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Critical behavior of mean-field spin glasses on a dilute random graph

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Abstract
We provide a rigorous strategy to find the critical exponents of the overlaps for dilute spin glasses, in the absence of an external field. Such a strategy is based on the expansion of a suitably perturbed average of the overlaps, which is used in the formulation of the free energy as the difference between a cavity part and the derivative of the free energy itself, considered as a function of the connectivity of the model. We assume the validity of certain reasonable approximations, equivalent to assuming a second-order transition, e.g. that higher powers of overlap monomials are of smaller magnitude near the critical point, of which we do not provide a rigorous proof.

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1. Introduction

Dilute spin glasses are important due to at least two reasons. Despite their mean-field nature, they share with finite-dimensional models the fact that each spin interacts with a finite number of other spins. Secondly, they are mathematically equivalent to some random optimization problems. The stereotypical model of dilute spin glasses is the Viana–Bray model [11, 13], which is equivalent to the random X-OR-SAT optimization problem in computer science, and the model we use as a guiding example here. In the original paper [13] the equilibrium of the model was studied, even in the presence of an external field, but the critical behavior was not obtained. In fact, the replica trick used in [13] only allows us to find, in a non-rigorous way, the critical point of the overlap between two replicas. It does not provide information about the critical exponents, and suggests wrong critical points for overlaps among several replicas. In the case of fully connected Gaussian models, the critical exponents were computed in a recent mathematical study [1]. Here, we use the techniques developed in [4] for finite connectivity spin glasses to extend the methodology of [1] to the case of dilute spin glasses:

using rigorous techniques we find that there is only one critical point for all overlaps and we compute the critical exponents of the overlaps among any number of replicas (whose distributions constitute the order parameter of the model [5, 8, 12]). Let us emphasize though that, roughly speaking, we do not prove for instance the statement 'the critical exponent of the (squared) overlap is two'. In fact, we do not prove that the overlap is continuous and we only assume it. Therefore what we prove is, roughly speaking, that 'if the (squared) overlap is continuous then its critical exponent is two.' This does not mean that our procedure is not rigorous, such as those procedures based for instance on replica methods or other possible approximations; it only means that we prove a weaker claim.

2. Model and notations

Given N points and families $\{i_\nu, j_\nu, k_\nu\}$ of i.i.d random variables uniformly distributed on these points, the (random) Hamiltonian of the Viana–Bray model is defined on Ising N-spin configurations $\sigma = (\sigma_1, \ldots, \sigma_N)$ through

$$H_N(\sigma, \alpha) = -\sum_{\nu=1}^{P_{\alpha N}} J_\nu \sigma_{i_\nu} \sigma_{j_\nu}$$

where P_ζ is a Poisson random variable with mean ζ, $\{J_\nu = \pm 1\}$ are i.i.d. symmetric random variables and $\alpha > 1/2$ is the connectivity. The expectation with respect to all the (*quenched*) random variables defined so far will be denoted by $\mathbb{E}$, while the Gibbs expectation at inverse temperature β with respect to this Hamiltonian will be denoted by Ω, and depends clearly on α and β. We also define $\langle\cdot\rangle = \mathbb{E}\Omega(\cdot)$. The pressure, i.e. minus β times the free energy, is by definition

$$A_N(\alpha) = \frac{1}{N}\mathbb{E}\ln\sum_\sigma \exp(-\beta H_N(\sigma, \alpha)).$$

When we omit the dependence on N we mean to have taken the thermodynamic limit. The quantities encoding the thermodynamic properties of the model are the overlaps, which are defined on several configurations (*replicas*) $\sigma^{(1)}, \ldots, \sigma^{(n)}$ by

$$q_{1\cdots n} = \frac{1}{N}\sum_{i=1}^{N} \sigma_i^{(1)} \cdots \sigma_i^{(n)}.$$

When dealing with several replicas, the Gibbs measure is simply the product measure, with the same realization of the quenched variables, but the expectation $\mathbb{E}$ destroys the factorization. We define β_{c} as the inverse temperature such that $2\alpha\tanh^2\beta_{\mathrm{c}} = 1$.

3. Previous results

We report here some known results which will be needed in the remainder of the paper. We refer to [4] for more information about the content of this section, but we report in the appendix some of the basic ideas that lead to the study of the stability of the system under certain stochastic perturbations we are about to introduce.

We are going to need the *cavity* function given by

$$\psi_N(\alpha', \alpha) = \mathbb{E}\ln\left\{\Omega\left[\exp\left(\beta\sum_{\nu=1}^{P_{2\alpha'}} J'_\nu \sigma_{k_\nu}\right)\right]\right\}$$

where the quenched variables appearing explicitly in this expression are independent copies of those in Ω. When the perturbation $\sum_{\nu=1}^{P_{2\alpha' t}} J'_\nu \sigma_{k_\nu}$ is added to the Hamiltonian, the corresponding Boltzmann factor will give place to Gibbs and quenched expectations denoted by $\Omega'_t(\cdot)$, $\langle\cdot\rangle'_t$, and the subindex t is simply omitted when $t = 1$. This perturbation, appearing in ψ, when $\alpha' = \alpha$, is equivalent to the addition of a new spin to the system; we refer to [4] for detailed explanations, which are summarized in the appendix anyway. The parameter t therefore allows us to interpolate between a system of N spins and one of $N+1$ spins, when $\alpha' = \alpha$; the case of generic $\alpha' \neq \alpha$ is only employed to have an independent variable without using t, whenever this is convenient to shorten the expressions. A consequence of the response of the system to the perturbation introduced, studied in [4] and reported in the appendix, is that monomials such that each replica appears an even number of times in them are stochastically stable: their average does not depend on the perturbation in the thermodynamic limit. The other overlap monomials are not stochastically stable, but their perturbed average can be expressed in terms of a power series in t, with (t-independent) stochastically stable (or invariant) averaged overlap polynomials as coefficients, in the thermodynamic limit. This is done by an iterative use of the following proposition, proven in [4].

Proposition 1. *Let Φ be a function of s replicas. Then the following cavity streaming equation holds*

$$\frac{\mathrm{d}\langle\Phi\rangle'_t}{\mathrm{d}t} = -2\alpha'\langle\Phi\rangle'_t + 2\alpha'\mathbb{E}\Bigg[\Omega'_t\Phi\Bigg\{1 + J\sum_{a}^{1,s}\sigma_{i_1}^{(a)}\theta + \sum_{a<b}^{1,s}\sigma_{i_1}^{(a)}\sigma_{i_1}^{(b)}\theta^2 + J\sum_{a<b<c}^{1,s}\sigma_{i_1}^{(a)}\sigma_{i_1}^{(b)}\sigma_{i_1}^{(c)}\theta^3 + \cdots\Bigg\} \times\left\{1 - sJ\theta\omega + \frac{s(s+1)}{2!}\theta^2\omega^2 - \frac{s(s+1)(s+2)}{3!}J\theta^3\omega^3 + \cdots\right\}\Bigg] \tag{1}$$

where $\omega = \Omega'_t(\sigma_{i_1})$, $\theta = \tanh\beta$.

In the following section we will consider explicitly our case of interest: that of $\Phi = q_{1\cdots 2n}$.

4. The expansion

Let $\Phi = q_{12}, q_{1234}, \ldots$. On the right-hand side of (1), consisting of the product of two factors in which each term brings a new overlap multiplying Φ, there is only one spin-flip invariant overlap: $q_{1\cdots 2n}^2$. But for the other terms we can use again the streaming equation, and each non-invariant overlap will be multiplied by a suitable overlap so that the number of replicas appearing an odd number of times decreases (by two). Integrating back in dt once the thermodynamic limit is taken, one can easily obtain

$$\langle q_{12}\rangle'_t = \tau' t\langle q_{12}^2\rangle - 2\tau'^2 t^2\langle q_{12}q_{23}q_{31}\rangle + O(q^4) \tag{2}$$

$$\vdots$$

$$\langle q_{1\cdots 2n}\rangle'_t = \tau'\theta^{2n-2}t\langle q_{1\cdots 2n}^2\rangle + t^2 O(q^3) + \cdots \tag{3}$$

where $\tau' = 2\alpha'\theta^2 = 2\alpha'\tanh^2\beta$ and we neglected monomials with the products of at least four overlaps. As an example, we gave the explicit form of the monomial of order three for $n = 2$. These expansions will be used to expand ψ in terms of averaged stable overlap monomials. If we take $t = 1$ and let β be very close to β_c, we know [4] that we can replace $\langle q_{12}\rangle'$ by $\langle q_{12}^2\rangle$, on the left-hand side of (2). This provides a relation, valid at least sufficiently close to the critical temperature, between $\langle q_{12}^2\rangle$ and $\langle q_{12}q_{23}q_{31}\rangle$, as we neglect the higher-order monomials in (2):

$$(\tau - 1)\langle q_{12}^2\rangle = 2\langle q_{12}q_{23}q_{31}\rangle \tag{4}$$

with $\tau = 2\alpha\theta^2 = 2\alpha\tanh^2\beta$. Note incidentally that this relation is compatible with the well-known fact [10] that the fluctuations of the rescaled overlap Nq_{12}^2 diverge only when $\tau \to 1$ (and not at higher temperatures), being $N\langle q_{12}q_{23}q_{31}\rangle$ small (due to the central limit theorem) as is the sum of N^3 bounded variables dived by N^2 instead of $N^{3/2}$.

5. Orders of magnitude

In the expansions of the previous section, we need to understand which terms are small near the critical point. We know that above the critical temperature all the overlaps are zero, and that those which are not zero by symmetry become non-zero below the critical temperature; therefore we assume that slightly below such a temperature the overlaps are very small. More precisely, we know that for instance

$$\langle q_{12}^2\rangle = \mathbb{E}\,\Omega^2(\sigma_{i_1}\sigma_{i_2})$$

is very small, and so is therefore $\Omega^2(\sigma_{i_1}\sigma_{i_2})$. This means that for temperatures sufficiently close to the critical one $\Omega^4(\sigma_{i_1}\sigma_{i_2})$ is negligible as compared to $\Omega^2(\sigma_{i_1}\sigma_{i_2})$. In other words, $\langle q_{1234}^2\rangle$ is assumed to be of a smaller order of magnitude than $\langle q_{12}^2\rangle$. Furthermore, if q_{12}^2 is small q_{12}^4 has to be of an even smaller order of magnitude. In fact we reasonably assume that $\langle q_{12}^4\rangle = \mathbb{E}\,\Omega^2(\sigma_{i_1}\sigma_{i_2}\sigma_{i_3}\sigma_{i_4})$, which is of order two in Ω, is of a smaller order than $\langle q_{12}^2\rangle$, which is also of order two in Ω. An explanation comes from the self-averaging discussed in [6], which tells us that $\mathbb{E}\Omega(\sigma_{i_1}\sigma_{i_2}\sigma_{i_3}\sigma_{i_4})$ is of the same order as $\mathbb{E}\Omega(\sigma_{i_1}\sigma_{i_2})\Omega(\sigma_{i_3}\sigma_{i_4})$, which is of order two in Ω, and hence increasing the number of spins in the expectation Ω is basically equivalent to increasing the order in Ω. This is actually proven in a perturbed system [6], but it is reasonable to assume that the consequences of self-averaging (not the self-averaging itself) on the orders of magnitude of the considered quantities is not lost when the perturbation is removed, and the monomials we have are the result of the streaming equation, in which the measure is perturbed. Consistently, (4) implies that near the critical point$\langle q_{12}q_{23}q_{31}\rangle$ is smaller than $\langle q_{12}^2\rangle$, and the two critical exponents differ by one. All these observations lead to the following criterion, basically equivalent to assuming a second-order transition. We define the degree of an averaged overlap monomial as the sum of the degrees of each overlap in it, where the degree of an overlap is its exponent times its number of replicas. For instance $\langle q_{1234}^2q_{12}^2q_{34}^2\rangle$ is of order $4\times2+2\times2+2\times2 = 16$. The definition we just gave coincides with the one that can be given in terms of Ω expectations, provided one multiplies the exponent of each Ω expectation by the number of randomly chosen spins appearing in it. For example $\langle q_{1234}^2q_{12}^2q_{34}^2\rangle = \mathbb{E}\,\Omega^2(\sigma_{i_1}\sigma_{i_2}\sigma_{i_3}\sigma_{i_4})\Omega^2(\sigma_{i_1}\sigma_{i_2}\sigma_{i_5}\sigma_{i_6})$ is of order $2\times4+2\times4 = 16$. Given an integer m, a monomial of order $2m + 2$ will be considered negligible, near the critical point—where all overlaps are very small, with respect to a monomial of order $2m$.

6. The transition

It is well known that all the overlaps are zero above the critical temperature $1/\beta_c$ where the replica symmetric solution holds, and that below this temperature the overlap between two replicas fluctuates and its square becomes non-zero identifying a replica symmetry breaking.

A detailed rigorous study of the fact that the critical temperature of the model is determined by the equation $2\alpha \tanh^2 \beta_c = 1$ was performed in [10]. In the same article, the reader can find a description of the breaking of replica symmetry occurring at all temperatures lower than the critical one, emerging as the loss of self-averaging of the overlap. As pointed out in [13], the use of the replica trick within a quadratic approximation can only provide the correct transition for the overlap between two replicas, while overlaps of more replicas would seem to be zero down to lower temperatures before starting fluctuating. Moreover within that method no information about the critical exponents was found. Our method allows us to compute the critical exponent of the overlap between two replicas and to gain information about the critical point and critical exponents of all overlap monomials. Let us start by showing that there is only one critical point for all overlap monomials.

By convexity, we have

$$\langle q_{1\cdots 2n}^2 \rangle = \mathbb{E}\Omega^{2n}(\sigma_{i_1}\sigma_{i_2}) \geqslant (\mathbb{E}\,\Omega^2(\sigma_{i_1}\sigma_{i_2}))^n = \langle q_{12}^2 \rangle^n$$

so that all overlaps are non-zero whenever $\langle q_{12}^2 \rangle$ is, i.e. below the critical temperature $1/\beta_c$. As a further example, a slightly more accurate use of convexity yields immediately $\langle q_{1234}^2 \rangle \geqslant \langle q_{12}^2 q_{34}^2 \rangle \geqslant \langle q_{12} \rangle^2$. This means that the critical exponents of q_{1234}^2 and $q_{12}^2 q_{34}^2$ cannot be larger than twice the critical exponent of q_{12}^2, but cannot be smaller than this critical exponent itself either, as $\langle q_{1234}^2 \rangle \leqslant \langle q_{12}^2 \rangle$.

7. Critical exponents

We will now relate the free energy to its derivative and to the cavity function. The following theorem follows easily from the results of [5], and here we only sketch the proof, based on standard convexity arguments.

Theorem 1. *In the thermodynamic limit, we have*

$$A(\alpha) = \ln 2 + \psi(\alpha, \alpha) - \alpha A'(\alpha)$$

for all values of α, β, *where* A' *is the derivative of* A.

Proof. sketched proof. It was proven in [5] that

$$A(\alpha) = \lim_N \left[\mathbb{E} \ln \Omega \left(\sum_{\sigma_{N+1}} \exp \left(\beta \sum_{\nu=1}^{P_{2\alpha}} J'_\nu \sigma_{k_\nu} \sigma_{N+1} \right) \right) - \mathbb{E} \ln \Omega (\exp -\beta (H'_N(\alpha/N))) \right] \tag{5}$$

where the quenched variables in H' are independent of those in Ω, just like for the first term on the right-hand side. The second term on the right-hand side is easy to compute, at least in principle [5], and it is the derivative of A multiplied by α, because

$$\mathbb{E} \ln \Omega (\exp -\beta (H'_N(\alpha/N))) = NA(\alpha(1 + 1/N) - NA(\alpha).$$

This leads to the result to prove, as the gauge invariance of Ω allows us to take out the sum over σ_{N+1} as $\ln 2$, and therefore the first term on the right-hand side of (5) is precisely ψ.

□

It is easy to see [5] by direct calculation that

$$\partial_1 \psi_N(\alpha', \alpha) = 2 \sum_n \frac{\theta^{2n}}{2n} (1 - \langle q_{1\cdots 2n} \rangle') \tag{6}$$

$$A'(\alpha) = \sum_n \frac{\theta^{2n}}{2n} \left(1 - \langle q_{1\cdots 2n}^2 \rangle\right). \tag{7}$$

From the theorem above we have then

$$A'(\alpha) = \partial_1\psi(\alpha,\alpha) + \partial_2\psi(\alpha,\alpha) - A'(\alpha) - \alpha A''(\alpha).$$

But we know [4] that near the critical point *saturation* $\langle q_{2n}\rangle' \to \langle q_{2n}^2\rangle$ occurs in the thermodynamic limit, so that $\partial_1\psi(\alpha,\alpha) \to 2A'(\alpha)$ and therefore we have just proven the next.

Proposition 2. *In the thermodynamic limit*

$$\partial_2\psi(\alpha,\alpha) - \alpha A''(\alpha) = 0. \tag{8}$$

Note that if in the statement of theorem 1 we assumed saturation $\langle q_{1\cdots 2n}\rangle_t' \to \langle q_{1\cdots 2n}^2\rangle$ not just for $t = 1$ but for all t (once $\psi(t\alpha,\alpha)$ is written using (6) as the integral of its derivative with respect to t), we would obtain $\psi = 2A'$ and

$$A(\alpha) = \alpha A'(\alpha) + \ln 2$$

which, as the initial condition is easily checked to be $A'(0) = \ln\cosh\beta$, gives the well-known replica symmetric solution $A(\alpha) = \ln 2 + \alpha\ln\cosh\beta$. This means that stability and saturation of the overlaps are equivalent to the replica symmetry.

Now let us analyze (8). We consider $\psi(\alpha',\alpha)$ as the integral of its derivative with respect to its first argument. The derivative, given in (6), contains the perturbed averaged overlaps, which we expand using (2) and (3), etc. In these expansions the variable α' appears only explicitly in front of the averaged overlap monomials, which do not depend on α', they only depend on α. Therefore we can perform explicitly the integration of these simple power series in α'. The dependence on α of $\psi(\alpha',\alpha)$ is hence only in the averaged overlap monomials, and the same holds for $A'(\alpha)$, because of (7). Therefore the derivatives of $\psi(\alpha',\alpha)$ and $A'(\alpha)$ with respect to α in (8) involve only the averaged overlap monomials. In other words if we define $\tilde{A}(\alpha',\alpha) = \ln 2 + \psi(\alpha',\alpha) - \alpha' A'(\alpha)$, so that $A(\alpha) = \tilde{A}(\alpha,\alpha)$ thanks to theorem 1, equation (8) amounts to say that $\partial_2\tilde{A}(\alpha,\alpha) = 0$. But since the second argument appears only in the averaged overlap monomials, we can consider $A(\alpha) = \tilde{A}(\alpha,\alpha) \equiv \hat{A}(\alpha, p_1(\alpha), p_2(\alpha),\ldots)$ as a function of the averaged overlap monomials, here called $p_1(\alpha), p_2(\alpha),\ldots$, such that

$$\partial_2\tilde{A} = \sum_m \frac{\partial\hat{A}}{\partial p_m}\frac{\mathrm{d}p_m}{\mathrm{d}\alpha} = 0. \tag{9}$$

We can now use (2) and (3), etc to have an explicit expansion of $A(\alpha)$ and deal with the differential equation (9). The result is easy to obtain and reads

$$\begin{aligned} A(\alpha) &= \ln 2 + \frac{\tau}{2} - \frac{\tau}{4}(\tau-1)\langle q_{12}^2\rangle + \frac{\tau^3}{3}\langle q_{12}q_{23}q_{13}\rangle + O(q^4) \\ &\quad + \theta^2\left(\frac{\tau}{4} - \frac{\tau}{8}(\tau\theta^2-1)\langle q_{1234}^2\rangle - \frac{3\tau^3}{4}\langle q_{1234}q_{12}q_{34}\rangle + O(q^4)\right) + O(\theta^4). \end{aligned} \tag{10}$$

Note that this expansion extends that found in [13]. In a first approximation, assuming a second-order transition so to have small overlaps, we may consider, for small $\tau - 1$,

$$A(\alpha) \sim \ln 2 + \frac{\tau}{2} - \frac{\tau}{4}(\tau-1)\langle q_{12}^2\rangle + \frac{\tau^3}{3}\langle q_{12}q_{23}q_{13}\rangle$$

and (9) becomes

$$-\frac{1}{4}(\tau-1)\frac{\mathrm{d}\langle q_{12}^2\rangle}{\mathrm{d}\alpha} + \frac{1}{3}\frac{\mathrm{d}\langle q_{12}q_{23}q_{31}\rangle}{\mathrm{d}\alpha} = 0 \tag{11}$$

because

$$\frac{\hat{A}}{\partial\langle q_{12}^2\rangle} = -\frac{\tau}{4}(\tau-1) \sim -\frac{1}{4}(\tau-1)$$

$$\frac{\hat{A}}{\partial\langle q_{12}q_{23}q_{31}\rangle} = \frac{\tau^3}{3} \sim \frac{1}{3}.$$

But now the use of (4) in (11) offers

$$-\frac{1}{4}(\tau-1)\frac{\mathrm{d}\langle q_{12}^2\rangle}{\mathrm{d}\alpha} + \frac{1}{3}\frac{1}{2}\frac{\mathrm{d}(\tau-1)\langle q_{12}^2\rangle}{\mathrm{d}\alpha} = 0$$

from which, after a couple of elementary steps

$$(\tau-1)\frac{\mathrm{d}\langle q_{12}^2\rangle}{\mathrm{d}(\tau-1)} - 2\langle q_{12}^2\rangle = 0.$$

This equation becomes exact when the temperature is sufficiently close to the critical one, $\tau \sim 1$, and the solution is easy to find:

$$\langle q_{12}^2\rangle \propto (\tau-1)^2$$

describing the critical behavior of the overlap slightly below the critical temperature. The critical exponent is hence two.

Note that (4) implies that $\langle q_{12}q_{23}q_{31}\rangle$ is zero above the temperature $1/\beta_2$ and positive slightly below. Moreover, (4) gives the critical exponent for $\langle q_{12}q_{23}q_{31}\rangle$: three.

From our analysis in the previous sections, we conclude that the critical exponent of q_{1234}^2 is strictly larger than three, but no larger than four. The criterion explained in the section on the order of magnitudes, together with 4 and the critical exponent of q_{12}^2, provides a relation between the degree of an overlap monomial and its critical exponent: degree $2m$ corresponds to critical exponent m. So for instance the critical exponent of $q_{1\cdots 2n}^2$, which is of order $4n$, is $2n$. In the infinite connectivity limit we recover all the critical exponents for the fully connected Gaussian SK model [1].

Remark. If we extended the use of $\langle q_{1\cdots 2n}\rangle' \to \langle q_{1\cdots 2n}^2\rangle$ to lower temperatures, such that $2\alpha\theta^{2n} \equiv \tau_{2n} \sim 1$, we would obtain for $q_{1\cdots 2n}^2$, for all n, the same identical differential equation we got for q_{12}^2. We would then get the same approximated behavior one gets using the replica method in a quadratic approximation [13]: q_{2n}^2 would be zero above the temperature such that $\tau_{2n} = 1$, then it starts fluctuating, with critical exponent two. It is therefore interesting to note that in this sense the replica method with the quadratic approximation is equivalent to extending stochastic stability below the critical point.

8. Summary and conclusions

Our strategy requires the expansion of the averaged overlaps in powers of a perturbing parameter with stochastically stable overlap monomials as coefficient (similarly to the expansion exhibited in [2] for Gaussian models). This allowed us to write the free energy in terms of overlap fluctuations and to discover that it does not depend on a certain family of these monomials. As a consequence, we obtained a differential equation whose solution, once all small terms are neglected, gave the critical behavior of the overlaps. The strategy can clearly be adapted, in an even simpler form, when the interaction is ferromagnetic, i.e. in the case of dilute mean-field ferromagnet [7], and such a simplified approach easily recovers—once further simplified [3], in the infinite connectivity limit—the well-known results for the Curie–Weiss model of a fully connected mean-field ferromagnet [9].

Our method works for second-order transitions and is ultimately based on stochastic stability, but such a stability is proven or at least believed to hold in several contexts, therefore generalizations of our method to finite-dimensional spin glasses, to the traveling salesman problem, to the K-SAT problem, to neural networks and to other cases are not to be excluded and are being studied. We plan on reporting soon on these topics.

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Appendix A. Cavity approach

In this appendix we summarize some of the basic ideas of the cavity approach, which is at the basis of the tools we introduce in section 3 to be then used to split the free energy into two pieces in section 7. Here we just provide a brief summary, the interested reader may find more on the cavity approach to dilute mean-field spin glasses in [4].

The thermodynamic limit of the free energy density exists if and only if the sequence of the increments (due to the addition of a particle to the system) is convergent in the Cesàro sense (indicated by a boldface **C**):

$$\lim_{N\to\infty}\frac{1}{N}\mathbb{E}\ln Z_N \equiv \lim_{N\to\infty}\frac{1}{N}\sum_{n=0}^{N-1}\mathbb{E}\ln\frac{Z_{n+1}}{Z_n} \equiv \mathbf{C}\lim_{N\to\infty}\mathbb{E}\ln\frac{Z_{N+1}}{Z_N}.$$

The idea at the basis of the cavity approach is in fact to measure the effect on the free energy of the addition of one spin to the system. We will denote by σ the configuration of N given spin, and we will add a spin σ_{N+1} to this system. Now, following [5], we can write, in distribution,

$$-H_{N+1}(\sigma,\sigma_{N+1};\alpha) \sim \sum_{\nu=1}^{P_{\alpha\frac{N^2}{N+1}}} J_\nu\sigma_{i_\nu}\sigma_{j_\nu} + \sum_{\nu=1}^{P_{\alpha\frac{2N}{N+1}}} \tilde{J}_\nu\sigma_{k_\nu}\sigma_{N+1} \tag{A.1}$$

where we have neglected a third term which does not contribute when N is large [5], $\{\tilde{J}_\nu\}$ are independent copies of J; $\{i_\nu\}$, $\{j_\nu\}$, and $\{k_\nu\}$ are independent random variables all uniformly distributed over $\{1,\ldots,N\}$. Note that we can also write, in distribution,

$$H_{N+1}(\sigma,\sigma_{N+1};\alpha) \sim H_N(\sigma;\tilde{\alpha}) + \tilde{h}_\sigma\sigma_{N+1} \tag{A.2}$$

where

$$\tilde{\alpha} = \alpha\frac{M}{M+1} \qquad \tilde{h}_\sigma = -\sum_{\nu=1}^{P_{2\tilde{\alpha}}}\tilde{J}_\nu\sigma_{k_\nu}.$$

Clearly $\tilde{\alpha}\to\alpha$ as $N\to\infty$. The field $\tilde{h}$ is called *cavity field*, and note that the connectivity degree appearing in it is twice the degree of connectivity of the random graph α. It is now clear in what sense the field $\tilde{h}$, which can be seen as a perturbation, gives place to the addition of a new spin, as discussed in section 3. It is interesting to study how the systems react to a field of a slightly more general form: $h' = -\sum_{\nu=1}^{P_{2\alpha'}}\tilde{J}_\nu\sigma_{k_\nu}$, which is precisely what appears in the function ψ introduced in section 3. In this paper we make use of a result that is presented in [4], and then used in [6] to explore its consequences from the point of view of stochastic stability and more generally self-averaging (see [6] and references therein for a study of stochastic stability and self-averaging in dilute spin glasses). Here we report for convenience only the main statement which is of interest for the present work. The expectation of an overlap monomial Q such that each replica appears an even number of times in it is not affected by the perturbation introduced above: $\langle Q\rangle' = \langle Q\rangle$, according to the notations introduced in section 3. Overlaps of the type just described are called *filled*, and we just stated that they are stochastically stable under the cavity perturbation. The result we just mentioned does not hold for other overlap monomials, but their perturbed average can be computed through an expansion in terms of stochastically stable overlap monomials. This is studied in details in [4], but the main formula is reported in section 3.

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