.

Proposition 2.4. *The WKB approximation of equations (2.22), standardly performed by the substitution $Z = e^{\alpha/\nu}$, gives, in the thermodynamic limit $\nu \to 0$ (i.e. $N \to \infty$), one of the following three equations for the free energy α:*

$$\alpha_t^2 + \alpha_x^2 = 1, \tag{2.23a}$$

$$\alpha_t^2 - \alpha_x^2 = 1 \tag{2.23b}$$

and

$$\alpha_t - \alpha_x^2 = 0. \tag{2.23c}$$

Equations (2.23) show that the free energy α plays the same role as the Hamilton–Jacobi function in classical mechanics.

Moreover, equations (2.23) are completely integrable and can be solved via the method of characteristics. Differentiating equations (2.23) w.r.t. x, we obtain the following

Riemann–Hopf-type equation:

$$u_t = (V(u))_x, \tag{2.24}$$

where $u = \alpha_x$ and the function $V(u)$ is given as follows:

$$\begin{aligned} &\text{P-type} \quad V(u) = -\sqrt{1-u^2} \\ &\text{K-type} \quad V(u) = \sqrt{1+u^2} \\ &\text{F-type} \quad V(u) = u^2. \end{aligned}$$

In particular, based on the classical method of characteristics, we have the following theorem.

Theorem 2.5. *The general solution u to equation (2.24) is readily obtained via the method of characteristics and is given by the formula*

$$x + V'(u)t = f(u), \tag{2.25}$$

where $f(u)$ is an arbitrary function of its argument that is locally fixed by the initial condition on u. In particular, given the initial datum

$$u(x, 0) = U(x),$$

we have that $f = U^{-1}$ is the inverse function of $U(x)$. The free energy, i.e. the solution to the corresponding equation in (2.23), is obtained by direct integration as follows:

$$\alpha = \int_0^x u(\xi, t)\, \mathrm{d}\xi + \Phi(t),$$

where function $\Phi(t)$ is such that $\Phi' = V(u(0, t))$.

It is well known that the generic solution to the conservation laws of the form (2.24) fails in finite time by developing a gradient catastrophe. At the point of the gradient catastrophe that is the analogue of caustics in the geometric optics limit and in the semiclassical limit of quantum mechanics, the WKB approximation fails and the classical solution develops a multi-valuedness. The appropriate description of the system beyond the region where the classical solution is multi-valued requires the study of equations (2.22). However, the critical point of the gradient catastrophe is signified by a phase transition from a disordered (‘classical’) to an ordered (‘quantum’) state. Clearly, whether or not a phase transition will occur depends on the particular model that is specified by the initial datum via the function $f(u)$ in (2.25). More specifically, we have the following theorem.

Theorem 2.6. *The critical point (x_c, t_c, u_c) is given, if it exists, as a solution to the following equations:*

$$x_c + V'(u_c)t_c = f(u_c), \quad V''(u_c)t = f'(u_c) \quad \text{and} \quad V'''(u_c)t = f''(u_c), \tag{2.26}$$

such that

$$\frac{f^{(3)}(u_c)}{V''(u_c)} - \frac{V^{(4)}(u_c)f'(u_c)}{V''(u_c)^2} > 0.$$

3. Examples

(a) Fourier scenario

The mechanical interpretation of the Curie–Weiss model, which is associated with the F-type normal, has already been extensively discussed in a number of papers (e.g. [4]). Let us briefly recall the main leading to the definition of such an analogy.

Definition 3.1. The Curie–Weiss Hamiltonian is defined by the Hamiltonian of form

$$\frac{1}{N} H_N(m_N) = -\frac{1}{2} m_N^2 + h m_N. \tag{3.1}$$

We are interested in an explicit expression of the free energy in terms of the order parameter. A number of methods have been proposed over the decades and are currently available (see, for example, [4] for a recent review) to evaluate the free energy including a solution method based on a *mechanical analogy*.

Following the interpolation procedure introduced in [8], let us consider the interpolating free energy (or interpolating *action*)

$$\alpha_N(t,x) = \frac{1}{N}\ln\sum_{\{\sigma\}}^{2^N}\exp\left(-t\cdot\frac{Nm_N^2}{2}+x\cdot m_N\right) = \frac{1}{N}\ln\sum_{\{\sigma\}}^{2^N}\exp(\mathbf{X}\cdot\mathbf{E}), \tag{3.2}$$

such that $\alpha(t=-\beta, x=0) = \lim_{N\to\infty}\alpha_N(t=-\beta, x=0)$, i.e. it returns to the thermodynamical free energy in the absence of an external field.

Note that, in the last term of equation (3.2), we have introduced the two-vector space–time as $\mathbf{X} = (t, -x)$ and the two-vector energy–momentum as $\mathbf{E}/N = (\langle m_N^2\rangle/2, \langle m_N\rangle)$.

Theorem 3.2 ([8]). *The free energy* (3.2) *satisfies the following Hamilton–Jacobi-type equation:*

$$\frac{\partial\alpha_N(t,x)}{\partial t} + \frac{1}{2}\left(\frac{\partial\alpha_N(t,x)}{\partial x}\right)^2 - V_N(t,x) = 0, \tag{3.3}$$

where $V_N(t,x) = N^{-1}\partial_x^2\alpha(x,t) = \frac{1}{2}(\langle m_N^2\rangle - \langle m_N\rangle^2)$.

Proof. By a direct calculation, it is straightforward to show that expression (3.2) for the free energy solves equation (3.3). ■

In the domain where the function $\alpha(x,t)$ is sufficiently smooth (i.e. smooth enough to have a unique maximizer in the variational problem of theorem 2.1), in the thermodynamic limit, we have

$$\lim_{N\to\infty} V_N(t,x) = \lim_{N\to\infty}\frac{1}{2}(\langle m_N^2\rangle - \langle m_N\rangle^2) = 0,$$

and the corresponding free energy

$$\alpha(t,x) = \ln 2 + \ln\cosh(x + m(t,x)t) - \frac{m(t,x)^2}{2}t$$

is the solution to the Hamilton–Jacobi equation (3.3) with the initial datum

$$\alpha(0,x) = \ln 2 + \ln\cosh x \tag{3.4}$$

that is obtained via a direct evaluation of the sum in (3.2) and where $m(t,x)$ is the unique maximizer in the variational problem defined by theorem 2.1. In particular, at zero external field where the phase transition occurs, we have [7]

$$\alpha(\beta) = \ln 2 + \ln\cosh(\beta m) - \tfrac{1}{2}\beta m^2, \tag{3.5}$$

where we recall that $t = \beta$.

Remarkably, the principles of thermodynamics (such as the free energy minimization) come into play here as the Maupertius minimum action principle and imply the extremization of this expression w.r.t. the order parameter giving the celebrated self-consistency equation $\langle m\rangle = \tanh(\beta\langle m\rangle)$.

As is well known, the self-consistency equation predicts a paramagnetic phase at $\beta < 1$, with $\langle m\rangle \equiv 0$ and a bifurcation at the critical noise level $\beta_c = 1$, from which two branches of the magnetization (symmetric around zero) arise and the system undergoes a phase transition towards a ferromagnetic phase. As figure 1 shows, the magnetization develops a gradient catastrophe at the origin $x = 0$, where m vanishes, and at $t = 1$. The critical values are obtained via equations (2.26).

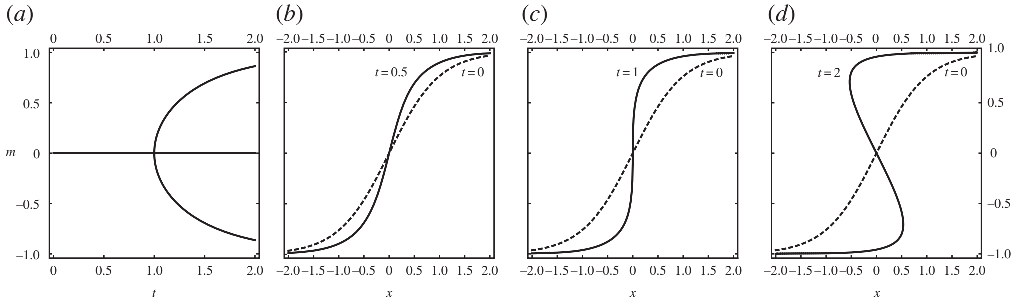

Figure 1. Analysis of the F-type. (*a*) Magnetization profile at $x = 0$ versus t. Magnetization profile versus x at, respectively, (*b*) $t = 0.5 < t_c$, (*c*) $t = 1.0 = t_c$, and (*d*) $t = 2 > t_c$. Beyond the gradient catastrophe that occurs at $t = 1.0$, the solution exhibits a multi-valued solution associated with metastable states of the system. The initial datum (at $t = 0$) is also reported for visual comparison.

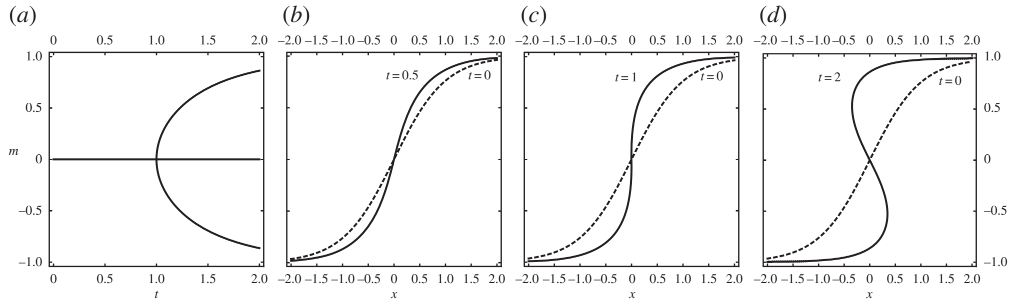

Figure 2. Analysis of the K-type. (*a*) Magnetization profile at $x = 0$ versus t. Magnetization profile versus x at, respectively, (*b*) $t = 0.5 < t_c$, (*c*) $t = 1.0 = t_c$, and (*d*) $t = 2 > t_c$. Similarly to the Curie–Weiss model, the magnetization profile fails in the origin at $t = 1$ and develops multi-valuedness for $t > 1$. The initial datum (at $t = 0$) is also reported for visual comparison.

(b) Klein–Gordon scenario

As discussed above, the Curie–Weiss Hamiltonian is an F-type normal form (2.23*c*) associated with the classical (Euclidean) kinetic energy. Let us now focus on the K-type normal form (2.23*b*) whose mechanical analogue can be viewed as a relativistic extension of the Curie–Weiss model (figure 2).

Definition 3.3. The Hamiltonian of the K-type model is defined as follows:

$$\frac{-H_N(m_N)}{N} = \sqrt{1 + m_N^2} + h m_N. \tag{3.6}$$

Let us observe that introducing the variable v (the relativistic speed) via $m = \gamma v$ with $\gamma = (1 - v^2)^{-1/2}$, we have $\sqrt{1 + m^2} = (1 - v^2)^{-1/2}$. By direct calculation, we can prove the following theorem.

Theorem 3.4. *The interpolating action/free energy is*

$$\alpha_N(t, x) = \frac{1}{N} \ln \sum_{\sigma}^{2^N} \exp\left(t\sqrt{1 + m_N^2} + x \cdot N m_N\right) = \frac{1}{N} \ln \sum_{\sigma}^{2^N} \exp(\mathbf{X} \cdot \mathbf{E}) \tag{3.7}$$

and obeys the following relativistic Hamilton–Jacobi equation:

$$\left.\begin{aligned}\left(\frac{\partial\alpha_N(t,x)}{\partial t}\right)^2-\left(\frac{\partial\alpha_N(t,x)}{\partial x}\right)^2+V_N(t,x)=1\\ \text{and}\qquad V_N(t,x)=\frac{1}{N}((\partial^2_{tt}\alpha_N(t,x))-(\partial^2_{xx}\alpha_N(t,x))).\end{aligned}\right\}\tag{3.8}$$

Note that the potential is given, up to a scale factor $1/N$, by the *D'Alambertian* of the action, that is, a relativistic invariant, and, consequently, the left-hand side of the Hamilton–Jacobi equation is also Lorentz-invariant.

As observed above, the thermodynamic free energy is obtained via the identification $t=\beta$ and $x=\beta h$.

(i) Generalized free energy by the minimum action principle

Introducing the standard notation of covariant and contravariant vectors, equation (3.8) becomes

$$\frac{\partial\alpha_N}{\partial x^\mu}\frac{\partial\alpha_N}{\partial x_\mu}+\frac{1}{N}\Box\alpha_N=1,\tag{3.9}$$

where *square* represents the D'Alambert operator, and (3.9) can be interpreted as the Hamilton–Jacobi equation describing the motion of a relativistic particle in the potential $V_N(t,x)=(\Box\alpha_N(t,x))/N$.

We observe that, as in the Curie–Weiss case, the potential vanishes in the thermodynamic limit as long as the function $\alpha_N(t,x)$ is smooth.

Hence, in the thermodynamic limit, equation (3.8) gives

$$\frac{\partial\alpha_N}{\partial x^\mu}\frac{\partial\alpha_N}{\partial x_\mu}=m_0c^2\equiv 1,\tag{3.10}$$

which, from a field theory perspective, gives the semi-classical Klein–Gordon scenario [19].

Remark 3.5. In relativistic mechanics, the generalized momentum is defined as

$$P^\mu=\left(\frac{E}{c},\gamma m v\right),$$

where v is the classical velocity of the particle, $\gamma=1/\sqrt{1-v^2}$ is the Lorentz factor and $E=\gamma$ (we set the rest energy $m_0c^2=1$) is the relativistic energy; hence, consistent with our findings, we have

$$\left(\frac{E}{c}\right)^2-(\gamma m v)^2=\frac{1}{1-v^2}-\frac{v^2}{1-v^2}=1.\tag{3.11}$$

Moreover, observing that the covariant gradient of the action is the contravariant momentum (e.g. [20])

$$\frac{\partial\alpha}{\partial x_\mu}=(\alpha_t,-\alpha_x)=P^\mu$$

we have the following identification between the statistical mechanical and relativistic dynamical variables:

$$P^\mu=(\gamma,\gamma v)=\left(\sqrt{1+m^2},m\right).\tag{3.12}$$

Remark 3.6. Let us observe that the expansion of the energy in the Taylor series around $m=0$, i.e.

$$E=\sqrt{1+m^2}=1-\frac{1}{2}m^2+O(m^4),$$

corresponds to the non-relativistic limit, where the leading-order constant is identified with the rest energy (normalized as $m_0c^2=1$) and the first-order contribution is the Curie–Weiss potential associated with the Euclidean kinetic energy.

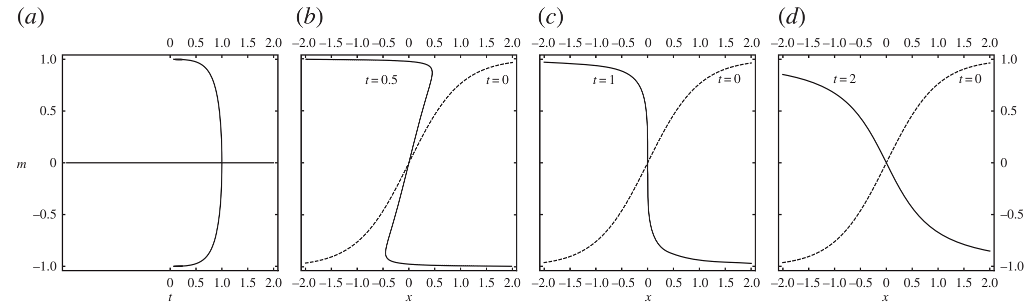

Figure 3. Analysis of the P-type. (*a*) Magnetization profile at $x = 0$ versus t. Magnetization profile versus x at (*b*) $t = 0.5 < t_c$, (*c*) $t = 1.0 = t_c$, and (*d*) $t = 2 > t_c$, respectively. The initial datum (at $t = 0$) is also reported for a visual comparison. Note that in the high noise region, in addition to the (stable) solution, say $m = 0$, two additional (instable) extremal points (maxima) for the free energy appear as a consequence of the infinite ferromagnetic contributions.

Proposition 3.7. *The free energy of the K-type model at zero external field is*

$$\alpha(\beta) = \ln 2 + \ln\cosh\left(\frac{m}{\sqrt{1+m^2}}\beta\right) + \frac{\beta}{\sqrt{1+m^2}}. \tag{3.13}$$

The associated self-consistency condition $\partial\alpha/\partial m = 0$ becomes

$$m = \tanh\left(\beta\frac{m}{\sqrt{1+m^2}}\right). \tag{3.14}$$

Proof. Let us note that equation (3.10) describes the free motion of a relativistic particle and can be readily integrated. Observing that the relativistic Lagrangian $\mathscr{L} = -\gamma^{-1}$ is preserved along the characteristics $x + vt$, then the action is computed as follows:

$$\begin{aligned}\alpha(t,x) &= \alpha(0,x) + \int_0^t \frac{dt'}{\gamma} = \ln 2 + \ln\cosh(-x) + \frac{t}{\gamma}\\ &= \ln 2 + \ln\cosh(vt - x) + \frac{t}{\gamma} = \ln 2 + \ln\cosh\left(\frac{m}{\sqrt{1+m^2}}t - x\right) + \frac{t}{\sqrt{1+m^2}}.\end{aligned} \tag{3.15}$$

Evaluating $\alpha(\beta, 0)$, one obtains the solution (3.13). ■

Remark 3.8. Let us observe that the free energy and the self-consistency equation for the Curie–Weiss model are readily recovered from the Taylor expansion around $m = 0$ of equations (3.13) and (3.14), respectively.

(c) Poisson scenario

We finally discuss the case of the elliptic dispersion curve.

Definition 3.9. The Hamiltonian of the P-type model is defined as follows:

$$\frac{-H_N(m_N)}{N} = -\sqrt{1 - m_N^2} + hm_N. \tag{3.16}$$

As discussed above, the partition function is obtained as a solution to the Poisson equation (2.22*a*). Moreover, the free energy $\alpha = -\nu \log Z$ in the thermodynamic limit satisfies equation (2.23*a*) and is given according to the following theorem.

Theorem 3.10. *Fixing* $h = 0$*, the free energy of the coupled generalized ferromagnetic P-type model is*

$$\alpha(\beta) = \ln 2 + \ln\cosh\left(\beta \frac{m}{\sqrt{1-m^2}}\right) - \frac{\beta}{\sqrt{1-m^2}}. \tag{3.17}$$

Moreover, the self-consistency equation is as follows:

$$m = \tanh\left(\beta \frac{m}{\sqrt{1-m^2}}\right). \tag{3.18}$$

Remark 3.11. Similarly to the K-type case, the free energy and the self-consistency equation for the Curie–Weiss model are readily recovered from the Taylor expansion around $m = 0$ of equations (3.17) and (3.18), respectively.

As shown in figure 3, owing to the ill-posedness of the initial value problem, the solutions do not evolve continuously from the initial datum producing a multi-valued solution due to the occurrence of an additional two instable extremal points for the free energy as a consequence of the infinite ferromagnetic contributions.

4. Conclusion

In this paper, we have discussed in detail a *formal analogy* between the thermodynamic evolution of mean-field spin systems and one-dimensional Hamiltonian systems.

We focused our attention on the class of spin models associated with an algebraic dispersion curve that contains the celebrated Curie–Weiss model as a particular case. The partition function for a finite number N of particles plays the role of the *quantum* wave function and obeys a linear PDE. The thermodynamic limit is obtained via the standard WKB analysis, where the Hamilton principal function is identified with the free energy of the thermodynamic system. The Hamilton–Jacobi equation can be treated via standard techniques and it is shown that the magnetization is a solution to a Riemann–Hopf-type equation. Hence, the model is completely integrable and solvable by the characteristics method.

Within this framework, thermodynamic phase transitions are associated with the occurrence of caustics in the semiclassical approximation. In particular, the critical point is identified as the point of gradient catastrophe where the magnetization satisfies the Riemann–Hopf equation.

All these features are discussed in detail for the class of models associated with a second-order dispersion curve. The reduction of the dispersion curve to the canonical form leads to three families of models associated with the conics: F-type parabolic, K-type hyperbolic and P-type elliptic. F-type models are associated with the semiclassical dynamics of a non-relativistic particle. Such models are reduced to the Curie–Weiss model that has been extensively studied in the literature (e.g. [3,7]). K-type models give a class of infinitely many p-spin contributions (namely higher order interactions in the Hamiltonian, e.g. from m^2 to $m^4, m^6, \ldots, m^p$) to the interaction and the thermodynamic limit is associated with the semiclassical limit of a relativistic particle. P-type models describe infinitely many ferromagnetic p-spin contributions to the interaction associated with elliptic dynamics. In particular, we observe that, due to the ill-posedness of the initial value problem, ferromagnetic contributions sum up to produce two meta-stable states (local maxima of the free energy) in the ergodic region.

We observe that both K-type and P-type extensions of the Curie–Weiss model can be viewed as 'relativistic' extensions of the Curie–Weiss model as the speed remains bounded, although only the K-type is associated with a Lorentz invariant Hamiltonian system.

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