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Dilution robustness for mean field ferromagnets

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Abstract. In this work we compare two different random dilutions on a mean field ferromagnet: the first model is built on a Bernoulli diluted graph while the second lives on a Poisson diluted one. While it is known that the two models have, in the thermodynamic limit, the same free energy, we investigate the structural constraints that the two models must fulfill. We rigorously derive for each model the set of identities for the multi-overlap distribution, using different methods for the two dilutions: constraints in the former model are obtained by studying the consequences of the self-averaging of the internal energy density, while in the latter they are obtained by a stochastic stability technique. Finally we prove that the identities emerging in the two models are the same, showing *robustness* of the ferromagnetic properties of diluted graphs with respect to the details of dilution.

Keywords: rigorous results in statistical mechanics, cavity and replica method, disordered systems (theory)

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Contents

1. Introduction

In the past decade increasing interest has been shown in statistical mechanics built on diluted graphs (see i.e. [3, 10, 12, 27, 28, 30]). For diluted spin glasses [24, 7] this interest is at least doubly motivated: despite their mean field nature, they share with finite dimensional models the fact that each spin interacts with a finite number of other spins. Moreover they are mathematically equivalent to some random optimization problems (i.e. K-SAT or X-OR-SAT depending on the size of the instantaneous interaction [25, 26]).

Although the cases of simpler ferromagnetic models [4, 5, 19] are not interesting from the point of view of the hard satisfiability interpretation, they are still interesting because of their finite connectivity nature and for testing different variations on the topology of the graph that they are based on.

With this in mind we consider two different ways of diluting the graph [13]: the first ferromagnet has its links distributed according to a Bernoulli probability distribution, the second according to a Poisson one.

For these models we compared the properties of a family of linear constraints for the order parameters (known as Aizenman–Contucci polynomials [2, 6] in the case of spin glasses).

These relations were investigated earlier [2, 21] in the spin glass framework, where they were obtained as a result of the stability of the quenched measure with respect to random perturbation, or equivalently through the bound on the fluctuation of the internal energy. Here we propose them as a test for robustness under dilution.

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The methods used to approach the identities for the two models are structurally different. For the Poisson case, in fact, the additive law of Poissonian random variables makes possible direct exploitation of the stochastic stability property. The same strategy is not applicable for the Bernoulli random variables but, for those, we derive the set of identities from the general bound on the quenched fluctuations (even though, for the sake of completeness, in the appendix we derive the constraints within this general framework for the Poisson case too). The methods that we use are generalizations of those appearing in [8, 11, 16, 17, 23, 29, 15, 20, 6, 1].

Our main result is a rigorous proof of the identities and, especially, the fact that they coincide for the two dilutions.

2. The mean field diluted ferromagnet

We introduce a large number N of sites, labeled by the index $i = 1, \ldots, N$, and associate with each of them an Ising variable $\sigma_i = \pm 1$.

We introduce furthermore two families of discrete independent random variables $\{i_\nu\}$, $\{j_\nu\}$, uniformly distributed on $1, 2, \ldots, N$.

The Hamiltonian $H_N(\sigma)$ of the diluted ferromagnet is expressed through

$$H_N(\sigma) = -\sum_{\nu=1}^{x} \sigma_{i_\nu} \sigma_{j_\nu}, \tag{1}$$

where x depends on the dilution probability distribution.

For the Bernoullian dilution case the variable x is called k, and defined by

$$E_{\rm B}[\cdot] = \sum_{k=0}^{M} \frac{M!}{(M-k)!k!} \left(\frac{\alpha}{N}\right)^k \left(1 - \frac{\alpha}{N}\right)^{M-k} [\cdot], \tag{2}$$

$M = N(N-1)/2$ being the maximum amount of couples $\sigma_i \sigma_j$ existing in the model and α/N the probability that two spins interact.

$\alpha > 0$ plays the role of the connectivity.

The mean and the variance of k are obtained as

$$E_{\rm B}[k] = \frac{M\alpha}{N}, \tag{3}$$

$$E_{\rm B}[k^2] - E_{\rm B}^2[k] = \frac{M\alpha}{N}\left(1 - \frac{\alpha}{N}\right). \tag{4}$$

We will make use in the sequel of following properties of a Bernoulli distribution:

$$E_{\rm B}[kg(k)] = \frac{M\alpha}{N} E_{\rm B}[g(k+1)], \tag{5}$$

$$E_{\rm B}[k^2 g(k)] = \frac{M(M-1)\alpha^2}{N^2} E_{\rm B}[g(k+2)] - \frac{M\alpha}{N} E_{\rm B}[g(k+1)], \tag{6}$$

$$\frac{\rm d}{{\rm d}\alpha} E_{\rm B}[g(k)] = \frac{M}{N} E_{\rm B}[g(k+1) - g(k)]. \tag{7}$$

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For the Poissonian dilution case, we denote x as $\xi_{\alpha N}$, which is a random variable of mean αN, for some $\alpha > 0$ (again defining the connectivity of the model), such that

$$P(\xi_{\alpha N} = k) = \pi(k, \alpha N) = \exp(-\alpha N) \frac{(\alpha N)^k}{k!}, \qquad k = 0, 1, 2, \ldots . \tag{8}$$

Furthermore, we stress that the Poisson distribution obeys the following properties:

$$k\pi(k, \lambda) = \lambda \pi(k-1, \lambda), \tag{9}$$

$$\frac{\mathrm{d}}{\mathrm{d}\lambda} \pi(k, \lambda) = -\pi(k, \lambda) + \pi(k-1, \lambda)(1 - \delta_{k,0}). \tag{10}$$

As for the Bernoulli case, the average with respect to the Poisson measure will be denoted by the index

$$E_{\mathrm{P}} = \sum_{k=0}^{\infty} \frac{\mathrm{e}^{\alpha N} (\alpha N)^k}{k!}. \tag{11}$$

The expectation with respect to all the quenched variables will be denoted by $\mathbf{E}$ and represents the product of the expectation over the dilution distribution and the expectation over the uniformly distributed variables

$$\mathbf{E} = E_{\mathrm{B,P}} \cdot \frac{1}{N^2} \sum_{i,j}^{1,N}.$$

The thermodynamic objects that we deal with are the partition function

$$Z_N(\alpha, \beta) = \sum_{\{\sigma\}} \mathrm{e}^{-\beta H_N(\alpha)}, \tag{12}$$

the quenched intensive free energy

$$A_N(\alpha, \beta) = \frac{1}{N} \mathbf{E} \ln Z_N(\alpha, \beta), \tag{13}$$

the Boltzmann state

$$\omega(g(\sigma)) = \frac{1}{Z_N(\alpha, \beta)} \sum_{\{\sigma_N\}} g(\sigma) \mathrm{e}^{-\beta H_N(\alpha)}, \tag{14}$$

the replicated Boltzmann state

$$\Omega(g(\sigma)) = \prod_s \omega^{(s)}(g(\sigma^{(s)})) \tag{15}$$

and the global average $\langle g(\sigma) \rangle$ defined as

$$\langle g(\sigma) \rangle = \mathbf{E}[\Omega(g(\sigma))]. \tag{16}$$

The functional order parameter of the theory is the infinite family of multi-overlaps, defined as

$$q_{1\cdots n} = \frac{1}{N} \sum_{i=1}^{N} \sigma_i^{(1)} \cdots \sigma_i^{(n)},$$

where particular emphasis is given to the magnetization $m = q_1 = (1/N) \sum_{i=1}^{N} \sigma_i$ and to the two-replica overlap $q_{12} = (1/N) \sum_{i=1}^{N} \sigma_i^1 \sigma_i^2$.

3. Bernoullian diluted case

Identities in the Bernoullian model will be obtained as a consequence of the internal energy self-average [14, 20]; before focusing on this procedure, let us recall:

Definition 1. *A quantity $A(\sigma)$ is called self-averaging if*

$$\lim_{N\to\infty} \left\langle (A(\sigma) - \langle A(\sigma)\rangle)^2 \right\rangle = \lim_{N\to\infty} \left(\langle A^2(\sigma)\rangle - \langle A(\sigma)\rangle^2 \right) = 0, \tag{17}$$

through which we can recall:

Proposition 1. *Given two regular functions $A(\sigma)$ and $B(\sigma)$, if at least one of them is self-averaging, then the following relation holds:*

$$\lim_{N\to\infty} \langle A(\sigma)B(\sigma)\rangle = \lim_{N\to\infty} \langle A(\sigma)\rangle \lim_{N\to\infty} \langle B(\sigma)\rangle. \tag{18}$$

Proof. The proof is straightforward. Let us suppose the self-averaging quantity is $B(\sigma)$ and use $A(\sigma)$ as a trial function. Then we have

$$\begin{aligned} 0 &\le |\langle A(\sigma)B(\sigma)\rangle - \langle A(\sigma)\rangle\langle B(\sigma)\rangle| \\ &= |\langle A(\sigma)B(\sigma) - A(\sigma)\langle B(\sigma)\rangle + \langle A(\sigma)\rangle B(\sigma)\rangle - \langle A(\sigma)\rangle\langle B(\sigma)\rangle| \\ &\le \sqrt{\langle A^2(\sigma)\rangle}\sqrt{\langle (B(\sigma) - \langle B(\sigma)\rangle)^2\rangle}, \end{aligned} \tag{19}$$

where, in the last part, we used the Cauchy–Schwartz relation.

In the thermodynamic limit the proof becomes complete. □

The scheme to follow is then clear: using the above proposition as the underlying backbone in the derivation of the constraints in this section, we must, first, show that the internal energy density of the model self-averages. Then we use as trial functions suitably chosen quantities for the order parameters.

The identities follow on evaluating explicitly both the terms of equation (18): this operation produces several contributions, all involving the order parameters, among which cancelations occur and the remaining part gives the identities.

3.1. Self-averaging of the internal energy density

Once we have defined $h_l = H(\sigma^{(l)})/N$ as the density of the Hamiltonian evaluated on the generic lth replica and

$$\theta = \tanh(\beta), \tag{20}$$

$$\alpha' = M\alpha/N^2 \stackrel{N\to\infty}{\longrightarrow} \alpha/2, \tag{21}$$

for simplicity, we start with the following:

Theorem 1. *In the thermodynamic limit, and in the β-average, the internal energy density self-averages:*

$$\lim_{N\to\infty} \int_{\beta_1}^{\beta_2} \mathbb{E}\left(\Omega(h^2) - \Omega(h)^2\right) \mathrm{d}\beta = 0. \tag{22}$$

This theorem has already been proved in [18] by using essentially the existence of the thermodynamic limit for the free energy density. To make the paper self-contained we provide an alternative proof in the appendix.

We can now introduce the following lemma.

Lemma 1. *Let us consider for simplicity the quantity*

$$\Delta G = \sum_{l=1}^{s} \left[E(\Omega(h_l G) - \Omega(h_l)\Omega(G)) \right]. \tag{23}$$

For every regular, smooth function G, in the β-average, we have

$$\lim_{N\to\infty} \int_{\beta_1}^{\beta_2} |\Delta G| d\beta = 0. \tag{24}$$

Proof.

$$\int_{\beta_1}^{\beta_2} |\Delta G| d\beta \leq \int_{\beta_1}^{\beta_2} \sum_{l=1}^{s} |\mathbf{E}[\Omega(h_l G) - \Omega(h_l)\Omega(G)]| \, \mathrm{d}\beta \tag{25}$$

$$\leq \int_{\beta_1}^{\beta_2} \sum_{l=1}^{s} \sqrt{\mathbf{E}[(\Omega(h_l G) - \Omega(h_l)\Omega(G))^2]} \, \mathrm{d}\beta \tag{26}$$

$$\leq s \int_{\beta_1}^{\beta_2} \sqrt{\mathbf{E}[\Omega(h^2) - \Omega^2(h)]} \, \mathrm{d}\beta \tag{27}$$

$$\leq s\sqrt{\beta_2 - \beta_1} \sqrt{\int_{\beta_1}^{\beta_2} \mathbf{E}[\Omega(h^2) - \Omega^2(h)] \, \mathrm{d}\beta} \stackrel{N\to\infty}{\longrightarrow} 0, \tag{28}$$

where (25) comes from triangle inequality; (26) is obtained via the Jensen inequality applied to the measure $\mathbf{E}[\cdot]$. In the same way (27) comes from Schwarz inequality applied to the measure $\Omega(\cdot)$ (G being bounded), while (28) is obtained via Jensen the inequality applied to the measure $(\beta_2 - \beta_1)^{-1} \int_{\beta_1}^{\beta_2} (\cdot) \, \mathrm{d}\beta$. □

Now we can state the main theorem for the linear constraints.

We are going to introduce a specific trial function that we call $f_G(\alpha, \beta)$.

Theorem 2. *Let us consider the following series of multi-overlap functions G acting, in complete generality, on s replicas:*

$$\begin{aligned} f_G(\alpha, \beta) = \alpha' \Bigg[& \left(\sum_{l=1}^{s} \langle G m_l^2 \rangle - s \langle G m_{s+1}^2 \rangle \right) (1 - \theta^2) \\ & + 2\theta \left(\sum_{a<l}^{1,s} \langle G q_{al}^2 \rangle - s \sum_{l}^{1,s} \langle G q_{l,s+1}^2 \rangle + \frac{s(s+1)}{2} \langle G q_{s+1,s+2}^2 \rangle \right) \\ & + 3\theta^2 \left(\sum_{l<a<b}^{1,s} \langle G q_{l,a,b}^2 \rangle - s \sum_{l<a}^{1,s} \langle G q_{l,a,s+1}^2 \rangle + \frac{s(s+1)}{2} \sum_{l}^{1,s} \langle G q_{l,s+1,s+2}^2 \rangle \right. \\ & \left. - \frac{s(s+1)(s+2)}{3!} \langle G q_{s+1,s+2,s+3}^2 \rangle \right) + \mathrm{O}(\theta^3) \Bigg]. \end{aligned} \tag{29}$$

In the thermodynamic limit the following generator of linear constraints holds:

$$\lim_{N\to\infty}\int_{\beta_1}^{\beta_2} \mathrm{d}\beta |f_G(\alpha,\beta)| = 0. \tag{30}$$

Proof. Let us consider explicitly the quantities encoded in (23). For the sake of clarity all the calculations are reported in the appendix; here we present just the results.

$$\begin{aligned}\mathbf{E}[\Omega(h_l G)] = -\alpha'\Bigg[&\langle Gm_l^2\rangle + \theta\left(\sum_{a=1}^{s}\langle Gq_{a,l}^2\rangle - s\langle Gq_{l,s+1}^2\rangle\right)\\ &+ \theta^2\left(\sum_{a<b}^{1,s}\langle Gq_{l,a,b}^2\rangle - s\sum_{a}^{1,s}\langle Gq_{l,a,s+1}^2\rangle + \frac{s(s+1)}{2}\langle Gq_{l,s+1,s+2}^2\rangle\right) + \mathrm{O}(\theta^2)\Bigg],\end{aligned} \tag{31}$$

$$\begin{aligned}\mathbf{E}[\Omega(h_l)\Omega(G)] = -\alpha'\Bigg[&\langle Gm_l^2\rangle + \theta\left(\sum_{a=1}^{s+1}\langle Gq_{a,l}^2\rangle - (s+1)\langle Gq_{l,s+1}^2\rangle\right)\\ &+ \theta^2\Bigg(\sum_{a}^{1,s}\langle Gm_a^2\rangle - (s+1)\langle Gm_l^2\rangle + \sum_{a<b}^{1,s}\langle Gq_{l,a,b}^2\rangle\\ &- (s+1)\sum_{a}^{1,s}\langle Gq_{l,a,s+1}^2\rangle + \frac{(s+1)(s+2)}{2}\langle Gq_{l,s+1,s+2}^2\rangle\Bigg) + \mathrm{O}(\theta^2)\Bigg].\end{aligned} \tag{32}$$

Subtracting the last equation from the former, immediately we conclude that

$$\Delta G = -f_G(\alpha,\beta), \tag{33}$$

from which theorem thesis follows. □

3.2. Linear constraints for multi-overlaps

We outline here the first-order identities, as it is customary to do in the spin glasses counterpart [6] or for neural networks [9].

Proposition 2. *The first class of multi-overlap constraints is obtained by choosing* $G = m^2$.

In fact, if we set $G = q_1^2 = m^2$, the function $f_G(\alpha,\beta)$ becomes

$$\begin{aligned}f_{m^2}(\alpha,\beta) = \alpha'[&\left(\langle m_1^4\rangle - \langle m_1^2 m_2^2\rangle\right)\left(1-\theta^2\right) + -2\theta\left(\langle m_1^2 q_{12}^2\rangle - \langle m_1^2 q_{23}^2\rangle\right)\\ &+ 3\theta^2\left(\langle m_1^2 q_{123}^2\rangle - \langle m_1^2 q_{234}^2\rangle\right) + \mathrm{O}(\theta^3)],\end{aligned}$$

from which, changing the Jacobian via $\mathrm{d}\theta = (1-\theta^2)\,\mathrm{d}\beta$, we get

$$\begin{aligned}\lim_{N\to\infty}\int_{\beta_1}^{\beta_2} |f_{m^2}(\alpha,\beta)|\,\mathrm{d}\beta = \frac{\alpha}{2}\int_{\theta_1}^{\theta_2}\mathrm{d}\frac{\theta}{(1-\theta^2)}\Big[\,\big|&\left(\langle m_1^4\rangle - \langle m_1^2 m_2^2\rangle\right) - 2\theta\left(\langle m_1^2 q_{12}^2\rangle - \langle m_1^2 q_{23}^2\rangle\right)\\ &+ 3\theta^2\left(\langle m_1^2 q_{123}^2\rangle - \langle m_1^2 q_{234}^2\rangle\right) + \mathrm{O}(\theta^3)\big|\,\Big] = 0,\end{aligned} \tag{34}$$

where the (not interesting) breakdown at $\theta = 1$ of the expression above reflects the lack of convergence of the harmonic series that we used in equation (A.9).

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Proposition 3. *The second class of multi-overlap constraints is obtained by choosing* $G = q_{12}^2$.

In fact, if we set $G = q_{12}^2$, the function $f_G(\alpha, \beta)$ becomes

$$f_{q_{12}^2}(\alpha,\beta) = \alpha'[(2\langle m_1^2 q_{12}^2\rangle - 2\langle m_3^2 q_{12}^2\rangle)(1-\theta^2) + 2\theta(\langle q_{12}^4\rangle - 4\langle q_{12}^2 q_{23}^2\rangle + 3\langle q_{12}^2 q_{34}^2\rangle) \\ - 6\theta^2(\langle q_{12}^2 q_{123}^2\rangle - 3\langle q_{12}^2 q_{234}^2\rangle + 2\langle q_{12}^2 q_{345}^2\rangle) + \mathrm{O}(\theta^3)].$$

Again

$$\lim_{N\to\infty}\int_{\beta_1}^{\beta_2} |f_{q^2}(\alpha,\beta)|\,\mathrm{d}\beta = \frac{\alpha}{2}\int_{\theta_1}^{\theta_2} d\frac{\theta}{(1-\theta^2)}[|(\langle m_1^2 q_{12}^2\rangle - \langle m_3^2 q_{12}^2\rangle) \\ + \theta(\langle q_{12}^4\rangle - 4\langle q_{12}^2 q_{23}^2\rangle + 3\langle q_{12}^2 q_{34}^2\rangle) - 3\theta^2(\langle q_{12}^2 q_{123}^2\rangle - 3\langle q_{12}^2 q_{234}^2\rangle + 2\langle q_{12}^2 q_{345}^2\rangle) \\ + \mathrm{O}(\theta^3)|] = 0, \tag{35}$$

from which the constraints are obtained for the rhs of (34) and (35) set to zero.

4. Poissonian diluted case

To tackle the Poisson diluted ferromagnet, we are going to use the cavity field approach.

The idea behind this technique is that, calling $F(\beta)$ the extensive free energy and $f(\beta)$ the intensive one, a bridge between the two, in the large N limit, is offered simply by the relation

$$(-F_{N+1}(\beta) - F_N(\beta)) = f(\beta) + \mathrm{O}(N^{-1}). \tag{36}$$

As our system has a topologically quenched disorder the $N+1$ spins can be seen as an 'external random cavity field' for the former system of N particles.

The identities derived by tuning that field are called the 'stochastic stability'. The simplest way to find them is to consider monomials which are left invariant by the random field: the derivative with respect to it, being zero, will produce the desired polynomial.

4.1. Cavity field decompositions for the pressure density

To start applying the sketched plan let us decompose (in distribution) a Poissonian random Hamiltonian of $N+1$ spins into two Hamiltonians [1]: the former that of the 'inner' N interacting spins; the latter that of the remaining spin interacting with the internal N spins of the cavity.

Up to negligible corrections that go to zero in the thermodynamic limit, we can write in distribution

$$H_{N+1}(\alpha) = -\sum_{\nu=1}^{P_{\alpha(N+1)}} \sigma_{i_\nu}\sigma_{j_\nu} \sim -\sum_{\nu=1}^{P_{\tilde{\alpha}N}} \sigma_{i_\nu}\sigma_{j_\nu} - \sum_{\nu=1}^{P_{2\tilde{\alpha}}} \sigma_{i_\nu}\sigma_{N+1}, \tag{37}$$

or simply for compactness

$$H_{N+1}(\alpha) \sim H_N(\tilde{\alpha}) + \hat{H}_N(\tilde{\alpha})\sigma_{N+1}, \tag{38}$$

where

$$\tilde{\alpha} = \frac{N}{N+1}\alpha \stackrel{N\to\infty}{\longrightarrow} \alpha, \qquad \hat{H}_N(\tilde{\alpha}) = -\sum_{\nu=1}^{P_{2\tilde{\alpha}}} \sigma_{i_\nu}. \tag{39}$$

It is useful now to introduce an interpolating parameter $t \in [0,1]$ in the term encoding the linear connectivity shift so as to obtain the derivative with respect to the random field by differentiating with respect to this parameter.

Definition 2. *We define the t-dependent Boltzmann state $\tilde{\omega}_t$ as*

$$\tilde{\omega}_t(g(\sigma)) = \frac{1}{Z_{N,t}(\alpha,\beta)} \sum_{\{\sigma\}} g(\sigma) \mathrm{e}^{\beta \sum_{\nu=1}^{P_{\tilde{\alpha}N}} \sigma_{i_\nu}\sigma_{j_\nu} + \beta \sum_{\nu=1}^{P_{2\tilde{\alpha}t}} \sigma_{i_\nu}}. \tag{40}$$

We stress the simplicity with which the t parameter switches between the system of $N+1$ spins and the one built just from the former N, in the large N limit. In fact, the two-body Hamiltonian being left invariant by the gauge symmetry, $\sigma_i \to \epsilon\sigma_i$ for all $i \in (1,\ldots,N)$ with $\epsilon = \pm 1$, on choosing $\epsilon = \sigma_{N+1}$ we have

$$Z_{N,t=1}(\tilde{\alpha},\beta) = Z_{N+1}(\alpha,\beta), \tag{41}$$

$$Z_{N,t=0}(\tilde{\alpha},\beta) = Z_N(\tilde{\alpha},\beta). \tag{42}$$

Note that $Z_{N,t}(\tilde{\alpha},\beta)$ is defined according to (40) and consistently, dealing with the perturbed Boltzmann measure, we introduce an index t also into the global averages $\langle\cdot\rangle \to \langle\cdot\rangle_t$.

4.2. Stochastic stability via cavity fields

We are now ready to attack the problem.

We divide the ensemble of overlap monomials into two large categories: stochastically stable monomials and (as side results) ones that are not stochastically stable. Then we find explicitly the family of stochastically stable monomials, and by putting their derivative with respect to t equal to zero we obtain the identities. To follow the plan let us introduce:

Definition 3. *We define as stochastically stable monomials those multi-overlap monomials where each replica appears an even number of times.*

We are ready to introduce the main theorem, which offers, as a straightforward consequence, a useful corollary, stated immediately after.

Theorem 3. *At $t=1$ the Boltzmann factor of the perturbed measure is comparable with the canonical Boltzmann factor, and, in the thermodynamic limit, we get*

$$\lim_{N\to\infty} \mathbb{E}\tilde{\Omega}_{N,t=1}(\sigma_{i_1}\sigma_{i_2}\cdots\sigma_{i_n}) = \lim_{N\to\infty} \mathbb{E}\Omega_{N+1}(\sigma_{i_1}\sigma_{i_2}\cdots\sigma_{i_n}\sigma_{N+1}^n). \tag{43}$$

Corollary 1. *In the thermodynamic limit, the averages $\langle\cdot\rangle_t$ of the stochastically stable monomials become independent of t in the β-average.*

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Proof. Let us focus on the proof of theorem 3. Corollary 1 will be produced as a straightforward application of theorem 3 to stochastically stable monomials.

Let us start the proof. Let us assume for a generic multi-overlap monomial the following representation:

$$Q = \sum_{i_l^1} \dots \sum_{i_l^s} \prod_{l=1}^{n^a} \sigma^a_{i_l^a} I(\{i_l^a\}),$$

where a labels replicas; the inner product accounts for the spins indicated by the indices l which belong to the Boltzmann state a of the product state Ω for the multi-overlap $q_{a,a'}$ and runs over the integers from 1 to the amount of times the replica a appears in the expression.

The external product multiplies all the terms coming from the internal one. The factor I fixes replica–bond constraints.

For example the monomial $Q = q_{12}q_{23}$ has $s = 3, n^1 = n^3 = 1, n^2 = 2$ and $I = N^{-2}\delta_{i_1^1,i_1^2}\delta_{i_1^2,i_2^3}$, where the δ-functions give the correlations $1,2 \rightarrow q_{1,2}$ and $2,3 \rightarrow q_{2,3}$.

By applying the Boltzmann and quenched disordered expectations we get

$$\langle Q \rangle_t = \mathbf{E} \sum_{i_l^a} I(\{i_l^a\}) \prod_{a=1}^{s} \omega_t \left(\prod_{l=1}^{n^a} \sigma^a_{i_l^a} \right).$$

Let us suppose now that Q is not stochastically stable (for otherwise the proof will be simply ended) and let us decompose it by factorizing the Boltzmann state ω and splitting the terms involving replicas appearing an even number of times from the ones involving replicas appearing an odd number of times. Then, evaluate the whole outcome at $t = 1$:

$$\langle Q \rangle_t = \mathbf{E} \sum_{i_l^a, i_l^b} I(\{i_l^a\}, \{i_l^b\}) \prod_{a=1}^{u} \omega_a \left(\prod_{l=1}^{n^a} \sigma^a_{i_l^a} \right) \prod_{b=u+1}^{s} \omega_b \left(\prod_{l=1}^{n^b} \sigma^b_{i_l^b} \right),$$

where u stands for the amount of replicas which appear an odd number of times inside Q.

In this way we split the measure Ω into two ensembles ω_a and ω_b. Replicas belonging to ω_b are even in number while the ones in ω_a are odd in number.

At this point, as the Hamiltonian has a two-body interaction and consequently is left unchanged by the symmetry $\sigma_i^a \rightarrow \sigma_i^a \sigma_{N+1}^a, \forall i \in (1, N)$ (as $\sigma_{N+1}^2 \equiv 1$), we apply such a symmetry globally to the whole set of N spins. The even measure is left unchanged by this symmetry while the odd one take on a multiplying term σ_{N+1}:

$$\langle Q \rangle = \sum_{i_l^a, i_l^b} I(\{i_l^a\}, \{i_l^b\}) \prod_{a=1}^{u} \omega \left(\sigma^a_{N+1} \prod_{l=1}^{n^a} \sigma^a_{i_l^a} \right) \prod_{b=u+1}^{s} \omega \left(\sigma^b_{N+1} \prod_{l=1}^{n^b} \sigma^b_{i_l^b} \right).$$

The last trick is that, noticing the arbitrariness of the $N+1$ label in σ_{N+1}, we can change it to a generic label k for each $k \neq \{i_l^a\}$ and multiply by $1 = N^{-1}\sum_{k=1}^{N}$. At finite N the theorem is recovered, forgetting terms $O(1/N)$, and becomes exact in the thermodynamic limit. □

It is straightforward to check that the effect of theorem 3 is not felt by stochastically stable multi-overlap monomials (corollary 1) thanks to the dichotomy of the Ising spins ($\sigma_{N+1}^{2n} \equiv 1 \forall n \in \mathbf{N}$).

The last point missing for obtaining the identities is finding a streaming equation for working out the derivatives with respect to the random field of the stochastically stable monomials. For this task we introduce the following:

Proposition 4. *Given F_s as a generic function of the spins of s replicas, the following streaming equation holds:*

$$\begin{aligned}\frac{\partial\langle F_s\rangle_{t,\tilde{\alpha}}}{\partial t} &= 2\tilde{\alpha}\theta\left[\sum_{a=1}^{s}\langle F_s\sigma_{i_0}^a\rangle_{t,\tilde{\alpha}} - s\langle F_s\sigma_{i_0}^{s+1}\rangle_{t,\tilde{\alpha}}\right]\\ &+ 2\tilde{\alpha}\theta^2\left[\sum_{a<b}^{1,s}\langle F_s\sigma_{i_0}^a\sigma_{i_0}^b\rangle_{t,\tilde{\alpha}} - s\sum_{a=1}^{s}\langle F_s\sigma_{i_0}^a\sigma_{i_0}^{s+1}\rangle_{t,\tilde{\alpha}}\right.\\ &+\left.\frac{s(s+1)}{2!}\langle F_s\sigma_{i_0}^{s+1}\sigma_{i_0}^{s+2}\rangle_{t,\tilde{\alpha}}\right]\\ &+ 2\tilde{\alpha}\theta^3\left[\sum_{a<b<c}^{1,s}\langle F_s\sigma_{i_0}^a\sigma_{i_0}^b\sigma_{i_0}^c\rangle_{t,\tilde{\alpha}} - s\sum_{a<b}^{1,s}\langle F_s\sigma_{i_0}^a\sigma_{i_0}^b\sigma_{i_0}^{s+1}\rangle_{t,\tilde{\alpha}}\right.\\ &+ \frac{s(s+1)}{2!}\sum_{a=1}^{s}\langle F_s\sigma_{i_0}^a\sigma_{i_0}^{s+1}\sigma_{i_0}^{s+2}\rangle_{t,\tilde{\alpha}}\\ &+\left.\frac{s(s+1)(s+2)}{3!}\langle F_s\sigma_{i_0}^{s+1}\sigma_{i_0}^{s+2}\sigma_{i_0}^{s+3}\rangle_{t,\tilde{\alpha}}\right] + \mathrm{O}(\theta^3). \end{aligned} \qquad (44)$$

Proof. The proof follows by direct calculations:

$$\begin{aligned}\frac{\partial\langle F_s\rangle_{t,\tilde{\alpha}}}{\partial t} &= \frac{\partial}{\partial t}\mathbf{E}\left[\frac{\sum_{\{\sigma\}}F_s\mathrm{e}^{\sum_{a=1}^{s}(\beta\sum_{\nu=1}^{P_{\tilde{\alpha}N}}\sigma_{i_\nu}^a\sigma_{j_\nu}^a+\beta\sum_{\nu=1}^{P_{2\tilde{\alpha}t}}\sigma_{i_\nu}^a)}}{\sum_{\{\sigma\}}\mathrm{e}^{\sum_{a=1}^{s}(\beta\sum_{\nu=1}^{P_{\tilde{\alpha}N}}\sigma_{i_\nu}^a\sigma_{j_\nu}^a+\beta\sum_{\nu=1}^{P_{2\tilde{\alpha}t}}\sigma_{i_\nu}^a)}}\right]\\ &= 2\tilde{\alpha}\mathbf{E}\left[\frac{\sum_{\{\sigma\}}F_s\mathrm{e}^{\sum_{a=1}^{s}(\beta\sigma_{i_0}^a+\beta\sum_{\nu=1}^{P_{\tilde{\alpha}N}}\sigma_{i_\nu}^a\sigma_{j_\nu}^a+\beta\sum_{\nu=1}^{P_{2\tilde{\alpha}t}}\sigma_{i_\nu}^a)}}{\sum_{\{\sigma\}}\mathrm{e}^{\sum_{a=1}^{s}(\beta\sigma_{i_0}^a+\beta\sum_{\nu=1}^{P_{\tilde{\alpha}N}}\sigma_{i_\nu}^a\sigma_{j_\nu}^a+\beta\sum_{\nu=1}^{P_{2\tilde{\alpha}t}}\sigma_{i_\nu}^a)}}\right] - 2\tilde{\alpha}\langle F_s\rangle_{t,\tilde{\alpha}}\\ &= 2\tilde{\alpha}\mathbf{E}\left[\frac{\tilde{\Omega}_t(F_s\mathrm{e}^{\sum_{a=1}^{s}\beta\sigma_{i_0}^a})}{\tilde{\Omega}_t(\mathrm{e}^{\sum_{a=1}^{s}\beta\sigma_{i_0}^a})}\right] - 2\tilde{\alpha}\langle F_s\rangle_{t,\tilde{\alpha}}\\ &= 2\tilde{\alpha}\mathbf{E}\left[\frac{\tilde{\Omega}_t(F_s\Pi_{a=1}^{s}(\cosh\beta+\sigma_{i_0}^a\sinh\beta))}{\tilde{\Omega}_t(\Pi_{a=1}^{s}(\cosh\beta+\sigma_{i_0}^a\sinh\beta))}\right] - 2\tilde{\alpha}\langle F_s\rangle_{t,\tilde{\alpha}}\\ &= 2\tilde{\alpha}\mathbf{E}\left[\frac{\tilde{\Omega}_t(F_s\Pi_{a=1}^{s}(1+\sigma_{i_0}^a\theta))}{(1+\tilde{\omega}_t(\sigma_{i_0}^a)\theta)^s}\right] - 2\tilde{\alpha}\langle F_s\rangle_{t,\tilde{\alpha}}, \end{aligned} \qquad (45)$$

and then, by noting that

$$\Pi_{a=1}^{s}(1+\sigma_{i_0}^a\theta) = 1+\sum_{a=1}^{s}\sigma_{i_0}^a\theta+\sum_{a<b}^{1,s}\sigma_{i_0}^a\sigma_{i_0}^b\theta^2+\sum_{a<b<c}^{1,s}\sigma_{i_0}^a\sigma_{i_0}^b\sigma_{i_0}^c\theta^3+\cdots,$$

$$\frac{1}{(1+\tilde{\omega}_t\theta)^s} = 1-s\tilde{\omega}_t\theta+\frac{s(s+1)}{2!}\tilde{\omega}_t^2\theta^2-\frac{s(s+1)(s+2)}{3!}\tilde{\omega}_t^3\theta^3+\cdots,$$

we get

$$\frac{\partial\langle F_s\rangle_{t,\tilde\alpha}}{\partial t} = 2\tilde\alpha\mathbf{E}\Bigg[\tilde\Omega_t\left(F_s\left(1+\sum_{a=1}^{s}\sigma_{i_0}^a\theta+\sum_{a<b}^{1,s}\sigma_{i_0}^a\sigma_{i_0}^b\theta^2+\sum_{a<b<c}^{1,s}\sigma_{i_0}^a\sigma_{i_0}^b\sigma_{i_0}^c\theta^3+\cdots\right)\right)$$
$$\times\left(1-s\tilde\omega_t\theta+\frac{s(s+1)}{2!}\tilde\omega_t^2\theta^2-\frac{s(s+1)(s+2)}{3!}\tilde\omega_t^3\theta^3+\cdots\right)\Bigg]-2\tilde\alpha\langle F_s\rangle_{t,\tilde\alpha},$$

from which the theorem follows. □

4.3. Linear constraints for multi-overlaps

We saw that the stochastically stable multi-overlap monomials become asymptotically independent of the t parameter upon increasing the size of the system. Calling, for simplicity, $G_N(q)$ a stochastically stable multi-overlap monomial, identities follow as a consequence of corollary 1 and are encoded in the following relation:

$$\lim_{N\to\infty}\partial_t\langle G_N(q)\rangle_t=0.$$

As we did when we investigated the Bernoullian model, we analyze the stability of $\langle m^2\rangle$ and $\langle q_{12}^2\rangle$, up to the third order in θ, so as to compare the results at the end.

$$\partial_t\langle m_1^2\rangle_t = 2\tilde\alpha\theta\left(\langle m_1^3\rangle_t-\langle m_1^2m_2\rangle_t\right)-2\tilde\alpha\theta^2\left(\langle m_1^2q_{12}\rangle_t-\langle m_1^2q_{23}\rangle_t\right)$$
$$+2\tilde\alpha\theta^3\left(\langle m_1^2q_{123}\rangle_t-\langle m_1^2q_{234}\rangle_t\right)+\mathrm{O}(\theta^3)$$
$$\Rightarrow\ [2\alpha\theta(\langle m_1^4\rangle-\langle m_1^2m_2^2\rangle)-2\alpha\theta^2(\langle m_1^2q_{12}^2\rangle-\langle m_1^2q_{23}^2\rangle)$$
$$+2\alpha\theta^3(\langle m_1^2q_{123}^2\rangle-\langle m_1^2q_{234}^2\rangle)+\mathrm{O}(\theta^3)]=0, \tag{46}$$

$$\partial_t\langle q_{12}^2\rangle_t = 4\tilde\alpha\theta\left(\langle m_1q_{12}^2\rangle_t-\langle m_3q_{12}^2\rangle_t\right)+2\tilde\alpha\theta^2\left(\langle q_{12}^3\rangle_t-4\langle q_{12}^2q_{13}\rangle_t+3\langle q_{12}^2q_{34}\rangle_t\right)$$
$$-4\tilde\alpha\theta^3\left(\langle q_{12}^2q_{123}\rangle_t-3\langle q_{12}^2q_{134}\rangle_t+4\langle q_{12}^2q_{345}\rangle_t\right)+\mathrm{O}(\theta^3)$$
$$\Rightarrow\ [4\alpha\theta\left(\langle m_1^2q_{12}^2\rangle-\langle m_3^2q_{12}^2\rangle\right)+2\alpha\theta^2\left(\langle q_{12}^4\rangle-4\langle q_{12}^2q_{13}^2\rangle+3\langle q_{12}^2q_{34}^2\rangle\right)$$
$$-4\alpha\theta^3\left(\langle q_{12}^2q_{123}^2\rangle-3\langle q_{12}^2q_{134}^2\rangle+2\langle q_{12}^2q_{345}^2\rangle\right)+\mathrm{O}(\theta^3)]=0. \tag{47}$$

5. Discussion and outlook

Let us start this section by comparing the results that we get from the two models.

We have to compare equations (34) versus (46) and (35) versus (47).

We see that the details of the dilution do not affect the constraints: the series show the same set of identities.

In fact, despite the fact that we are not generally allowed to set to zero each term in the expressions (46), (47), (34), (35) (as we do to obtain the identities (48)–(53) alone), at least close to the critical line, where different multi-overlaps have different scaling laws [1], i.e. $q_n^2\propto(\alpha\theta-1)^n$, such a spreading is possible and we can forget each single coefficient of (α,β) as it does not affect the identities (it is never involved in the averages $\langle\cdot\rangle$).

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We get

$$0 = \langle m_1^4 \rangle - \langle m_1^2 m_2^2 \rangle, \tag{48}$$

$$0 = \langle m_1^2 q_{12}^2 \rangle - \langle m_1^2 q_{23}^2 \rangle, \tag{49}$$

$$0 = \langle m_1^2 q_{123}^2 \rangle - \langle m_1^2 q_{234}^2 \rangle, \tag{50}$$

when investigating the magnetization as a trial function and

$$0 = \langle m_1^2 q_{12}^2 \rangle - \langle m_3^2 q_{12}^2 \rangle, \tag{51}$$

$$0 = \langle q_{12}^4 \rangle - 4\langle q_{12}^2 q_{23}^2 \rangle + 3\langle q_{12}^2 q_{34}^2 \rangle, \tag{52}$$

$$0 = \langle q_{12}^2 q_{123}^2 \rangle - 3\langle q_{12}^2 q_{134}^2 \rangle + 2\langle q_{12}^2 q_{345}^2 \rangle, \tag{53}$$

when investigating the two-replica overlap.

Although it is a minor point, we stress that the differences among the global coefficients of (α, β) (not shown here) are clearly related to the differences of the two methods involved (in the former the constraints appear in the integration over the temperature, while in the latter this β-average is already worked out); furthermore the limiting connectivity in the Bernoulli dilution is $\alpha/2$, while it is 2α in the Poisson model; so there is an overall factor 4 difference among the results (for the sake of clarity we worked out in the appendix also the constraints in the Poisson diluted case via the first method, to check explicitly the consistency of the two model coefficients).

Then, by looking explicitly at the constraints, several physical features can be recognized; in fact every term is well known.

The first class (equations (48)–(50)) is the standard first momentum self-averaging on replica symmetric systems. In fact, on assuming replica equivalence, equation (48) turns out to be the standard internal energy self-averaging of the Curie–Weiss model. Equations (49) and (50) contribute as higher order internal energy self-averages on taking into account the dilution (in fact, they go to zero whenever $\alpha \to \infty$ because powers of θ higher than 1 go to zero and only the Curie–Weiss self-averaging for the internal energy survives, as it should).

With a glance at the identities coming from the second constraint series (equations (51)–(53)) we recognize immediately the replica symmetry ansatz for the magnetization in the first identity, followed by the first and the second Aizenman–Contucci relations for systems with quenched disorder [2, 4].

Interestingly these series are in agreement even with other models, apparently quite different, such as spin glasses with Gaussian coupling $\mathcal{N}[1, 1]$ instead of $\mathcal{N}[0, 1]$ [15]. A very interesting conjecture may be that these constraints hold for systems whose interaction has on average positive strength and are affected by quenched disorder, independently of whether the disorder affects the strength of the interaction or the topology of the interaction. We plan to report soon on this topic.

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Appendix

A.1. Alternative proof of energy self-averaging

Starting from the thermodynamic relation

$$\mathbf{E}[\Omega(h^2)-\Omega^2(h)] = -\frac{1}{N}\frac{\mathrm{d}}{\mathrm{d}\beta}\mathbf{E}[\Omega(h)], \tag{A.1}$$

we evaluate explicitly the term $E[\Omega(h)]$ as

$$\mathbf{E}[\Omega(h)] = -\frac{1}{N}\mathbf{E}\left[\frac{\sum_{\{\sigma\}}\sum_{\nu=1}^{k}\sigma_{i_\nu}\sigma_{j_\nu}\mathrm{e}^{-\beta H}}{Z_N(\alpha,\beta)}\right] \tag{A.2}$$

$$= -\frac{1}{N}\mathbf{E}\left[\frac{\sum_{\{\sigma\}}k\sigma_{i_0}\sigma_{j_0}\mathrm{e}^{-\beta H}}{Z_N(\alpha,\beta)}\right] \tag{A.3}$$

$$= -\alpha'\mathbf{E}\left[\frac{\sum_{\{\sigma\}}\sigma_{i_0}\sigma_{j_0}\mathrm{e}^{\beta\sigma_{i_0}\sigma_{j_0}}\mathrm{e}^{-\beta H}}{\sum_{\{\sigma\}}\mathrm{e}^{\beta\sigma_{k_0}\sigma_{l_0}}\mathrm{e}^{-\beta H}}\right] \tag{A.4}$$

$$= -\alpha'\mathbf{E}\left[\frac{\sum_{\{\sigma\}}\sigma_{i_0}\sigma_{j_0}(\cosh\beta+\sigma_{i_0}\sigma_{j_0}\sinh\beta)\mathrm{e}^{-\beta H}}{\sum_{\{\sigma\}}(\cosh\beta+\sigma_{k_0}\sigma_{l_0}\sinh\beta)\mathrm{e}^{-\beta H}}\right] \tag{A.5}$$

$$= -\alpha'\mathbf{E}\left[\frac{\sum_{\{\sigma\}}\sigma_{i_0}\sigma_{j_0}(1+\sigma_{i_0}\sigma_{j_0}\theta)\mathrm{e}^{-\beta H}}{\sum_{\{\sigma\}}(1+\sigma_{k_0}\sigma_{l_0}\theta)\mathrm{e}^{-\beta H}}\right] \tag{A.6}$$

$$= -\alpha'\mathbf{E}\left[\frac{\omega(\sigma_{i_0}\sigma_{j_0})+\theta}{1+\omega(\sigma_{k_0}\sigma_{l_0})\theta}\right], \tag{A.7}$$

where in (A.3) we fixed the index ν, in (A.4) we used the property (5) of the Bernoulli distribution and we introduced two further families of random variables: $\{k_\nu\}$, $\{l_\nu\}$, and in (A.5) we used $\mathrm{e}^{\beta\sigma_{i_0}\sigma_{j_0}} = \cosh\beta+\sigma_{i_0}\sigma_{j_0}\sinh\beta$.

Let us now expand the denominator of (A.7), keeping in mind the relation

$$\frac{1}{(1+\tilde{\omega}_t\theta)^p} = 1 - p\tilde{\omega}_t\theta + \frac{p(p+1)}{2!}\tilde{\omega}_t^2\theta^2 - \frac{p(p+1)(p+2)}{3!}\tilde{\omega}_t^3\theta^3 + \cdots,$$

such that, by imposing $p=1$, we obtain

$$\mathbf{E}[\Omega(h)] = -\alpha'\mathbf{E}\left[\theta + \sum_{n=1}^{\infty}(-1)^n\theta^n(1-\theta^2)\langle q_{1\cdots n}^2\rangle\right]. \tag{A.8}$$

By applying the modulus function to the equation above we can proceed further with the following bound:

$$|\mathbf{E}[\Omega(h)]| \le \alpha'\mathbf{E}\left[|\theta| + \sum_{n=1}^{\infty}|\theta^n(1-\theta^2)\langle q_{1\cdots n}^2\rangle|\right]. \tag{A.9}$$

Both $|\theta|$ and $|\langle q^2_{1\cdots n}\rangle|$ belong to $[0,1]$ so we get

$$|E[\Omega(h)]| \leq \alpha' \left[1 + (1-\theta^2)\sum_{n=1}^{\infty}\theta^n\right], \tag{A.10}$$

whose harmonic series converges to $1/(1-\theta)$, $|\theta|<1$.

The fact that the convergence is not guaranteed at zero temperature with this technique is not a problem because, first, the identities that we are looking for hold in the β-average, and, secondly, the zero-temperature case has been intensively investigated elsewhere [22].

For each finite β, then, we can write

$$|\mathbf{E}[\Omega(h)]| \leq \alpha' \left[1 + \frac{(1-\theta^2)}{1-\theta}\right] \tag{A.11}$$

$$= \alpha' \left[1 + \frac{(1-\theta)(1+\theta)}{1-\theta}\right] \tag{A.12}$$

$$= \alpha' \left[1 + (1+\theta)\right] \tag{A.13}$$

$$\leq 3\alpha', \tag{A.14}$$

and consequently

$$\int_{\beta_1}^{\beta_2} \mathbf{E}[\Omega(h^2) - \Omega^2(h)]\, \mathrm{d}\beta \leq \int_{\beta_1}^{\beta_2} |\mathbf{E}[\Omega(h^2) - \Omega^2(h)]|\, \mathrm{d}\beta$$

$$= \frac{1}{N}\int_{\beta_1}^{\beta_2} \left|\frac{\mathrm{d}}{\mathrm{d}\beta}\mathbf{E}[\Omega(h)]\right| \mathrm{d}\beta$$

$$\leq 3\frac{\alpha'}{N} \tag{A.15}$$

$$\Rightarrow \lim_{N\to\infty}\int_{\beta_1}^{\beta_2} \mathbf{E}[\Omega(h^2) - \Omega^2(h)]\, \mathrm{d}\beta = 0, \tag{A.16}$$

and the proof is closed. □

A.2. Details in Bernoulli dilution calculations

Here some technical calculations concerning the self-averaging technique applied to the Bernoulli diluted graph are reported.

$$\mathbf{E}[\Omega(h_l G)] = -\frac{1}{N}\mathbf{E}\left[\frac{\sum_{\{\sigma\}}\sum_{\nu=1}^{k}\sigma^l_{i_\nu}\sigma^l_{j_\nu} G \mathrm{e}^{(\beta\sum_{a=1}^{s}\sum_{\nu=1}^{k}\sigma^a_{i_\nu}\sigma^a_{j_\nu})}}{\sum_{\{\sigma\}}\mathrm{e}^{(\beta\sum_{a=1}^{s}\sum_{\nu=1}^{k}\sigma^a_{i_\nu}\sigma^a_{j_\nu})}}\right]$$

$$= -\frac{1}{N}\mathbf{E}\left[\frac{\sum_{\{\sigma\}} k\sigma^l_{i_0}\sigma^l_{j_0} G \mathrm{e}^{(\beta\sum_{a=1}^{s}\sum_{\nu=1}^{k}\sigma^a_{i_\nu}\sigma^a_{j_\nu})}}{\sum_{\{\sigma\}}\mathrm{e}^{(\beta\sum_{a=1}^{s}\sum_{\nu=1}^{k}\sigma^a_{i_\nu}\sigma^a_{j_\nu})}}\right],$$

and recalling the properties of the Bernoullian distribution (5) we can write

$$
\begin{aligned}
&= -\frac{\alpha M}{N^2}\mathbf{E}\left[\frac{\sum_{\{\sigma\}}\sigma^l_{i_0}\sigma^l_{j_0}G\prod_{a=1}^s \mathrm{e}^{\beta\sigma^a_{i_0}\sigma^a_{j_0}}\mathrm{e}^{-\beta H_s}}{\sum_{\{\sigma\}}\prod_{a=1}^s \mathrm{e}^{\beta\sigma^a_{i_0}\sigma^a_{j_0}}\mathrm{e}^{-\beta H_s}}\right]\\
&= -\frac{\alpha M}{N^2}\mathbf{E}\left[\frac{\sum_{\{\sigma\}}\sigma^l_{i_0}\sigma^l_{j_0}G\prod_{a=1}^s [\cosh\beta+\sigma^a_{i_0}\sigma^a_{j_0}\sinh\beta]\mathrm{e}^{-\beta H_s}}{\sum_{\{\sigma\}}\prod_{a=1}^s [\cosh\beta+\sigma^a_{i_0}\sigma^a_{j_0}\sinh\beta]\mathrm{e}^{-\beta H_s}}\right]\\
&= -\frac{\alpha M}{N^2}\mathbf{E}\left[\frac{\sum_{\{\sigma\}}\sigma^l_{i_0}\sigma^l_{j_0}G\prod_{a=1}^s [1+\sigma^a_{i_0}\sigma^a_{j_0}\theta]\mathrm{e}^{-\beta H_s}}{\sum_{\{\sigma\}}\prod_{a=1}^s [1+\sigma^a_{i_0}\sigma^a_{j_0}\theta]\mathrm{e}^{-\beta H_s}}\right]\\
&= -\frac{\alpha M}{N^2}\mathbf{E}\left[\frac{\Omega\left(\sigma^l_{i_0}\sigma^l_{j_0}G\prod_{a=1}^s [1+\sigma^a_{i_0}\sigma^a_{j_0}\theta]\right)}{\left(1+\omega(\sigma^a_{i_0}\sigma^a_{j_0})\theta\right)^s}\right].
\end{aligned}
$$

Let us expand both the numerator and the denominator up to the second order in θ:

$$
\begin{aligned}
&= -\frac{\alpha M}{N^2}\mathbf{E}\Bigg[\Omega\bigg((\sigma^l_{i_0}\sigma^l_{j_0}G)(1+\sum_a^{1,s}\sigma^a_{i_0}\sigma^a_{j_0}\theta+\sum_{a<b}^{1,s}\sigma^a_{i_0}\sigma^a_{j_0}\sigma^b_{i_0}\sigma^b_{j_0}\theta^2)\bigg)\\
&\quad\times\left(1-s\omega(\sigma_{i_0}\sigma_{j_0})\theta+\frac{s(s+1)}{2}\omega^2(\sigma_{i_0}\sigma_{j_0})\theta^2\right)\Bigg]\\
&= -\frac{\alpha M}{N^2}\mathbf{E}\Bigg[\Omega\bigg(G\sigma^l_{i_0}\sigma^l_{j_0}+G\sum_a^{1,s}\sigma^a_{i_0}\sigma^a_{j_0}\sigma^l_{i_0}\sigma^l_{j_0}\theta+G\sum_{a<b}^{1,s}\sigma^a_{i_0}\sigma^a_{j_0}\sigma^b_{i_0}\sigma^b_{j_0}\sigma^l_{i_0}\sigma^l_{j_0}\theta^2\bigg)\\
&\quad\times\left(1-s\omega(\sigma^a_{i_0}\sigma^a_{j_0})\theta+\frac{s(s+1)}{2}\omega^2(\sigma_{i_0}\sigma_{j_0})\theta^2\right)\Bigg]\\
&= -\frac{\alpha M}{N^2}\mathbf{E}\Bigg[\Omega(G\sigma^l_{i_0}\sigma^l_{j_0})+\theta\bigg(\sum_a^{1,s}\Omega(G\sigma^a_{i_0}\sigma^a_{j_0}\sigma^l_{i_0}\sigma^l_{j_0})-s\Omega(G\sigma^l_{i_0}\sigma^l_{j_0})\omega(\sigma_{i_0}\sigma_{j_0})\bigg)\\
&\quad+\theta^2\bigg(\sum_{a<b}^{1,s}\Omega(G\sigma^a_{i_0}\sigma^a_{j_0}\sigma^b_{i_0}\sigma^b_{j_0}\sigma^l_{i_0}\sigma^l_{j_0})-s\sum_a^{1,s}\Omega(G\sigma^a_{i_0}\sigma^a_{j_0}\sigma^l_{i_0}\sigma^l_{j_0})\omega(\sigma_{i_0}\sigma_{j_0})\\
&\quad+\frac{s(s+1)}{2}\Omega(G\sigma^l_{i_0}\sigma^l_{j_0})\omega^2(\sigma^a_{i_0}\sigma^a_{j_0})\bigg)\Bigg]\\
&= -\frac{\alpha M}{N^2}\Bigg[\langle Gm_l^2\rangle+\theta\bigg(\sum_{a=1}^{s}\langle Gq^2_{a,l}\rangle-s\langle Gq^2_{l,s+1}\rangle\bigg)\\
&\quad+\theta^2\bigg(\sum_{a<b}^{1,s}\langle Gq^2_{l,a,b}\rangle-s\sum_a^{1,s}\langle Gq^2_{l,a,s+1}\rangle+\frac{s(s+1)}{2}\langle Gq^2_{l,s+1,s+2}\rangle\bigg)\Bigg].
\end{aligned}
$$

The other term $\mathbf{E}[\Omega(h_l)\Omega(G)]$ can be worked out as follows:

$\mathbf{E}[\Omega(h_l)\Omega(G)]$

$$
= -\frac{1}{N}\mathbf{E}\left[\frac{\sum_{\{\sigma\}}\sum_{\nu=1}^k\sigma^l_{i_\nu}\sigma^l_{j_\nu}G\mathrm{e}^{\beta\sum_{\nu=1}^k\sigma^l_{i_\nu}\sigma^l_{j_\nu}}\mathrm{e}^{(\beta\sum_{a=1}^s\sum_{\nu=1}^k\sigma^a_{i_\nu}\sigma^a_{j_\nu})}}{\sum_{\{\sigma\}}\mathrm{e}^{\beta\sum_{\nu=1}^k\sigma^l_{i_\nu}\sigma^l_{j_\nu}}\mathrm{e}^{(\beta\sum_{a=1}^s\sum_{\nu=1}^k\sigma^a_{i_\nu}\sigma^a_{j_\nu})}}\right]
$$

$$= -\frac{1}{N}\mathbf{E}\left[\frac{\sum_{\{\sigma\}} k\sigma_{i_0}^l\sigma_{j_0}^l G e^{\beta\sum_{\nu=1}^k \sigma_{i\nu}^l\sigma_{j\nu}^l} e^{(\beta\sum_{a=1}^s\sum_{\nu=1}^k \sigma_{i\nu}^a\sigma_{j\nu}^a)}}{\sum_{\{\sigma\}} e^{\beta\sum_{\nu=1}^k \sigma_{i\nu}^l\sigma_{j\nu}^l} e^{(\beta\sum_{a=1}^s\sum_{\nu=1}^k \sigma_{i\nu}^a\sigma_{j\nu}^a)}}\right]$$

$$= -\frac{\alpha M}{N^2}\mathbf{E}\left[\frac{\sum_{\{\sigma\}} \sigma_{i_0}^l\sigma_{j_0}^l G e^{\beta\sigma_{i_0}^l\sigma_{j_0}^l}[\prod_{a=1}^s e^{\beta\sigma_{i_0}^a\sigma_{j_0}^a}] e^{-\beta H_{s+1}}}{\sum_{\{\sigma\}} e^{\beta\sigma_{i_0}^l\sigma_{j_0}^l}[\prod_{a=1}^s e^{\beta\sigma_{i_0}^a\sigma_{j_0}^a}] e^{-\beta H_{s+1}}}\right]$$

$$= -\frac{\alpha M}{N^2}\mathbf{E}\left[\frac{\Omega(\sigma_{i_0}^l\sigma_{j_0}^l G(1+\sigma_{i_0}^l\sigma_{j_0}^l\theta)[\prod_{a=1}^s(1+\sigma_{i_0}^a\sigma_{j_0}^a\theta)])}{(1+\omega(\sigma_{i_0}\sigma_{j_0})\theta)^{s+1}}\right]$$

$$= -\frac{\alpha M}{N^2}\mathbf{E}\left[\Omega\left((\sigma_{i_0}^l\sigma_{j_0}^l G + G\theta)\left(1+\sum_a^{1,s}\sigma_{i_0}^a\sigma_{j_0}^a\theta + \sum_{a<b}^{1,s}\sigma_{i_0}^a\sigma_{j_0}^a\sigma_{i_0}^b\sigma_{j_0}^b\theta^2\right)\right)\right.$$

$$\left.\times\left(1-(s+1)\omega(\sigma_{i_0}\sigma_{j_0})\theta + \frac{(s+1)(s+2)}{2}\omega^2(\sigma_{i_0}\sigma_{j_0})\theta^2\right)\right]$$

$$= -\frac{\alpha M}{N^2}\mathbf{E}\left[\Omega(G\sigma_{i_0}^l\sigma_{j_0}^l) + \theta\left(\Omega(G) + \sum_a^{1,s}\Omega(G\sigma_{i_0}^a\sigma_{j_0}^a\sigma_{i_0}^l\sigma_{j_0}^l)\right)\right.$$

$$+\theta^2\left(\sum_a^{1,s}\Omega(G\sigma_{i_0}^a\sigma_{j_0}^a) + \sum_{a<b}^{1,s}\Omega\left(G\sigma_{i_0}^a\sigma_{j_0}^a\sigma_{i_0}^b\sigma_{j_0}^b\sigma_{i_0}^l\sigma_{j_0}^l\right)\right)$$

$$\left.\times\left(1-(s+1)\omega(\sigma_{i_0}\sigma_{j_0})\theta + \frac{(s+1)(s+2)}{2}\omega^2(\sigma_{i_0}\sigma_{j_0})\theta^2\right)\right]$$

$$= -\frac{\alpha M}{N^2}\mathbf{E}\left[\Omega(G\sigma_{i_0}^l\sigma_{j_0}^l) + \theta\left(\Omega(G) + \sum_a^{1,s}\Omega(G\sigma_{i_0}^a\sigma_{j_0}^a\sigma_{i_0}^l\sigma_{j_0}^l)\right.\right.$$

$$\left.-(s+1)\Omega(G\sigma_{i_0}^l\sigma_{j_0}^l)\omega(\sigma_{i_0}\sigma_{j_0})\right)$$

$$+\theta^2\left(\sum_a^{1,s}\Omega(G\sigma_{i_0}^a\sigma_{j_0}^a) + \sum_{a<b}^{1,s}\Omega(G\sigma_{i_0}^a\sigma_{j_0}^a\sigma_{i_0}^b\sigma_{j_0}^b\sigma_{i_0}^l\sigma_{j_0}^l)\right.$$

$$-(s+1)\Omega(G)\omega(\sigma_{i_0}\sigma_{j_0})$$

$$-(s+1)\sum_a^{1,s}\Omega(G\sigma_{i_0}^a\sigma_{j_0}^a\sigma_{i_0}^l\sigma_{j_0}^l)\omega(\sigma_{i_0}\sigma_{j_0})$$

$$\left.\left.+\frac{(s+1)(s+2)}{2}\Omega(G\sigma_{i_0}^l\sigma_{j_0}^l)\omega^2(\sigma_{i_0}^a\sigma_{j_0}^a)\right)\right]$$

$$= -\frac{\alpha M}{N^2}\left[\langle Gm_l^2\rangle + \theta\left(\langle G\rangle + \sum_{a=1}^s\langle Gq_{a,l}^2\rangle - (s+1)\langle Gq_{l,s+1}^2\rangle\right)\right.$$

$$+ \theta^2\left(\sum_a^{1,s}\langle Gm_a^2\rangle - (s+1)\langle Gm_l^2\rangle + \sum_{a<b}^{1,s}\langle Gq_{l,a,b}^2\rangle - (s+1)\sum_a^{1,s}\langle Gq_{l,a,s+1}^2\rangle + \frac{(s+1)(s+2)}{2}\langle Gq_{l,s+1,s+2}^2\rangle\right)\Bigg]. \quad (A.17)$$

A.3. Poisson identities via the self-averaging technique

For the sake of completeness we report also the constraints in the Poisson diluted model obtained by using the first method:

$$f_G(\alpha,\beta) = \alpha\Bigg[\left(\sum_{l=1}^{s}\langle Gm_l^2\rangle - s\langle Gm_{s+1}^2\rangle\right)(1-\theta^2) + 2\theta\left(\sum_{a<l}^{1,s}\langle Gq_{al}^2\rangle - s\sum_l^{1,s}\langle Gq_{l,s+1}^2\rangle + \frac{s(s+1)}{2}\langle Gq_{s+1,s+2}^2\rangle\right) + 3\theta^2\Bigg(\sum_{l<a<b}^{1,s}\langle Gq_{l,a,b}^2\rangle - s\sum_{l<a}^{1,s}\langle Gq_{l,a,s+1}^2\rangle + \frac{s(s+1)}{2}\sum_l^{1,s}\langle Gq_{l,s+1,s+2}^2\rangle - \frac{s(s+1)(s+2)}{3!}\langle Gq_{s+1,s+2,s+3}^2\rangle\Bigg) + \mathrm{O}(\theta^3)\Bigg], \quad (A.18)$$

from which, choosing as a trial function m^2, we have

$$f_{m^2}^P(\alpha,\beta) = \alpha[\left(\langle m_1^4\rangle - \langle m_1^2m_2^2\rangle\right)(1-\theta^2) - 2\theta\left(\langle m_1^2q_{12}^2\rangle - \langle m_1^2q_{23}^2\rangle\right) + 3\theta^2\left(\langle m_1^2q_{123}^2\rangle - \langle m_1^2q_{234}^2\rangle\right) + \mathrm{O}(\theta^3)],$$

from which, changing the Jacobian using $\mathrm{d}\theta = (1-\theta^2)\,\mathrm{d}\beta$, we get

$$\lim_{N\to\infty}\int_{\beta_1}^{\beta_2}|f_{m^2}^P(\alpha,\beta)|\,\mathrm{d}\beta = \alpha\int_{\theta_1}^{\theta_2}\mathrm{d}\theta\Big[|\left(\langle m_1^4\rangle - \langle m_1^2m_2^2\rangle\right) - 2\frac{\theta}{(1-\theta^2)}\left(\langle m_1^2q_{12}^2\rangle - \langle m_1^2q_{23}^2\rangle\right) + 3\frac{\theta^2}{(1-\theta^2)}\left(\langle m_1^2q_{123}^2\rangle - \langle m_1^2q_{234}^2\rangle\right) + \mathrm{O}(\theta^3)|\Big] = 0. \quad (A.19)$$

If we set $G = q_{12}^2$ as the trial function, $f_G^P(\alpha,\beta)$ becomes

$$f_{q^2}^P(\alpha,\beta) = \alpha[\left(2\langle m_1^2q_{12}^2\rangle - 2\langle m_3^2q_{12}^2\rangle\right)(1-\theta^2) + 2\theta\left(\langle q_{12}^4\rangle - 4\langle q_{12}^2q_{23}^2\rangle + 3\langle q_{12}^2q_{34}^2\rangle\right) - 6\theta^2\left(\langle q_{12}^2q_{123}^2\rangle - 3\langle q_{12}^2q_{234}^2\rangle + 2\langle q_{12}^2q_{345}^2\rangle\right) + \mathrm{O}(\theta^3)].$$

Again,

$$\lim_{N\to\infty}\int_{\beta_1}^{\beta_2}|f_{q^2}^P(\alpha,\beta)|\,\mathrm{d}\beta = 2\alpha\int_{\theta_1}^{\theta_2}\mathrm{d}\theta\Big[|\left(\langle m_1^2q_{12}^2\rangle - \langle m_3^2q_{12}^2\rangle\right) + \frac{\theta}{(1-\theta^2)}\left(\langle q_{12}^4\rangle - 4\langle q_{12}^2q_{23}^2\rangle + 3\langle q_{12}^2q_{34}^2\rangle\right)$$

$$- \; 3\frac{\theta^2}{(1-\theta^2)}\left(\langle q_{12}^2 q_{123}^2\rangle - 3\langle q_{12}^2 q_{234}^2\rangle + 2\langle q_{12}^2 q_{345}^2\rangle\right)$$
$$+ \; \mathrm{O}(\theta^3)|\Big] = 0. \qquad (A.20)$$

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