Criticality in diluted ferromagnets

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Abstract. We perform a detailed study of the critical behavior of the mean field diluted Ising ferromagnet using analytical and numerical tools. We obtain self-averaging for the magnetization and write down an expansion for the free energy close to the critical line. The scaling of the magnetization is also rigorously obtained and compared with extensive Monte Carlo simulations. We explain the transition from an ergodic region to a non-trivial phase by commutativity breaking of the infinite volume limit and a suitable vanishing field. We find full agreement among theory, simulations and previous results.

Contents

1. Introduction

In the past few years a match between the study of systems defined on lattices by means of statistical mechanics [25] and the study of networks by means of graph theory [13] gave rise to very interesting models such as the small world magnets [34, 32] and the scale free networks [14]. However, a complete analysis starting from the simple fully connected mean field Ising model [6, 19] and going up to these recent complex models [4, 28] is still not complete (even though several important steps have been obtained, examples being provided by [27, 12, 21]) and important models have not yet been taken into account. Among these, a certain role is played by the mean field diluted Ising model: an Erdös–Renyi network [20] which has spins as nodes and their interactions as links. The interactions are encoded in a matrix connecting pairs of spins which, when the connectivity allows the link to be present, shares the same value for all the couples.

Despite their easy formalization [31], diluted ferromagnets are poorly investigated with rigorous tools [22]. Inspired by a recent work on these systems in which the authors presented a detailed analysis of the ergodic region and the zero-temperature line [18] we extend recent techniques developed in a series of papers [3, 5, 2, 10] to this model with the aim of analyzing its critical behavior. We systematically develop the interpolating cavity field method [5] and use it to sketch the derivation of a free energy expansion: the higher the order of the expansion, the deeper we could go beyond the ergodic region. Within this framework we perform a detailed analysis of the scaling of magnetization (and susceptibility) at the critical line. The critical exponents turn out to be the classical ones. At the end we perform extensive Monte Carlo (MC) simulations for different graph sizes and bond concentrations and we compare results with theory. Indeed, also numerically,

we provide evidence that the universality class of the diluted Ising model is independent of the dilution. In fact the critical exponents that we measured are consistent with those pertaining to the Curie–Weiss model, in agreement with analytical results. The critical line is also well reproduced.

The paper is organized as follows. In section 2 we describe the model, in section 3 we introduce the cavity field technique, which constitutes the framework that we are going to use in section 4 to investigate the free energy of the system at general values of temperature and dilution. Section 5 deals with the criticality of the model; there we find the critical line and the critical behavior of the main order parameter, i.e. magnetization, we provide its self-averaging and we work out a picture by means of which we explain the breaking of the ergodicity. Section 6 is devoted to numerical investigations, especially focused on criticality. Finally, section 7 is left for an outlook and conclusions.

2. Model and notation

Given N points and families $\{i_\nu, j_\nu\}$ of iid random variables uniformly distributed on these points, the (random) Hamiltonian of the diluted Curie–Weiss model is defined on Ising N-spin configurations $\sigma = (\sigma_1, \ldots, \sigma_N)$ through

$$H_N(\sigma, \alpha) = -\sum_{\nu=1}^{P_{\alpha N}} \sigma_{i_\nu} \sigma_{j_\nu} \tag{1}$$

where P_ζ is a Poisson random variable with mean ζ and $\alpha > 1/2$ is the connectivity. The expectation with respect to all the (*quenched*) random variables defined so far will be denoted by $\mathbb{E}$, while the Gibbs expectation at inverse temperature β with respect to this Hamiltonian will be denoted by Ω, and clearly depends on α and β. We also define $\langle \cdot \rangle = \mathbb{E}\Omega(\cdot)$. The pressure, i.e. minus β times the free energy, is by definition

$$A_N(\alpha) = \frac{1}{N}\mathbb{E}\ln Z_N(\beta) = \frac{1}{N}\mathbb{E}\ln\sum_\sigma \exp(-\beta H_N(\sigma, \alpha))$$

where we implicitly introduced the partition function $Z_N(\beta)$ too. When we omit the dependence on N we mean to have taken the thermodynamic limit which we assume to exist for all the observables that we deal with, in particular for the free energy [22, 11, 26] (however we will look for firmer ground on this point through numerical investigation in section 6). The quantities encoding the thermodynamic properties of the model are the overlaps, which are defined on several configurations (*replicas*) $\sigma^{(1)}, \ldots, \sigma^{(n)}$ by

$$q_{1\cdots n} = \frac{1}{N}\sum_{i=1}^{N} \sigma_i^{(1)} \cdots \sigma_i^{(n)}.$$

Particular attention must be paid to $q_1 = m = N^{-1}\sum_i^N \sigma_i$ which is called the *magnetization*.

When dealing with several replicas, the Gibbs measure is simply the product measure, with the same realization of the quenched variables, but the expectation $\mathbb{E}$ destroys the factorization. Sometimes for the sake of simplicity we will use $\theta = \tanh(\beta)$.

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3. Interpolating with the cavity field

In this section first we introduce the cavity field technique along the lines of [5] by expressing the Hamiltonian of a system made of $N+1$ spins through the Hamiltonian of N spins by scaling the connectivity degree α and neglecting vanishing terms in N as follows:

$$H_{N+1}(\alpha) = -\sum_{\nu=1}^{P_{\alpha(N+1)}} \sigma_{i_\nu}\sigma_{j_\nu} \sim -\sum_{\nu=1}^{P_{\tilde{\alpha}N}} \sigma_{i_\nu}\sigma_{j_\nu} - \sum_{\nu=1}^{P_{2\tilde{\alpha}}} \sigma_{i_\nu}\sigma_{N+1} \tag{2}$$

such that we can use the more compact expression

$$H_{N+1}(\alpha) \sim H_N(\tilde{\alpha}) + \hat{H}_N(\tilde{\alpha})\sigma_{N+1} \tag{3}$$

with

$$\tilde{\alpha} = \frac{N}{N+1}\alpha \xrightarrow{N\to\infty} \alpha, \qquad \hat{H}_N(\tilde{\alpha}) = -\sum_{\nu=1}^{P_{2\tilde{\alpha}}} \sigma_{i_\nu}. \tag{4}$$

So we see that we can express the Hamiltonian for $N+1$ particles via that for N particles, paying two prices: the first is a rescaling in the connectivity (vanishing in the thermodynamic limit), and the second is an added term, which will be encoded, at the level of the thermodynamics, by a suitable cavity function as follows: let us introduce an interpolating parameter $t \in [0,1]$ and the cavity function $\Psi_N(\tilde{\alpha}, t)$ given by

$$\Psi(\tilde{\alpha}, \beta; t) = \lim_{N\to\infty} \Psi_N(\tilde{\alpha}, \beta; t) \lim_{N\to\infty} \mathbb{E}\left[\ln \frac{\sum_{\{\sigma\}} e^{\beta \sum_{\nu=1}^{P_{\tilde{\alpha}N}} \sigma_{i_\nu}\sigma_{j_\nu} + \beta \sum_{\nu=1}^{P_{2\tilde{\alpha}t}} \sigma_{i_\nu}}}{\sum_\sigma e^{\beta \sum_{\nu=1}^{P_{\tilde{\alpha}N}} \sigma_{i_\nu}\sigma_{j_\nu}}} \right]$$

$$= \lim_{N\to\infty} \mathbb{E}\left[\ln \frac{Z_{N,t}(\tilde{\alpha}, \beta)}{Z_N(\tilde{\alpha}, \beta)} \right]. \tag{5}$$

The three terms appearing in the decomposition (3) give rise to the structure of the following theorem which we prove by assuming the existence of the thermodynamic limit. (Actually we still do not have a rigorous proof of the existence of the thermodynamic limit but we will provide strong numerical evidence in section 6.)

Theorem 1. *In the $N \to \infty$ limit, the free energy per spin is allowed to assume the following representation:*

$$A(\alpha, \beta) = \ln 2 - \alpha\frac{\partial A(\alpha, \beta)}{\partial \alpha} + \Psi(\alpha, \beta; t=1). \tag{6}$$

Proof. Consider the $N+1$-spin partition function $Z_{N+1}(\alpha, \beta)$ and let us split it as suggested by equation (3):

$$Z_{N+1}(\alpha, \beta) = \sum_{\{\sigma_{N+1}\}} e^{-\beta H_{N+1}(\alpha)} \sim \sum_{\{\sigma_{N+1}\}} e^{-\beta H_N(\tilde{\alpha}) - \beta\hat{H}_N(\tilde{\alpha})\sigma_{N+1}}$$

$$= \sum_{\{\sigma_{N+1}\}} e^{\beta \sum_{\nu=1}^{P_{\tilde{\alpha}N}} \sigma_{i_\nu}\sigma_{j_\nu} + \beta \sum_{\nu=1}^{P_{2\tilde{\alpha}}} \sigma_{i_\nu}\sigma_{N+1}} = 2\sum_{\{\sigma_N\}} e^{\beta \sum_{\nu=1}^{P_{\tilde{\alpha}N}} \sigma_{i_\nu}\sigma_{j_\nu} + \beta \sum_{\nu=1}^{P_{2\tilde{\alpha}}} \sigma_{i_\nu}} \tag{7}$$

where the factor 2 appears because of the sum over the hidden σ_{N+1} variable. Defining a perturbed Boltzmann state $\tilde{\omega}$ (and its replica product $\tilde{\Omega} = \tilde{\omega} \times \cdots \times \tilde{\omega}$) by

$$\tilde{\omega}(g(\sigma)) = \frac{\sum_{\{\sigma_N\}} g(\sigma) \mathrm{e}^{-\beta H_N(\tilde{\alpha})}}{\sum_{\{\sigma_N\}} \mathrm{e}^{-\beta H_N(\tilde{\alpha})}}, \qquad \tilde{\Omega}(g(\sigma)) = \prod_i \tilde{\omega}^{(i)}(g(\sigma^{(i)}))$$

where the tilde takes into account the shift in the connectivity $\alpha \to \tilde{\alpha}$, and multiplying and dividing the rhs of equation (7) by $Z_N(\tilde{\alpha}, \beta)$, we obtain

$$Z_{N+1}(\alpha, \beta) = 2 Z_N(\tilde{\alpha}, \beta) \tilde{\omega}(\mathrm{e}^{\beta \sum_{\nu=1}^{P_{2\tilde{\alpha}}}}). \tag{8}$$

Now taking the logarithm of both sides of equation (8), applying the average $\mathbb{E}$ and subtracting the quantity $[\ln Z_{N+1}(\tilde{\alpha}, \beta)]$, we get

$$\mathbb{E}[\ln Z_{N+1}(\alpha, \beta)] - \mathbb{E}[\ln Z_{N+1}(\tilde{\alpha}, \beta)] = \ln 2 + \mathbb{E}\left[\ln \frac{Z_N(\tilde{\alpha}, \beta)}{Z_{N+1}(\tilde{\alpha}, \beta)}\right] + \Psi_N(\tilde{\alpha}, \beta; t = 1). \tag{9}$$

In the large N limit the lhs of equation (9) becomes

$$\mathbb{E}[\ln Z_{N+1}(\alpha, \beta)] - \mathbb{E}[\ln Z_{N+1}(\tilde{\alpha}, \beta)] = (\alpha - \tilde{\alpha}) \frac{\partial}{\partial \alpha} \mathbb{E}[\ln Z_{N+1}(\alpha, \beta)]$$

$$= \alpha \frac{1}{N+1} \frac{\partial}{\partial \alpha} [\ln Z_{N+1}(\alpha, \beta)] = \alpha \frac{\partial A_{N+1}(\alpha, \beta)}{\partial \alpha} \tag{10}$$

and then by considering the thermodynamic limit the theorem follows. $\qquad \square$

Hence, we can express the free energy via an energy-like term and the cavity function. While it is well known how one deals with the energy-like form [18], the same cannot be said of the cavity function, and we want to develop its expansion via suitably chosen overlap monomials in a spirit close to that of stochastic stability [3, 15, 30], such that, at the end, we will not have the analytical solution for the free energy in the whole (α, β) plane, but we will manage its expansion close to (immediately below) the critical line. To see how the machinery works, let us start by giving some definitions and proving some simple theorems:

Definition 1. *We define the t-dependent Boltzmann state $\tilde{\omega}_t$ as*

$$\tilde{\omega}_t(g(\sigma)) = \frac{1}{Z_{N,t}(\alpha, \beta)} \sum_{\{\sigma\}} g(\sigma) \mathrm{e}^{\beta \sum_{\nu=1}^{P_{\tilde{\alpha}N}} \sigma_{i\nu} \sigma_{j\nu} + \beta \sum_{\nu=1}^{P_{2\tilde{\alpha}t}} \sigma_{i\nu}} \tag{11}$$

where $Z_N(\alpha, \beta)$ extends the classical partition function in the same spirit as the numerator of equation (11).

As we will often deal with several overlap monomials, let us divide them into two big categories.

Definition 2. *We can split the class of monomials of the order parameters into two families:*

- *We define as 'filled' or equivalently 'stochastically stable' all the overlap monomials built from an even number of the same replicas (i.e. q_{12}^2, m^2, $q_{12}q_{34}q_{1234}$).*
- *We define as 'fillable' or equivalently 'saturable' all the overlap monomials which are not stochastically stable (i.e. q_{12}, m, $q_{12}q_{34}$).*

We are going to show three theorems that will play a guiding role for our expansion: as this approach has been deeply developed in similar contexts (as a fully connected Ising model [6] or fully connected spin glasses [5] or diluted spin glasses [8], which are the *boundary models* of the subject of this paper) we will not show all the details of the proofs, but we sketch them as they are really intuitive. The interested reader can go deeper on this point by looking at the original works.

Theorem 2. *For large N, setting $t = 1$ we have*

$$\tilde{\omega}_{N,t}(\sigma_{i_1}\sigma_{i_2}\cdots\sigma_{i_n}) = \tilde{\omega}_{N+1}(\sigma_{i_1}\sigma_{i_2}\cdots\sigma_{i_n}\sigma_{N+1}^n) + \mathrm{O}\left(\frac{1}{N}\right) \tag{12}$$

such that in the thermodynamic limit, if $t = 1$, the Boltzmann average of a fillable multi-overlap monomial turns out to be the Boltzmann average of the corresponding filled multi-overlap monomial.

Theorem 3. *Let Q_{2n} be a fillable monomial of the overlaps (this means that there exists a multi-overlap q_{2n} such that $q_{2n}Q_{2n}$ is filled). We have*

$$\lim_{N\to\infty}\lim_{t\to 1}\langle Q_{2n}\rangle_t = \langle q_{2n}Q_{2n}\rangle \tag{13}$$

(example: for $N \to \infty$ we get $\langle m_1\rangle_t \to \langle m_1^2\rangle$, $\langle q_{12}\rangle_t \to \langle q_{12}^2\rangle$, $\langle q_{12}q_{34}\rangle_t \to \langle q_{12}q_{34}q_{1234}\rangle$).

Theorem 4. *In the $N \to \infty$ limit the averages $\langle\cdot\rangle$ of the filled monomials are independent of t in the β average.*

Proof. In this sketch we are going to show how to get theorem 2 in some detail; it automatically has as a corollary theorem 3 which ultimately gives, as a simple consequence when applied to filled monomials, theorem 4.

Let us assume for a generic overlap correlation function Q, of s replicas, the following representation:

$$Q = \prod_{a=1}^{s}\sum_{i_l^a}\prod_{l=1}^{n^a}\sigma_{i_l^a}^a I(\{i_l^a\})$$

where a labels the replicas, the internal product takes into account the spins (labeled by l) which contribute to the *shape a* part of the overlap $q_{a,a'}$ and runs for the number of times that the replica a appears in Q, the external product takes into account all the contributions of the internal one and the I factor fixes the constraints among different replicas in Q; so, for example, $Q = q_{12}q_{23}$ can be decomposed into this form noting that $s = 3$, $n^1 = n^3 = 1$, $n^2 = 2$, $I = N^{-2}\delta_{i_1^1,i_1^3}\delta_{i_1^2,i_1^3}$, where the δ functions fix the links between replicas $1, 2 \to q_{1,2}$ and $2, 3 \to q_{2,3}$. The averaged overlap correlation function is

$$\langle Q\rangle_t = \mathbb{E}\sum_{i_l^a}I(\{i_l^a\})\prod_{a=1}^{s}\omega_t\left(\prod_{l=1}^{n^a}\sigma_{i_l^a}^a\right).$$

Now if Q is a fillable polynomial, and we evaluate it at $t = 1$, let us decompose it, using the factorization of the ω state on different replicas, as

$$\langle Q\rangle_t = \mathbb{E}\sum_{i_l^a,i_l^b}I(\{i_l^a\},\{i_l^b\})\prod_{a=1}^{u}\omega_a\left(\prod_{l=1}^{n^a}\sigma_{i_l^a}^a\right)\prod_{b=u}^{s}\omega_b\left(\prod_{l=1}^{n^b}\sigma_{i_l^b}^b\right)$$

where u stands for the number of unfilled replicas inside the expression for Q. So we split the measure Ω into two different subsets ω_a and ω_b: in this way the replicas belonging to the b subset are always even in number, while the ones in the a subset are always odd in number. Applying the gauge $\sigma_i^a \to \sigma_i^a \sigma_{N+1}^a, \forall i \in (1, N)$, the even measure is unaffected by this transformation ($\sigma_{N+1}^{2n} \equiv 1$) while the odd measure takes σ_{N+1} inside the Boltzmann measure:

$$\langle Q \rangle = \sum_{i_l^a, i_l^b} I(\{i_l^a\}, \{i_l^b\}) \prod_{a=1}^{u} \omega\left(\sigma_{N+1}^a \prod_{l=1}^{n^a} \sigma_{i_l^a}^a\right) \prod_{b=u}^{s} \omega\left(\sigma_{N+1}^b \prod_{l=1}^{n^b} \sigma_{i_l^b}^b\right).$$

At the end we can replace in the last expression the subindex $N+1$ of σ_{N+1} by k for any $k \neq \{i_l^a\}$ and multiply by 1, as $1 = N^{-1} \sum_{k=0}^{N}$. Up to $O(1/N)$, which go to zero in the thermodynamic limit, we have the proof. $\qquad\square$

It is now immediately understood that the effect of theorem 2 on a fillable overlap monomial is to multiply it by its missing part, to be filled (theorem 3), while it has no effect if the overlap monomial is already filled (theorem 4) because of the Ising spins (i.e. $\sigma_{N+1}^{2n} \equiv 1 \, \forall n \in \mathbb{N}$).

Now the plan is as follows: we calculate the t-streaming of the Ψ function in order to derive it and then integrate it back once we have been able to express it as an expansion in power series of t with stochastically stable overlaps as coefficients. At the end we free the perturbed Boltzmann measure by setting $t = 1$ and in the thermodynamic limit we will have the expansion holding with the correct statistical mechanics weight:

$$\begin{aligned}\frac{\partial \Psi(\tilde{\alpha}, \beta, t)}{\partial t} &= \frac{\partial}{\partial t} \mathbb{E}[\ln \tilde{\omega}(e^{\beta \sum_{\nu=1}^{P_2 \tilde{\alpha} t} \sigma_{i_\nu}})] \\ &= 2\tilde{\alpha} \mathbb{E}[\ln \tilde{\omega}(e^{\beta \sum_{\nu=1}^{P_2 \tilde{\alpha} t} \sigma_{i_\nu} + \beta \sigma_{i_0}})] - 2\tilde{\alpha} \mathbb{E}[\ln \tilde{\omega}(e^{\beta \sum_{\nu=1}^{P_2 \tilde{\alpha} t} \sigma_{i_\nu}})] = 2\tilde{\alpha} \mathbb{E}[\ln \tilde{\omega}_t(e^{\beta \sigma_{i_0}})]. \quad (14)\end{aligned}$$

Now using the equality $e^{\beta \sigma_{i_0}} = \cosh\beta + \sigma_{i_0} \sinh\beta$, we can write the rhs of equation (14) as

$$\frac{\partial \Psi(\tilde{\alpha}, \beta, t)}{\partial t} = 2\tilde{\alpha} \mathbb{E}[\ln \tilde{\omega}_t(\cosh\beta + \sigma_{i_0} \sinh\beta)] = 2\tilde{\alpha} \log\cosh\beta - 2\tilde{\alpha} \mathbb{E}[\ln(1 + \tilde{\omega}_t(\sigma_{i_0})\theta)].$$

We can expand the function $\log(1 + \tilde{\omega}_t \theta)$ in powers of θ, obtaining

$$\frac{\partial \Psi(\tilde{\alpha}, t)}{\partial t} = 2\tilde{\alpha} \ln\cosh\beta - 2\tilde{\alpha} \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \theta^n \langle q_{1,\ldots,n} \rangle_t. \quad (15)$$

We learn by looking at equation (15) that the derivative of the cavity function is built from non-stochastically stable overlap monomials, and their averages depend on t, making their t-integration non-trivial (we stress that all the fillable terms are zero when evaluated at $t = 0$ due to the gauge invariance of the model). We can escape this constraint by iterating them again and again (and then integrating them back too) because their derivative, systematically, will develop stochastically stable terms, which turn out to be independent via the interpolating parameter and their integration is straightforwardly polynomial. For this task we introduce the following.

Proposition 1. *Let F_s be a function of s replicas. Then the following streaming equation holds:*

$$
\frac{\partial \langle F_s \rangle_{t,\tilde{\alpha}}}{\partial t} = 2\tilde{\alpha}\theta\Bigg[\sum_{a=1}^{s} \langle F_s \sigma_{i_0}^a \rangle_{t,\tilde{\alpha}} - s \langle F_s \sigma_{i_0}^{s+1} \rangle_{t,\tilde{\alpha}} \Bigg]
$$
$$
+ 2\tilde{\alpha}\theta^2 \Bigg[\sum_{a<b}^{1,s} \langle F_s \sigma_{i_0}^a \sigma_{i_0}^b \rangle_{t,\tilde{\alpha}} - s \sum_{a=1}^{s} \langle F_s \sigma_{i_0}^a \sigma_{i_0}^{s+1} \rangle_{t,\tilde{\alpha}} + \frac{s(s+1)}{2!} \langle F_s \sigma_{i_0}^{s+1} \sigma_{i_0}^{s+2} \rangle_{t,\tilde{\alpha}} \Bigg]
$$
$$
+ 2\tilde{\alpha}\theta^3 \Bigg[\sum_{a<b<c}^{1,s} \langle F_s \sigma_{i_0}^a \sigma_{i_0}^b \sigma_{i_0}^c \rangle_{t,\tilde{\alpha}} - s \sum_{a<b}^{1,s} \langle F_s \sigma_{i_0}^a \sigma_{i_0}^b \sigma_{i_0}^{s+1} \rangle_{t,\tilde{\alpha}}
$$
$$
+ \frac{s(s+1)}{2!} \sum_{a=1}^{s} \langle F_s \sigma_{i_0}^a \sigma_{i_0}^{s+1} \sigma_{i_0}^{s+2} \rangle_{t,\tilde{\alpha}} + \frac{s(s+1)(s+2)}{3!} \langle F_s \sigma_{i_0}^{s+1} \sigma_{i_0}^{s+2} \sigma_{i_0}^{s+3} \rangle_{t,\tilde{\alpha}} \Bigg]
\tag{16}
$$

where we neglected terms $\mathrm{O}(\theta^3)$.

Proof. The proof works by direct calculation:

$$
\frac{\partial \langle F_s \rangle_{t,\tilde{\alpha}}}{\partial t} = \frac{\partial}{\partial t} \mathbb{E} \Bigg[\frac{\sum_{\{\sigma\}} F_s \mathrm{e}^{\sum_{a=1}^{s}(\beta \sum_{\nu=1}^{P_{\tilde{\alpha}N}} \sigma_{i_\nu}^a \sigma_{j_\nu}^a + \beta \sum_{\nu=1}^{P_{2\tilde{\alpha}t}} \sigma_{i_\nu}^a)}}{\sum_{\{\sigma\}} \mathrm{e}^{\sum_{a=1}^{s}(\beta \sum_{\nu=1}^{P_{\tilde{\alpha}N}} \sigma_{i_\nu}^a \sigma_{j_\nu}^a + \beta \sum_{\nu=1}^{P_{2\tilde{\alpha}t}} \sigma_{i_\nu}^a)}} \Bigg]
$$
$$
= 2\tilde{\alpha} \mathbb{E} \Bigg[\frac{\sum_{\{\sigma\}} F_s \mathrm{e}^{\sum_{a=1}^{s}(\beta \sigma_{i_0}^a + \beta \sum_{\nu=1}^{P_{\tilde{\alpha}N}} \sigma_{i_\nu}^a \sigma_{j_\nu}^a + \beta \sum_{\nu=1}^{P_{2\tilde{\alpha}t}} \sigma_{i_\nu}^a)}}{\sum_{\{\sigma\}} \mathrm{e}^{\sum_{a=1}^{s}(\beta \sigma_{i_0}^a + \beta \sum_{\nu=1}^{P_{\tilde{\alpha}N}} \sigma_{i_\nu}^a \sigma_{j_\nu}^a + \beta \sum_{\nu=1}^{P_{2\tilde{\alpha}t}} \sigma_{i_\nu}^a)}} \Bigg] - 2\tilde{\alpha} \langle F_s \rangle_{t,\tilde{\alpha}}
$$
$$
= 2\tilde{\alpha} \mathbb{E} \Bigg[\frac{\tilde{\Omega}_t(F_s \mathrm{e}^{\sum_{a=1}^{s} \beta \sigma_{i_0}^a})}{\tilde{\Omega}_t(\mathrm{e}^{\sum_{a=1}^{s} \beta \sigma_{i_0}^a})} \Bigg] - 2\tilde{\alpha} \langle F_s \rangle_{t,\tilde{\alpha}}
$$
$$
= 2\tilde{\alpha} \mathbb{E} \Bigg[\frac{\tilde{\Omega}_t(F_s \Pi_{a=1}^{s}(\cosh\beta + \sigma_{i_0}^a \sinh\beta))}{\tilde{\Omega}_t(\Pi_{a=1}^{s}(\cosh\beta + \sigma_{i_0}^a \sinh\beta))} \Bigg] - 2\tilde{\alpha} \langle F_s \rangle_{t,\tilde{\alpha}}
$$
$$
= 2\tilde{\alpha} \mathbb{E} \Bigg[\frac{\tilde{\Omega}_t(F_s \Pi_{a=1}^{s}(1 + \sigma_{i_0}^a \theta))}{(1 + \tilde{\omega}_t(\sigma_{i_0}^a)\theta)^s} \Bigg] - 2\tilde{\alpha} \langle F_s \rangle_{t,\tilde{\alpha}}.
\tag{17}
$$

Now noting that

$$
\Pi_{a=1}^{s}(1 + \sigma_{i_0}^a \theta) = 1 + \sum_{a=1}^{s} \sigma_{i_0}^a \theta + \sum_{a<b}^{1,s} \sigma_{i_0}^a \sigma_{i_0}^b \theta^2 + \sum_{a<b<c}^{1,s} \sigma_{i_0}^a \sigma_{i_0}^b \sigma_{i_0}^c \theta^3 + \cdots
$$
$$
\frac{1}{(1 + \tilde{\omega}_t\theta)^s} = 1 - s\tilde{\omega}_t\theta + \frac{s(s+1)}{2!}\tilde{\omega}_t^2\theta^2 - \frac{s(s+1)(s+2)}{3!}\tilde{\omega}_t^3\theta^3 + \cdots
\tag{18}
$$

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we obtain

$$
\frac{\partial \langle F_s \rangle_{t,\tilde{\alpha}}}{\partial t} = 2\tilde{\alpha}\mathbb{E}\Bigg[\tilde{\Omega}_t\Bigg(F_s\Bigg(1 + \sum_{a=1}^{s}\sigma_{i_0}^a\theta + \sum_{a<b}^{1,s}\sigma_{i_0}^a\sigma_{i_0}^b\theta^2 + \sum_{a<b<c}^{1,s}\sigma_{i_0}^a\sigma_{i_0}^b\sigma_{i_0}^c\theta^3 + \cdots\Bigg)\Bigg)
$$
$$
\times\Bigg(1 - s\tilde{\omega}_t\theta + \frac{s(s+1)}{2!}\tilde{\omega}_t^2\theta^2 - \frac{s(s+1)(s+2)}{3!}\tilde{\omega}_t^3\theta^3 + \cdots\Bigg)\Bigg]
$$
$$
- 2\tilde{\alpha}\langle F_s\rangle_{t,\tilde{\alpha}}. \tag{19}
$$

from which our theorem follows. $\qquad\qquad\square$

4. Free energy analysis

Now that we have exploited the machinery we can start applying it to the free energy. Let us first work out its streaming with respect to the plane (α, β):

$$
\frac{\partial A(\alpha,\beta)}{\partial \beta} = -\frac{\langle H \rangle}{N} = \frac{1}{N}\mathbb{E}\Bigg(\frac{1}{Z_N}\sum_{\{\sigma\}}\sum_{\nu=1}^{P_{\alpha N}}\sigma_{i_\nu}\sigma_{j_\nu}\mathrm{e}^{-\beta H_N(\alpha)}\Bigg)
$$
$$
= \frac{1}{N}\sum_{k=1}^{\infty}k\pi(k-1,\alpha N)\mathbb{E}[\omega(\sigma_{i_k}\sigma_{j_k})_k]
$$
$$
= \alpha\sum_{k=1}^{\infty}\pi(k-1,\alpha N)\mathbb{E}\Bigg[\frac{\omega(\sigma_{i_k}\sigma_{j_k}\mathrm{e}^{\beta\sigma_{i_k}\sigma_{j_k}})_{k-1}}{\omega(\mathrm{e}^{\beta\sigma_{i_k}\sigma_{j_k}})_{k-1}}\Bigg]
$$
$$
= \alpha\mathbb{E}\Bigg[\frac{\omega(\sigma_{i_k}\sigma_{j_k}(\cosh\beta + \sigma_{i_k}\sigma_{j_k}\sinh\beta))}{\omega(\cosh\beta + \sigma_{i_k}\sigma_{j_k}\sinh\beta)}\Bigg] = \alpha\mathbb{E}\Bigg[\frac{\omega(\sigma_{i_k}\sigma_{j_k}) + \theta}{1 + \omega(\sigma_{i_k}\sigma_{j_k})\theta}\Bigg] \tag{20}
$$

by which we get (and with similar calculations for $\partial_\alpha A(\alpha,\beta)$ that we omit for the sake of simplicity)

$$
\frac{\partial A(\alpha,\beta)}{\partial \beta} = \alpha\theta - \alpha\sum_{n=1}^{\infty}(-1)^n(1-\theta^2)\theta^{n-1}\langle q_{1,\ldots,n}^2\rangle \tag{21}
$$

$$
\frac{\partial A(\alpha,\beta)}{\partial \alpha} = \ln\cosh\beta - \sum_{n=1}^{\infty}\frac{(-1)^n}{n}\theta^n\langle q_{1,\ldots,n}^2\rangle. \tag{22}
$$

Now remembering theorem 1 and assuming critical behavior (that we will verify *a fortiori* in section 5) we move to a different formulation of the free energy by considering the cavity function as the integral of its derivative. In a nutshell the idea is as follows: due to the second-order nature of the phase transition for this model (i.e. criticality that so far is assumed) we can expand the free energy in terms of the whole series of order parameters. Of course it is impossible to manage all of these infinite overlap correlation functions to get a full solution of the model in the whole (α, β) plane, but it is possible to show by means of rigorous bounds that close to the critical line (that we are going to find soon) higher order overlaps scale with higher order critical exponents, so we are allowed to neglect higher orders close to this line and we can investigate deeply criticality, which is the topic of the paper.

For this task let us expand the cavity functions as

$$\Psi(\tilde{\alpha}, \beta, t) = \int_0^t \frac{\partial \Psi}{\partial t'} \, \mathrm{d}t'$$

$$= 2\tilde{\alpha} t \log \cosh \beta + \tilde{\beta} \int_0^t \langle m \rangle_{t', \tilde{\alpha}} \, \mathrm{d}t' - \tfrac{1}{2} \tilde{\beta} \theta \int_0^t \langle q_{12} \rangle_{t', \tilde{\alpha}} \, \mathrm{d}t' + \mathrm{O}(\theta^3) \qquad (23)$$

where $\tilde{\beta} = 2\tilde{\alpha}\theta \to \beta' = 2\alpha\theta$ for $N \to \infty$. Now using the streaming equation as dictated by proposition 1 we can write the overlaps appearing in the expression of Ψ as polynomials of higher order filled overlaps so as to obtain a straightforward polynomial back-integration for the Ψ as they will no longer depend on the interpolating parameter thanks to theorem 4.

For the sake of simplicity the $\tilde{\alpha}$ dependence of the overlaps will be omitted, keeping in mind that our results are all taken in the thermodynamic limit and so we can quietly exchange $\tilde{\alpha}$ with α in these passages.

The first equation that we deal with is

$$\frac{\mathrm{d}\langle m \rangle_t}{\mathrm{d}t} = \tilde{\beta}[\langle m^2 \rangle - \langle m_1 m_2 \rangle_t] \qquad (24)$$

where $\langle m_1 m_2 \rangle$ is not filled and so we have to go further in the procedure and derive it in order to obtain filled monomials:

$$\frac{\mathrm{d}\langle m_1 m_2 \rangle_t}{\mathrm{d}t} = 2\tilde{\beta}[\langle m_1^2 m_2 \rangle_t - \langle m_1 m_2 m_3 \rangle_t] + \tilde{\beta}\theta[\langle m_1 m_2 q_{12} \rangle$$
$$- 4\langle m_1 m_2 q_{13} \rangle_t + 3\langle m_1 m_2 q_{34} \rangle_t]. \qquad (25)$$

In this expression we stress the presence of the filled overlap $\langle m_1 m_2 q_{12} \rangle$ and of $\langle m_1^2 m_2 \rangle_t$ which can be saturated in just one derivation. Wishing to have an expansion for $\langle m \rangle_t$ up to the third order in θ, it is easy to check that the saturation of the other overlaps in the last derivative would carry terms of higher order and so we can stop the procedure at the next step:

$$\frac{\mathrm{d}\langle m_1^2 m_2 \rangle_t}{\mathrm{d}t} = \tilde{\beta}[\langle m_1^2 m_2^2 \rangle] + \tilde{\beta}[\text{unfilled terms}] + \mathrm{O}(\theta^2) \qquad (26)$$

from which, integrating back in t,

$$\langle m_1^2 m_2 \rangle_t = \tilde{\beta}[\langle m_1^2 m_2^2 \rangle] t. \qquad (27)$$

Now inserting this result in the expression (25) and integrating again in t we find

$$\langle m_1 m_2 \rangle_t = \tilde{\beta}\theta \langle m_1 m_2 q_{12} \rangle t + \tilde{\beta}^2 \langle m_1^2 m_2^2 \rangle t^2 \qquad (28)$$

and coming back to $\langle m \rangle_t$ we get

$$\langle m \rangle_t = \tilde{\beta} \langle m^2 \rangle t - \frac{\tilde{\beta}^2 \theta}{2} \langle m_1 m_2 q_{12} \rangle t^2 - \frac{\tilde{\beta}^3}{3} \langle m_1^2 m_2^2 \rangle t^3 \qquad (29)$$

which is the attempted result. Let us move our attention to $\langle q_{12} \rangle_t$; analogously we can write

$$\frac{\mathrm{d}\langle q_{12} \rangle_t}{\mathrm{d}t} = 2\tilde{\beta}[\langle m_1 q_{12} \rangle_t - \langle m_3 q_{12} \rangle_t] + \tilde{\beta}\theta[\langle q_{12}^2 \rangle - 4\langle q_{12} q_{13} \rangle_t + 3\langle q_{12} q_{34} \rangle_t] \qquad (30)$$

and consequently obtain

$$\langle q_{12}\rangle_t = \tilde{\beta}\theta\langle q_{12}^2\rangle t + \tilde{\beta}^2\langle m_1 m_2 q_{12}\rangle t^2 + \mathrm{O}(\theta^4). \tag{31}$$

With the two expansions above, in the $N \to \infty$ limit, putting $t = 1$ we have

$$\Psi(\alpha, \beta, t = 1) = 2\alpha \ln \cosh \beta + \frac{\beta'}{2}\langle m^2\rangle - \frac{\beta'^4}{12}\langle m_1^2 m_2^2\rangle - \frac{\beta'^2\theta^2}{4}\langle q_{12}^2\rangle$$
$$- \frac{\beta'^3\theta}{3}\langle m_1 m_2 q_{12}\rangle + \mathrm{O}(\theta^6). \tag{32}$$

At this point we have all the ingredients for writing down the polynomial expansion for the free energy function as stated in the next part.

Proposition 2. *A general expansion via stochastically stable terms for the free energy of the diluted Ising model can be written as*

$$A(\alpha, \beta) = \ln 2 + \alpha \ln \cosh \beta + \frac{\beta'}{2}\left(\beta' - 1\right)\langle m_1^2\rangle$$
$$- \frac{\beta'^4}{12}\langle m_1^2 m_2^2\rangle - \frac{\beta'^2}{8\alpha}\left(\frac{\beta'^2}{2\alpha} - 1\right)\langle q_{12}^2\rangle - \frac{\beta'^4}{6\alpha}\langle m_1 m_2 q_{12}\rangle + \mathrm{O}(\theta^6). \tag{33}$$

It is immediately checked that the above expression, in the ergodic region where the averages of all the order parameters vanish, reduces to the well known high temperature (or high connectivity) solution [18] (i.e. $A(\alpha, \beta) = \ln 2 + \alpha \log \cosh \beta$).

Of course we are neglecting θ^6 and higher order terms because we are interested in an expansion holding close to the critical line, but we are not allowed to truncate the series for a general point in the phase space far beyond the ergodic region.

5. Critical behavior

Now we want to analyze the critical behavior of the model: we find the critical line where the ergodicity breaks, we obtain the critical exponent of the magnetization and the susceptibility, and at the end we show that within our framework the lack of ergodicity can be explained as the breaking of commutativity of the infinite volume limit against our cavity field, thought of as a properly chosen field, vanishing in the thermodynamic limit too, accordingly to the standard prescription of statistical mechanics [1].

5.1. Critical line

Let us firstly define the rescaled magnetization ξ_N as $\xi_N = \sqrt{N}m_N$. By applying the gauge transformation $\sigma_i \to \sigma_i\sigma_{N+1}$ in the expression for the quenched average of the magnetization (equation (29)) and multiplying it by N so to switch to ξ_N^2, setting $t = 1$ and sending $N \to \infty$, we obtain

$$\langle\xi_1^2\rangle = \frac{\beta'^3}{3(\beta' - 1)}\langle\xi_1\xi_2 m_1 m_2\rangle + \frac{\beta'^2\theta}{2(\beta' - 1)}\langle\xi_1\xi_2 q_{12}\rangle + \mathrm{O}\left(\frac{\theta^5}{\beta' - 1}\right) \tag{34}$$

by which we see (again remembering criticality and so forgetting higher order terms) that the only possible divergence of the (centered and rescaled) fluctuations of the
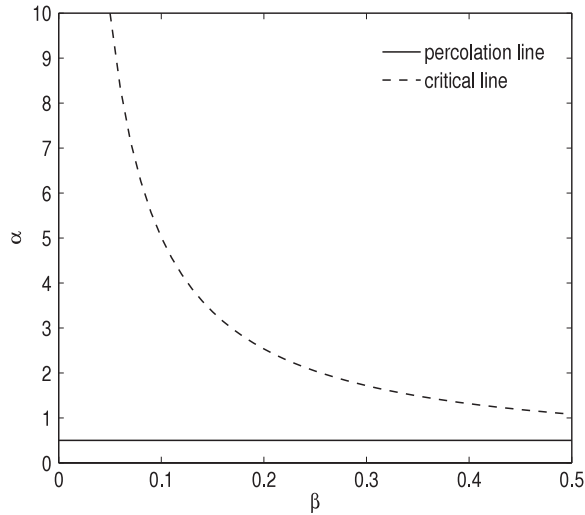
Figure 1. Phase diagram: below $\alpha_c = 0.5$ there is no giant component in the Erdös–Renyi graph; α_c defines the percolation threshold. Above, left of the critical line, the system behaves ergodically; conversely on the right, ergodicity is broken and the system displays magnetization.

magnetization happens at the value $\beta' = 1$ which gives $2\alpha\theta = 1$ as the critical line, in perfect agreement with [18] (see figure 1). The same critical line can be found more easily by simply looking at the expression (33) as follows: remembering that in the ergodic phase the minimum of the free energy corresponds to a order parameter of zero (i.e. $\sqrt{\langle m^2 \rangle} = 0$), this implies the coefficient of second order, $a(\beta') = \beta'/2(\beta' - 1)$, to be positive. Anyway, immediately below the critical line, values of the magnetization different from zero must be allowed (by definition; otherwise we were not crossing a critical line) and this can be possible if and only if $a(\beta') \leq 0$. Consequently (and using once more the second-order nature of the transition) on the critical line we must have $a(\beta') = 0$ and this gives again $2\alpha\theta = 1$.

5.2. Critical exponents and bounds

Now let us move to the critical exponents.

Critical exponents are needed to characterize singularities of the theory at the critical line and, for us, these indexes are the ones related to the magnetization $\langle m \rangle$ and to the susceptibility $\langle \chi \rangle \equiv \beta N[\langle m^2 \rangle - \langle m \rangle^2]$.

We define $\tau = (2\alpha \tanh \beta - 1)$ and we write $\langle m(\tau) \rangle \sim G_0 \cdot \tau^\delta$ and $\langle \chi(\tau) \rangle \sim G_0 \cdot \tau^\gamma$, where the symbol $\sim$ has the meaning that the term in the second member is the dominant one, but there are corrections of higher order.

Remembering the expansion of the squared magnetization that we rewrite for completeness:

$$\langle m^2 \rangle = \frac{\beta'^3}{3(\beta' - 1)} \langle m_1^2 m_2^2 \rangle + \frac{\beta'^2 \theta}{2(\beta' - 1)} \langle m_1 m_2 q_{12} \rangle + \mathrm{O}\left(\frac{\theta^5}{\beta' - 1} \right) \tag{35}$$

and considering that using the same gauge transformation $\sigma_i \rightarrow \sigma_i \sigma_{N+1}$ on equation (31), we have for the two-replica overlap the following representation:

$$\langle q_{12}^2 \rangle = -\frac{\beta'^2}{(\beta'\theta - 1)}\langle m_1 m_2 q_{12}\rangle + \mathrm{O}(\theta^6). \tag{36}$$

We can proceed, via simple algebraic calculations, to writing down the free energy, of course close to the critical line, depending only by the two parameters $\langle m^2 \rangle$ and $\langle q_{12}^2 \rangle$:

$$A(\alpha, \beta) = \ln 2 + \alpha \ln \cosh \beta + \frac{\beta'}{4}(\beta' - 1)\langle m_1^2 \rangle - \frac{\beta'^2}{48\alpha}\left(\frac{\beta'^2}{2\alpha} - 1\right)\langle q_{12}^2 \rangle + \mathrm{O}(\theta^6). \tag{37}$$

By a comparison of the formula obtained by deriving $A(\alpha, \beta)$ as expressed by equation (37) and the expression that we have previously found (equations (22)) that we report for the sake of readability:

$$\frac{\partial A(\alpha, \beta)}{\partial \alpha} = \ln \cosh \beta - \sum_{n=1}^{\infty} \frac{(-1)^n}{n}\theta^n \langle q_{1,\ldots,n}^2 \rangle \tag{38}$$

it is immediately seen that we have

$$\frac{\partial}{\partial \alpha}\left[\frac{\beta'}{4}(\beta' - 1)\langle m_1^2 \rangle\right] = \theta \langle m_1^2 \rangle. \tag{39}$$

If we put ourselves close to the value $\beta' = 1$ and make a change of variable $\tau = \beta' - 1$ with $\partial_\alpha = 2\theta \partial_\tau$ we get

$$\frac{\partial}{\partial \alpha}\left[\frac{\beta'}{4}(\beta' - 1)\langle m_1^2 \rangle\right] \sim \frac{\theta}{2}\frac{\partial}{\partial \tau}[\tau \langle m_1^2 \rangle] = \frac{\theta}{2}\langle m_1^2 \rangle + \frac{\theta \tau}{2}\frac{\partial \langle m_1^2 \rangle}{\partial \tau} = \theta \langle m_1^2 \rangle \tag{40}$$

by which we easily obtain

$$\frac{\partial \langle m_1^2 \rangle}{\langle m_1^2 \rangle} = \frac{\partial \tau}{\tau} \Rightarrow \langle m_1^2 \rangle \sim \tau \Rightarrow \sqrt{\langle m_1^2 \rangle} \sim \tau^{1/2}. \tag{41}$$

Therefore we get that the critical exponent for the magnetization, $\delta = 1/2$, which turns out to be the same as in the fully connected counterpart [6, 19], in agreement with the disordered extension of this model [10].

Again, by simple direct calculations, once we get the critical exponent for the magnetization it is straightforward to show that the susceptibility $\langle \chi \rangle$ obeys

$$\langle \chi \rangle \sim |\tau|^{-1} \tag{42}$$

close to the critical line, by which we find its critical exponent to be once again in agreement with the classical fully connected counterpart [1].

Now we want to show some wrong results which a naive calculation would suggest, so as to emphasize the importance of the bounds relating different monomials that we are going to discuss immediately after. Then in section 5.3, we explain what the physics is behind this picture by providing a mechanism for the breaking of the ergodicity.

The point on which we focus is the following: if we wish to perform the same procedure as we performed on $\langle m^2 \rangle$, applying blindly saturability below to the first critical line, to

the two-replica overlap $\langle q_{12}^2 \rangle$ we would obtain

$$\sqrt{\langle q_{12}^2 \rangle} \sim \tau_2 = (\beta'\theta - 1) \tag{43}$$

identifying $\theta_{c_2} = 1/(2\alpha)^{1/2}$ as another critical temperature, or better, the critical temperature typical of $\langle q_{12}^2 \rangle$. In the same way we could find θ_{c_n} for every $q_{1...n}^2$, obtaining

$$\theta_{c_n} = 1/(2\alpha)^{1/n}$$

such that, at the end, we would obtain a scenario with several transition lines, one for every order parameter.

This is not a possible scenario, as generally explained for instance in [7] and as dictated, in this model, by the following.

Proposition 3. *As soon as the first-order parameter (the magnetization) starts taking values different from zero, the same happens to all the other order parameters:*

$$\langle q_{1,...,n}^2 \rangle = \mathbb{E}_k \mathbb{E}_i [\omega^n(\sigma_{i_1}\sigma_{i_2})] \geq (\mathbb{E}_k \mathbb{E}_i [\omega(\sigma_{i_1}\sigma_{i_2})])^n = \langle m^2 \rangle^n \quad \forall n. \tag{44}$$

We omit the proof details as they are a simple application of the Jensen inequality (see e.g. [19]).

5.3. Saturability breaking

So far we have shown that it is not possible to have several transition lines, one for every order parameter. Now we want to understand why there is just one critical line by applying the theory developed in [9] to this model.

Starting from theorem 1 that we recall for simplicity:

$$A(\alpha, \beta) = \ln 2 - \alpha \frac{\partial A(\alpha, \beta)}{\partial \alpha} + \Psi(\alpha, \beta, t = 1) \tag{45}$$

we want to show the phase transition expressed by the non-commutativity of the thermodynamic limit and the vanishing perturbation.

Again for simplicity we report the expansion of $A(\alpha, \beta)$ that we have previously built:

$$A(\alpha, \beta) = \ln 2 + \alpha \ln \cosh \beta + \frac{\beta'}{4}(\beta' - 1)\langle m_1^2 \rangle - \frac{\beta'^2}{48\alpha}\left(\frac{\beta'^2}{2\alpha} - 1\right)\langle q_{12}^2 \rangle + \mathrm{O}(\theta^6) \tag{46}$$

which we obtained by considering the cavity function as the integral of its t-derivative:

$$\Psi(\tilde{\alpha}, t) = \int_o^t \frac{\partial \Psi(\tilde{\alpha}, t')}{\partial t'}\, \mathrm{d}t' = \int_o^t 2\tilde{\alpha} \ln \cosh \beta\, \mathrm{d}t' - 2\tilde{\alpha} \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \theta^n \int_o^t \langle q_{1,...,n} \rangle_t'\, \mathrm{d}t' \tag{47}$$

and performing, via the streaming equation, a procedure of saturating consecutive t-derivatives upon the overlaps in order to express them as functions of higher order filled terms. Then the only thing we had to do was send N to infinity, carrying out of the integral the overlaps, and then putting $t = 1$ to evaluate $\Psi(\alpha, \beta, t = 1)$. The result of this procedure brings us to equation (46).

But what if we exchanged the limit order by sending $t \to 1$ first and taking the thermodynamic limit after?

It is easy to note that all the overlaps appearing in equation (47) are fillable, such that we can avoid the saturation procedure simply by setting $t = 1$ first and then sending N to infinity. In this way, thanks to theorem 3, each fillable overlap is transformed into a filled t-independent one and this kills all the correlations among different replicas and allows us to write

$$A(\alpha, \beta) = \ln 2 + \alpha \ln \cosh \beta - \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \theta^n \langle q_{1,\ldots,n}^2 \rangle = \ln 2 + \alpha \frac{\partial A(\alpha, \beta)}{\partial \alpha} \quad (48)$$

where clearly

$$
\lim_{N \to \infty} \lim_{t \to 1} \Psi(\tilde{\alpha}, \beta, t) = \lim_{N \to \infty} \lim_{t \to 1} \int_o^t \frac{\partial \Psi(\tilde{\alpha}, \beta, t')}{\partial t'} \, dt'
$$

$$
= \lim_{N \to \infty} \lim_{t \to 1} \left[\int_o^t 2\tilde{\alpha} \log \cosh \beta \, dt' - 2\tilde{\alpha} \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \theta^n \int_o^t \langle q_{1,\ldots,n} \rangle_t' \, dt' \right]
$$

$$
= 2\alpha \ln \cosh \beta - 2\alpha \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \theta^n \langle q_{1,\ldots,n}^2 \rangle = 2\alpha \frac{\partial A(\alpha, \beta)}{\partial \alpha}. \quad (49)
$$

In particular, retaining just the first two terms of the expansions, we show the difference between the two results:

• $\lim_{t \to 1} \lim_{N \to \infty}$:

$$A(\alpha, \beta) = \ln 2 + \alpha \ln \cosh \beta + \frac{\beta'}{4}(\beta' - 1)\langle m_1^2 \rangle + \mathrm{O}(\theta^3). \quad (50)$$

• $\lim_{N \to \infty} \lim_{t \to 1}$:

$$A(\alpha, \beta) = \ln 2 + \alpha \ln \cosh \beta + \frac{\beta'}{2}\langle m_1^2 \rangle + \mathrm{O}(\theta^3). \quad (51)$$

We immediately recognize that equation (50) is the correct expression, holding also below the critical line. When the system lives in the ergodic region all the order parameters are zero and it reduces to $\alpha \ln \cosh(\beta)$ which is the ergodic solution; if we cross the critical line the formula takes into account the phase transition encoded in the coefficient of the second order and gives the correct expression immediately below.

It is also straightforward to recognize as the ergodic solution equation (51), which can be the correct one only above the critical line [18].

At the end we saw that there exists one and only one critical line for all the order parameters. We saw that this line can be depicted as the breaking of commutativity among the infinite limit operation and the setting of $t = 1$ (relaxing the Boltzmann factor from the interpolant, avoiding the trivial way of having $t = 0$). Hence, what could be the origin of the transition for all the other order parameters? The correlations among them and the magnetization, as clearly evident in the free energy expansion (see equation (33)). And lastly, what is the origin of these correlations? The *saturability property* that the model exhibits, as stated by theorem 3. In fact, at the critical line (which is the last line, from above, where gauge invariance is a symmetry of the Boltzmann state too, thanks to the continuity of the transition), saturability easily shows that

$$\lim_{N \to \infty} \langle m_1 q_{12} \rangle_t = \langle m_1 m_2 q_{12} \rangle, \quad (52)$$

explaining the birth of the correlations among the various order parameters (we reported just the first two, as an example).

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5.4. Self-averaging properties

We have previously shown how filled overlaps become asymptotically independent of t when N grows to infinity. Starting from this we can find identities stating the self-averaging of the order parameters as $\langle m^2 \rangle$.

In particular we are going to take these overlaps and calculate their derivative with respect to t; then by applying the gauge transformation and setting $t = 1, N \to \infty$ we can write down the self-averaging relations:

$$0 = \partial_t \langle m_1^2 \rangle_t = \beta'[\langle m_1^3 \rangle_t - \langle m_1^2 m_2 \rangle_t] \Rightarrow [\langle m_1^4 \rangle - \langle m_1^2 m_2^2 \rangle] = \mathbb{E}[\Omega(m^4) - \Omega^2(m^2)]. \tag{53}$$

We find, consistently with [18], standard self-averaging for the magnetization.

More interesting is the situation concerning the overlap, but we actually lack a complete mathematical control.

By applying the same trick as before we can equate to zero (in the large N limit) the t-derivative of the squared overlap (which is stochastically stable); consequently we get two terms:

$$0 = \partial_t \langle q_{12}^2 \rangle_t = 2\beta'[\langle m_1 q_{12}^2 \rangle_t - \langle m_3 q_{12}^2 \rangle_t] + \beta'\theta[\langle q_{12}^3 \rangle_t - 4\langle q_{12}^2 q_{13} \rangle_t + 3\langle q_{12}^2 q_{34} \rangle_t] \tag{54}$$

$$\Rightarrow [\langle m_1^2 q_{12}^2 \rangle - \langle m_3^2 q_{12}^2 \rangle] = 0 \qquad [\langle q_{12}^4 \rangle - 4\langle q_{12}^2 q_{13}^2 \rangle + 3\langle q_{12}^2 q_{34}^2 \rangle] = 0. \tag{55}$$

A priori we cannot assume factorization of the series so to put to zero each term separately (as we did in equation (55)); however, close to the critical line, surely the second term is a higher order and we can reasonably set to zero the first. Furthermore as the second term on the rhs has a pre-factor $\propto \alpha^{-1}$ differing from the first term, it would be difficult to imagine the opposite.

It is in fact very natural to assume that the two terms can be set to zero separately in the whole (α, β) plane and this is very interesting because the second term is a very well known relation in the field of spin glasses [16, 24, 3, 23, 5] suggesting a common structure among different kinds of disordered systems, the only sharing feature among diluted ferromagnets and spin glasses being some kind of disorder (topological in the former, frustrating in the latter).

We are not going to go deeper on this point as it is under investigation in [17] where the same set of relations (and more) are found and discussed.

6. Numerics

In this section we analyze, from the numerical point of view, the ferromagnetic system previously introduced by performing extensive Monte Carlo simulations with the Metropolis algorithm [29].

The Erdös–Renyi random graph is constructed by taking N sites and introducing a bond between each pair of sites with probability $p = \bar{\alpha}/(N-1)$, in such a way that the average coordination number per node is just $\bar{\alpha}$. Clearly, when $p = 1$ the complete graph is recovered.

The simplest version of the diluted Curie–Weiss Hamiltonian has a Poisson variable per bond of $H_N = -\sum_{ij} \sum_{\nu=0}^{P_{\bar{\alpha}/N}} \sigma_{i_\nu} \sigma_{j_\nu}$, and this gives the easiest approach when dealing with numerics.

For the analytical investigation we choose a slightly changed version (see equation (1)): each link gets a bond with probability close to α/N for large N; the probabilities of getting two and three bonds scale as $1/N^2, 1/N^3$ and are therefore negligible in the thermodynamic limit.

Working with directed links (as we do in the analytical framework) the probability of having a bond on any undirected link is twice the probability for a directed link (i.e. $2\alpha/N$). Hence, for large N, each site has average connectivity 2α. Finally in this way we allow self-loops, but they add just a σ-independent constant to H_N and are irrelevant, but we take the advantage of dealing with an H_N which is the sum of independent identically distributed random variables, which is useful for analytical investigation.

When comparing with numerics, consequently we must keep in mind that $\bar{\alpha} = 2\alpha$.

In the simulation, once the network has been diluted, we place a spin σ_i on each node i and allow it to interact with its nearest neighbors. Once the external parameter β is fixed, the system is driven by the single-spin dynamics and it eventually relaxes to a stationary state characterized by well defined properties. More precisely, after a suitable time lapse t_0 and for sufficiently large systems, measurements of a (specific) physical observable $x(\sigma, \bar{\alpha}, \beta)$ fluctuate around an average value only depending on the external parameters β^{-1} and $\bar{\alpha}$.

Moreover, for a system $(\bar{\alpha}, \beta)$ of a given finite size N, the extent of such fluctuations scales as $N^{-1/2}$ with the size of the system. The estimate of the thermodynamic observables $\langle x \rangle$ is therefore obtained as an average over a suitable number of (uncorrelated) measurements performed when the system is reasonably close to the equilibrium regime.

The estimate is further improved by averaging over different realizations of the same system $(\bar{\alpha}, \beta)$. In summary,

$$\langle x(\sigma, \bar{\alpha}, \beta) \rangle = \mathbb{E} \left[\frac{1}{M} \sum_{n=1}^{M} x(\sigma(t_n)) \right], \qquad t_n = t_0 + n\mathcal{T}$$

where $\sigma(t)$ denotes the configuration of the magnetic system at time step t and $\mathcal{T}$ is the decorrelation parameter (i.e. the time, in units of spin flips, needed to decorrelate a given magnetic arrangement).

In general, statistical errors during an MC run in a given sample turn out to be significantly smaller than those arising from the ensemble averaging (see also [33]). Figure 2 shows the dependence of the macroscopic observables $\langle m \rangle$ and $\langle e \rangle$ from the size of the system; values are obtained starting from a ferromagnetic arrangement, at the normalized inverse temperature $\beta/\bar{\alpha} = 1.67$. Notice that at this temperature the system composed of $N = 10^4$ parts is already very close to the asymptotic regime. Analogous results are found for different systems $(\bar{\alpha}, \beta)$, with β far enough from β_c.

In the following we focus on systems of sufficiently large size so as to permit discarding finite size effects. For a wide range of temperatures and dilutions, we measure the average magnetization $\langle m \rangle$ and energy $\langle e \rangle$, as well as the magnetic susceptibility χ, calculated as

$$\chi(\beta, \bar{\alpha}) \equiv \beta N \left[\langle m^2 \rangle - \langle m \rangle^2 \right].$$

Their profiles display the typical behavior expected for a ferromagnet and, consistently with the theory, highlight a phase transition at well defined temperatures $\beta_c(\alpha)$.

Now, we investigate in more detail the critical behavior of the system. We collect accurate data for the magnetization and susceptibility, for different values of $\bar{\alpha}$ and for
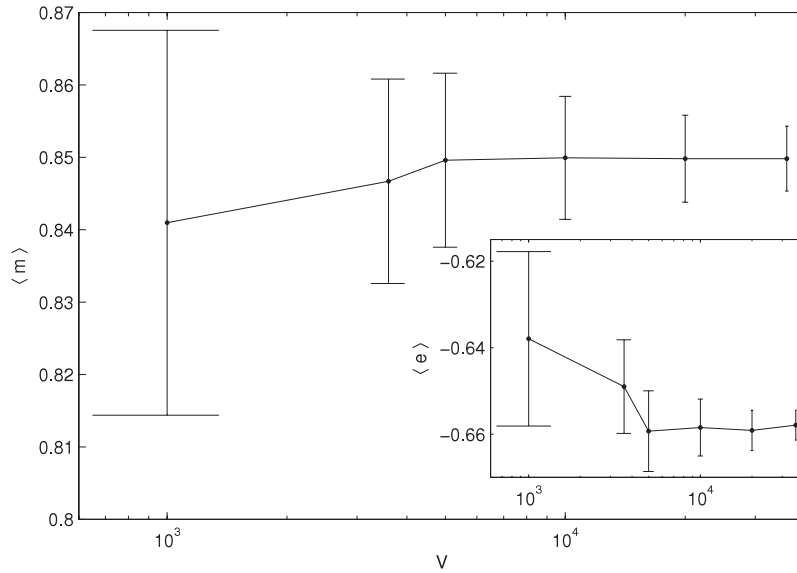
Figure 2. Finite size scaling for the magnetization and the internal energy (inset) for $\bar{\alpha} = 10$ and $(\beta/\bar{\alpha}) = 1.67$. All the measurements were carried out in the stationary regime and the error bars represent the fluctuations about the average values. We find good indications of the convergence of the quantities with the size of the system and thus of the existence of the thermodynamic limit.

Table 1. Estimates for the critical temperature and the critical exponents δ and γ obtained by a fitting procedure on data from numerical simulations concerning Ising systems of size $N = 36\,000$ and different dilutions (we stress that analytically we get $\delta = 0.5$ and $\gamma = -1$). Errors on temperatures are $<2\%$, while for exponents they are within 5%.

$\bar{\alpha}$	β_{c}^{-1}	δ	γ
10	9.93	0.48	-0.97
20	19.92	0.49	-1.04
30	29.98	0.48	-1.04
40	39.59	0.50	-1.02

temperatures approaching the critical one. These data are used to estimate both the critical temperature and the critical exponents for the magnetization and susceptibility. In figure 3 we show data as a function of the reduced temperature $\tau = (|\beta - \beta_{\mathrm{c}}|/\beta_{\mathrm{c}})^{-1}$ for $\bar{\alpha} = 10$ and $\bar{\alpha} = 20$. The best fit for observables is the power law

$$\langle m \rangle \sim \tau^{\delta}, \qquad \beta > \beta_{\mathrm{c}} \tag{56}$$

$$\chi \sim \tau^{\gamma}. \tag{57}$$

We obtain estimates for $\beta_{\mathrm{c}}(\bar{\alpha})$, $\delta(\bar{\alpha})$ and $\gamma(\bar{\alpha})$ by means of a fitting procedure. Results are gathered in table 1. Within the errors ($\leq 2\%$ for β_{c} and $\leq 5\%$ for the exponents), estimates for different values of $\bar{\alpha}$ agree and they are also consistent with the analytical results revealed in section 5
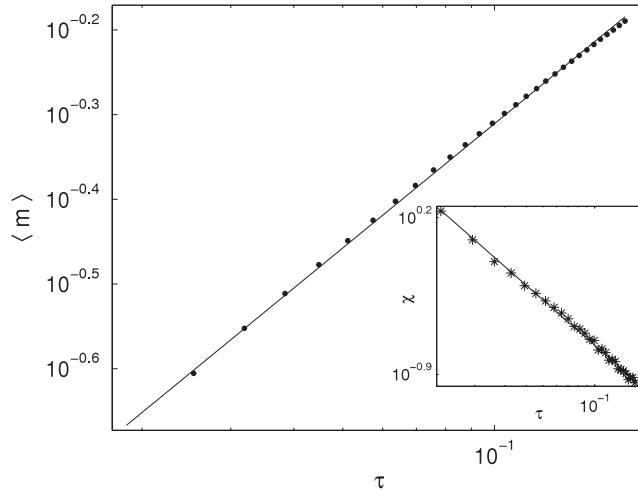
Figure 3. Log–log scale plot of the magnetization (main figure) and susceptibility (inset) versus the reduced temperature $\tau = (|\beta - \beta_c|/\beta_c)^{-1}$ for $\bar{\alpha} = 10$. Symbols represents data from numerical simulations performed on systems of size $N = 36\,000$, while lines represent the best fit.

We also checked the critical line for the ergodicity breaking, again finding optimal agreement with the criticality investigated by means of analytical tools.

7. Conclusions

In this paper we developed the *interpolating cavity field* technique for the mean field diluted ferromagnet. Once the general framework had been built we used it to analyze criticality: we found analytically the critical line and the critical exponent of the magnetization, whose self-averaging was also proved. We present an argument to explain the transition from an ergodic phase to a broken ergodicity phase via the breaking of commutativity of two limits, volume and applied field, as dictated by standard statistical mechanics. We furthermore showed the existence of only one critical line where all the multi-overlaps start taking positive values as soon as the magnetization becomes different from zero. We proved this both mathematically by means of a rigorous bound and physically via a mechanism that generates strong correlations among the magnetization and overlaps at the (unique) critical line: *shape saturability.*

At the end a detailed numerical analysis of the model was presented: by sharp Monte Carlo simulations the convergence of the energy density (and the magnetization) to its limit was investigated, obtaining monotonicity in the system size. The critical line, as well as scaling of the magnetization and the susceptibility, were also investigated, obtaining full agreement among theory and the simulations.

Future works should extend these techniques to several lateral models such as the bipartite diluted mean field Ising models, while the need for stronger techniques to go well beyond the critical line is also to be satisfied—as well as their practical applications to social science or biological networks. We plan to follow these research lines in the future.

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References

[1] Amit D J, 1992 *Modeling Brain Function: The World of Attractor Neural Network* (Cambridge: Cambridge University Press)
[2] Agostini A, Barra A and De Sanctis L, *Positive-overlap transition and critical exponents in mean field spin glasses*, 2006 *J. Stat. Mech.* P11015
[3] Aizenman M and Contucci P, *On the stability of the quenched state in mean field spin glass models*, 1998 *J. Stat. Phys.* **92** 765
[4] Albert R and Barabasi A-L, *Statistical mechanics of complex networks*, 2002 *Rev. Mod. Phys.* **74** 47
[5] Barra A, *Irreducible free energy expansion and overlap locking in mean field spin glasses*, 2006 *J. Stat. Phys.* **123** 601
[6] Barra A, *The mean field Ising model through interpolating techniques*, 2008 *J. Stat. Phys.* **132** 787
[7] Barra A, *Driven transitions at the onset of ergodicity breaking in gauge invariant complex systems*, 2008 *Adv. Complex Syst.* submitted
[8] Barra A and De Sanctis L, *Stability properties and probability distributions of multi-overlaps in diluted spin glasses*, 2007 *J. Stat. Mech.* P08025
[9] Barra A and De Sanctis L, *On the mean field spin glass transition*, 2008 *Eur. Phys. J.* B **64** 119
[10] Barra A, De Sanctis L and Folli V, *Critical behavior of a spin glass on a random graph*, 2008 *J. Phys. A: Math. Theor.* **41** 215005
[11] Bianchi A, Contucci P and Giardinà C, *Thermodynamic limit for mean-field spin models*, 2003 *Math. Phys. Electron. J.* **9** (6)
[12] Bovier A and Gayrard V, *The thermodynamics of the Curie–Weiss model with random couplings*, 1993 *J. Stat. Phys.* **72** 643
[13] Caldarelli G and Vespignani A, 2007 *Large Scale Structure and Dynamics of Complex Networks* (Singapore: World Scientific)
[14] Caldarelli G, 2008 *Scale Free Networks* (Oxford: Oxford Finance)
[15] Contucci P and Giardinà C, *Spin-glass stochastic stability: a rigorous proof*, 2005 *Ann. H. Poincaré* **6** (5) 915
[16] Contucci P and Giardinà C, *The Ghirlanda–Guerra identities*, 2007 *J. Stat. Phys.* **126** 917
[17] Contucci P, Giardinà C and Nishimori H, *Spin glass identities and the Nishimori line*, 2008 arXiv:0805.0754
[18] De Sanctis L and Guerra F, *Mean field dilute ferromagnet I. High temperature and zero temperature behavior*, 2008 *J. Stat. Phys.* **132** 759
[19] Ellis R S, 1985 *Large Deviations and Statistical Mechanics* (New York: Springer)
[20] Erdös P and Renyi A, 1959 *Publ. Math.* **290** 6
[21] Gallo I and Contucci P, *Bipartite mean field spin systems. existence and solution*, 2008 *Math. Phys. Electron. J.* **14**
[22] Gerschenfeld A and Montanari A, *Reconstruction for models on random graphs*, 2007 *Proc. Found. Comput. Sci.* 194
[23] Ghirlanda S and Guerra F, *General properties of overlap distributions in disordered spin systems. Towards Parisi ultrametricity*, 1998 *J. Phys. A: Math. Gen.* **31** 9149
[24] Guerra F, *About the overlap distribution in mean field spin glass models*, 1996 *Int. J. Mod. Phys.* B **10** 1675
[25] Guerra F, *An introduction to mean field spin glass theory: methods and results*, 2005 *Lecture at Les Houches Winter School*
[26] Guerra F and Toninelli F L, *The infinite volume limit in generalized mean field disordered models*, 2003 *Markov Proc. Rel. Fields* **9** 195
[27] Hase M O, de Almeida J R L and Salinas S R, *Replica-symmetric solutions of a dilute Ising ferromagnet in a random field*, 2005 *Eur. Phys. J.* B **47** 245

[28] Newman M, Watts D and Barabasi A L, 2006 *The Structure and Dynamics of Networks* (Princeton, NJ: Princeton University Press)

[29] Newman M E J and Barkema G T, 2001 *Monte Carlo methods in Statistical Physics* (Oxford: Oxford University Press)

[30] Parisi G, *Stochastic stability*, 2000 *Proc. Conf. Disordered and Complex Systems (London)*

[31] Semerjian G and Weigt M, *Approximation schemes for the dynamics of diluted spin models: the Ising ferromagnet on a Bethe lattice*, 2004 *J. Phys. A: Math. Gen.* **37** 5525

[32] Skantzos N, Perez Castillo I and Hatchett J, *Cavity approach for real variables on diluted graphs and application to synchronization in small-world lattices*, 2005 arXiv:cond-mat/0508609

[33] Szalma F and Iglói F, 1999 *J. Stat. Phys.* **95** 763

[34] Watts D J and Strogatz S H, *Collective dynamics of 'small-world' networks*, 1998 *Nature* **393** 409