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DRIVEN TRANSITIONS AT THE ONSET OF ERGODICITY BREAKING IN GAUGE-INVARIANT COMPLEX NETWORKS

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In the last few years, the statistical mechanics of spin glasses has become one of the major frameworks for analyzing the macroscopical equilibrium properties of complex systems starting from the microscopical dynamics of their components. Recently, many advances in its rigorous formulation without the replica trick have been achieved, highlighting the importance of this field of research in our understanding of complex systems. In this framework we analyze the critical behavior of a Poissonian diluted network with random competitive interactions among gauge-invariant dichotomic variables pasted on the nodes (i.e., a suitable version of the Viana–Bray diluted spin glass). The model is described by an infinite series of order parameters (the multioverlaps) and has two degrees of freedom: the temperature (which can be thought of as the noise level) and the connectivity (the averaged number of links per node in the underlying network).

In this paper, we show that there are not several transition lines, one for every order parameter, as a naive approach would suggest but just one corresponding to ergodicity breaking. We explain this scenario within a novel and simple mathematical technique: we show the existence of a driving mechanism such that, as the first order parameter (the two-replica overlap) becomes different from zero due to a real second order phase transition, it enforces all the other multioverlaps toward positive values thanks to the strong correlations which develop among themselves and the two-replica overlap at the critical line. These correlations are ultimately related — within our framework — to the breaking of the gauge invariance of the Boltzmann state at the boundary of the ergodic region. A discussion on the structure of the free energy, fundamental macroscopical observable by which the whole thermodynamic can be achieved, is also presented.

Keywords: Disordered systems; spin glasses; dilute networks.

1. Introduction

Among several different complex systems[1] and a large number of tools for their investigation,[2] the statistical mechanics of disordered systems has earned ever-increasing weight in the last two decades.[3–8]

In this paper, the *complex networks* we analyze by statistical mechanics can be understood as follows: they are networks because we allow the variables to live on the node of a nontrivial graph (a Poissonian Erdos–Renyi graph[9–11]), the links among the nodes being the interacting fields they exchange, and they are complex because, as opposed for example to the Ising model[12] (in which all the variables share the same coupling constants[13]) here the variables interact with equal probability with a positive coupling or a negative one, giving rise to frustration[14] and forming what in the language of physics is called a *spin glass*.[15,16]

We stress that our treatment is of great generality, since it applies to several mean field models with the only requirement of the global gauge invariance of the Hamiltonian describing the systems.

In the last few years, while an ever-increasing understanding of disordered statistical mechanics has been achievable — mainly due to the recent breakthrough in mathematical methods (see e.g., Refs. 8, 9, 17–21) and mainly focused on the paradigmatic (fully connected) Sherrington–Kirkpatrick (SK) model,[14] — the theory of networks, in particular random, small world and scalefree networks,[6,22–24] revealed surprisingly common features between apparently much different structures in Nature, turning to itself the attention of several scientists, such that, by merging these two fields of research (disordered statistical mechanics and network theory), our study on diluted spin glasses developed.

Random interactions on the Erdos–Renyi-like graph have been introduced early in the literature[25] and they are drawing an increasing interest as complex networks for several reasons:

In theoretical physics, it is still not clear how to deal with finite-dimensionality models[26] (such as a nearest neighbor interaction, of whatever kind, on a three-dimensional lattice), and diluted systems are a bridge among mean field theories and finite-dimensionality theories. Furthermore, diluted spin glasses provide a different and more abstract formulation of the X-OR-SAT optimization problems in computer science,[27] which belong to the class of hard combinatory optimization problems (NP-complete). Moreover, they are at the basis of a diluted (and more realistic) neural network theory[3,28,29] and one of the first attempts when trying to embed a spin glass on an arbitrary random graph.[30,31]

Obtaining a complete description of their behavior avoiding the replica trick[14] is therefore a primary challenge.[8]

As these models are not Gaussian, they need not just an (functional) order parameter (i.e., q_2) as their fully connected counterpart (i.e., the SK model[14,18]), but the whole series of even (due to the parity symmetry) multioverlaps (i.e., $q_2, q_4, \ldots, q_{2n}$[25,32]). In the fully connected models we have a phase transition related to the increase from zero of q_2,[9] therefore one may ask if in these diluted models there are several transition lines (in the connectivity–temperature plane) one for each multioverlap, or they share the unique transition line at which ergodicity breaks (the critical line for q_2). In a previous recent work,[30] we proved only mathematically, by bounds, that the latter scenario was the correct one, but the physics behind it was still rather obscure and in particular no ideas concerning the nature of this transition were presented.

In this paper, we first show how a naive calculation would suggest the first scenario, and then we show both mathematically (extending our previous results) and physically (offering a picture for the nature of the transition) that the latter scenario is the correct one and, in particular, our physical picture is as follows: At the boundaries of the ergodic region, the fluctuations of the first order parameter (i.e., q_2) start to diverge, according to a well-defined second order phase transition, while the fluctuations of all the other order parameters do not; this naively suggests the validity of the several transition alternatives; however, due to the strong correlations at the critical point among all the order parameters, this first "jumping" to a nonzero value for q_2 drives all the others toward positive values too, acting as an "*ad hoc*" field in the space of these parameters. So the transition for the multioverlaps is surprisingly neither first order nor second order, but is a transition driven by a coupling field, namely $q_2 > 0$, which stems at the broken ergodicity line. This does not close the discussion on the theory of spin glasses on networks but shows a clear behavior at criticality.

2. Model and Notation

Consider N nodes, indexed by Latin letters i, j, etc., with an Ising spin (up–down degrees of freedom) attached to each of them, so as to have spin configurations

$$\sigma : \{1, \dots, N\} \ni i \to \sigma_i = \pm 1 \,.$$

Hence we may consider $\sigma \in \{-1, +1\}^N$. For the sake of convenience we talk about spins but, as long as we deal with equilibrium properties (i.e., phase diagrams and criticality), whatever dichotomic variable (quiescent of firing neurons, red or green cross-lights, on–off computer, etc.) would be fine. Let P_ζ be a Poisson random variable of mean ζ, N and let $\{J_\nu\}$ be independent identically distributed copies of a random variable J with symmetric distribution. For the sake of simplicity we will assume that $J = \pm 1$, without loss of generality, the fundamental competitions in the interactions ensured by the $\pm$ signs, with no particular emphasis on the strength. We consider randomly chosen points, and therefore introduce $\{i_\nu\}$, $\{j_\nu\}$ as independent identically distributed random variables, with uniform distribution over $1, \dots, N$. Assuming that there is no external field, the (suitable for our purpose) Hamiltonian of the Viana–Bray (VB) model[25] for dilute mean field spin glass is the symmetric random variable

$$H_N(\sigma, \alpha; \mathcal{J}) = -\sum_{\nu=1}^{P_{\alpha N}} J_\nu \sigma_{i_\nu} \sigma_{j_\nu} \,, \quad \alpha \in \mathbb{R}_+ \,. \tag{1}$$

$\mathbb{E}$ will be the expectation with respect to all the (quenched) variables, i.e. all the random variables except the spins, collectively denoted by $\mathcal{J}$. The nonnegative parameter α is called the *degree of connectivity.*

The Hamiltonian (1) as written has the advantage that it is the sum of (a random number of) i.i.d. terms. To see the connection to the original VB Hamiltonian, note that the Poisson-distributed total number of bonds obeys $P_{\alpha N} = \alpha N + O(\sqrt{N})$ for large N. As there are N^2 ordered spin pairs (i, j), each gets a bond with probability

$\sim \alpha/N$ for large N. The probabilities of getting two, three (and so on) bonds scale as $1/N^2$, $1/N^3, \ldots$ and so can be neglected. The probability of having a bond between any unordered pair of spins is twice as large, i.e., $2\alpha/N$. For large N each site therefore has on average 2α bonds connecting to it and, more precisely, this number of bonds to each site has a Poisson distribution with mean 2α.

We will occasionally find it useful to switch to alternative representations of H_N like

$$H_N(\sigma, \alpha) = -\sum_{1 \le i,j \le N} \sum_{\nu=0}^{m_{ij}} J_{ij}^{\nu} \sigma_i \sigma_j \,, \tag{2}$$

where each $m_{ij} = P_{\alpha/N}$ is i.i.d.-Poisson-distributed, and for each (i, j) we have corresponding independently drawn coupling strengths J_{ij}^{ν}. The fact that sums of Poisson variables are Poisson variables with the sum of the means also allows us to write this as

$$H_N(\sigma, \alpha) = -\sum_{1 \le i<j \le N} \sum_{\nu=0}^{m'_{ij}} J_{ij}^{\nu} \sigma_i \sigma_j + \sum_{\nu=1}^{P_\alpha} J_\nu \,, \tag{3}$$

where now $m'_{ij} = P_{2\alpha/N}$ and the last term comes from self-interactions and of course plays no role in the following.

To show the equivalence of the two versions (1) and (2) of the Hamiltonian, we just have to work out the joint distribution of the number of bonds m_{ij} allocated to each ordered pair (i, j) in (1). We set $m = \sum_{ij} m_{ij}$ for the total number of bonds, which is Poisson-distributed; conditional on this, m_{ij} are then multinomial, each with mean m/N^2:

$$\begin{aligned} P(m_{11}, \ldots, m_{NN}) &= e^{-\alpha N} \frac{(\alpha N)^m}{m!} \times \left(\frac{1}{N^{2m}} \frac{m!}{m_{11}! \cdots m_{NN}!} \right) \\ &= \prod_{ij} \left[e^{-\alpha/N} \left(\frac{\alpha}{N} \right)^{m_{ij}} \frac{1}{m_{ij}!} \right]. \end{aligned}$$

This factorizes into i.i.d. Poisson distributions for each m_{ij}, of mean α/N, as claimed in (2). We will later use similar ways of writing the Hamiltonian also for groups of spins.

The Gibbs measure ω is defined by

$$\omega(\varphi) = \frac{1}{Z} \sum_{\sigma} \exp(-\beta H(\sigma)) \varphi(\sigma)$$

for any observable $\varphi : \{-1, +1\}^N \to \mathbb{R}$, and clearly

$$Z_N(\beta) = \sum_{\sigma} \exp(-\beta H_N(\sigma)) \,,$$

which is the well-known partition function. When one is dealing with more than one configuration, the product Gibbs measure is denoted by Ω, and various configurations taken from each product space are called "replicas." We preserve the symbol $\langle\cdot\rangle$ for $\mathbb{E}\Omega(\cdot)$. Sometimes we will deal with a perturbed Boltzmann measure, whose perturbation will be triggered by a tunable parameter t, and we will stress the dependence on such a perturbation with a subscript t on the averages $\langle\cdot\rangle \to \langle\cdot\rangle_t$.

As already done above, we will often omit the dependence on β and on the size of the system N of various quantities. The free energy density f_N is defined by

$$A_N(\beta,\alpha) = -\beta f_N(\beta,\alpha) = \frac{1}{N}\mathbb{E}\ln Z_N(\beta,\alpha)\,.$$

The whole physical behavior of the model is encoded by the even multioverlaps $q_{1\cdots 2n}$,[30] which are functions of several configurations $\sigma^{(1)},\sigma^{(2)},\ldots$ defined by

$$q_{1\cdots 2n} = \frac{1}{N}\sum_{i=1}^{N}\sigma_i^{(1)}\cdots\sigma_i^{(2n)}\,.$$

For the sake of simplicity, often we will denote by $\theta = \theta(\beta)$ the expression $\tanh(\beta J) = \tanh(\beta)$.

3. Interpolation with the Cavity Fields

In this section, we present a short summary of the technique we want to use to analyze the criticality of the model. This technique has already been developed in a series of papers, starting in Ref. 18 (and applied later on in Ref. 30 to the diluted network) and we are going to explain how it does work without showing any proof (new results apart). The interested reader can check the original papers reported in the bibliography.

Criticality is the behavior of the system when it crosses critical lines in the phase space, which usually correspond to macroscopic changes in thermodynamics (i.e., phase transitions[12]). The idea of investigating criticality in these complex systems in a nutshell is simple and relate between stochastic stability[34] and cavity fields,[14] which we recall briefly:

- The main purpose of the *cavity field* method[14] is to look for an explicit expression of $A_N(\beta,\alpha) = -\beta f_N(\beta,\alpha)$ upon increasing the size of the system from N particles (the cavity) to $N+1$ so that, in the limit of N that goes to infinity

$$\lim_{N\to\infty}\frac{(-\beta F_{N+1}(\beta,\alpha)) - (-\beta F_N(\beta,\alpha))}{N+1-N} = -\beta f(\beta,\alpha)\,, \tag{4}$$

  because the existence of the thermodynamic limit[5,35] implies only vanishing correction of the free energy density.
- The main idea of the stochastic stability technique relates to the standard perturbation by an external field of classical statistical mechanics[3]: by taking the freedom of adding to the Hamiltonian an extra term which enforces the system toward a particular state we look for suitable functions of the phase space which have a nonnegligible response to this field (i.e. enforcing the magnetization to take positive values by adding an external magnetic field in the Ising model[13]).

This technique, useful for finding the order parameters, is straightforward in simple systems such as the Ising model,[13,33] but in this context it is still not completely understood because the finding of the right coupling external field for a system with a number of minima of the free energy increasing with the size of the system is not intuitive.

In these systems one usually perturbs the system with a random field so as to have

$$\tilde{H}_N(\sigma, h) = H_N(\sigma) + t \sum_{i=1}^{N} h_i \sigma_i \,, \tag{5}$$

where the tilde stands for the perturbed Hamiltonian, h_i are the random fields acting on the spins, and t is a tuning of the amplitude of the perturbation, eventually sent to zero afterward.

Our method starts by matching the two ideas above:

Due to the randomness of the coupling J and the gauge invariance of the model (the transformation $\sigma \to \sigma\epsilon$, with $\epsilon \pm 1$, which leaves the Hamiltonian unaffected, being $\epsilon^2 = 1$), we can think at a random perturbation as a term $h_i \sim \sum_{\nu}^{P_{2\bar{\alpha}t}} \tilde{J}_\nu \sigma_{i_\nu}$ (such that for $t = 0$ the perturbation is absent, while for $t = 1$ it is fully experienced by the system). Then, by applying the gauge $\sigma_{i_\nu} \to \sigma_{i_\nu}\sigma_{N+1}$, we can turn the stochastic perturbation into a cavity field.

Of course, if the system is not gauge-invariant (i.e., a P spin model with odd interacting spins[33]), the whole construction fails.

We can write, in distribution,

$$-H_{N+1}(\sigma; \alpha) \sim \sum_{\nu=1}^{P_{\alpha \frac{N2}{N+1}}} J_\nu \sigma_{k_\nu} \sigma_{l_\nu} + \sum_{\nu=1}^{P_{\alpha \frac{2N}{N+1}}} \tilde{J}_\nu \sigma_{m_\nu} \sigma_{i_\nu} \,, \tag{6}$$

where we have neglected a term which does not contribute when N is large[32]; $\{\tilde{J}_\nu\}$ are independent copies of J; $\{k_\nu\}$, $\{m_\nu\}$ and $\{l_\nu\}$ are independent random variables all uniformly distributed over $\{1, \ldots, N\}$; and $\{i_\nu\}$ are independent random variables uniformly distributed over the set $\{1\}$, consisting of $\{1\}$ only (but other formulations with several added spins are allowed[17]). So $\sigma_{i_\nu} \equiv \sigma_1$. Notice that we can also write, in distribution,

$$H_{N+1}(\sigma; \alpha) \sim H_N(\sigma; \bar{\alpha}) + h_\sigma \sigma_1 \,, \tag{7}$$

where

$$\bar{\alpha} = \alpha \frac{N}{N+1} \,, \quad h_\sigma = -\sum_{\nu=1}^{P_{2\bar{\alpha}}} \tilde{J}_\nu \sigma_{k_\nu} \,.$$

Notice also that, similarly,

$$H_N(\sigma; \alpha) = H_N(\sigma; \bar{\alpha}, \mathcal{J}) + H_N(\sigma; \bar{\alpha}/N, \hat{\mathcal{J}}) \,, \tag{8}$$

thanks to the additivity property of Poisson variables, and the two Hamiltonians on the right hand side have independent quenched random variables $\mathcal{J}$ and $\hat{\mathcal{J}}$.

Now, as $\sigma_{N+1} \in \pm 1 = \epsilon$ and $\hat{J}$ are symmetrical random variables, we can think of the term encoding for the added $N+1$ spin as a perturbation on the system of the first N spins.

In this way the cavity field acts as a stochastic field. So we introduce a random perturbation by interpolating on a fictitious parameter t between an N system of connectivity α and an $N+1$ system with a small change in the connectivity $\alpha \to \bar{\alpha} = \alpha(1+N^{-1})$, and we found two main properties for all the overlap monomials as $\langle q_{2n}^p \rangle, p \in \mathbb{N}$ (recalls that for $p = 1$ we are dealing with the order parameters of the theory): *robustness* and *saturability*.

- *Robustness* states that all the multioverlaps which are "filled," i.e., they have each replica appearing an even number of times (like $\langle q_{12}^2 \rangle$, $\langle q_{1234}^2 \rangle$, $\langle q_{12}q_{34}q_{1234} \rangle$), are not affected by the perturbation in the $N \to \infty$ limit;
- *Saturability* states that, once called "fillable" the other multioverlap monomials, in the $t \to 1$ and $N \to \infty$ limits, fillable monomials become filled (i.e., $\lim_{N\to\infty} \lim_{t\to 1} \langle q_{12} \rangle_t = \langle q_{12}^2 \rangle$, $\lim_{N\to\infty} \lim_{t\to 1} \langle q_{1234} \rangle_t = \langle q_{1234}^2 \rangle$, $\lim_{N\to\infty} \lim_{t\to 1} \langle q_{12}q_{34} \rangle_t = \langle q_{12}q_{34}q_{1234} \rangle$).

By these two properties it has been possible to show several features of this kind of complex systems.[9,18,30,32]

The machinery we need relies on these two classes of overlap monomials and is built by the next simple statements.

Lemma 1. *In the $N \to \infty$ limit, the average $\langle \cdot \rangle_t$ of filled monomials is not affected by the presence of the perturbation modulated by t, for instance*

$$\int_{\bar{\alpha}_1}^{\bar{\alpha}_2} \langle q_{12}q_{23}q_{13} \rangle_t d\bar{\alpha} = \int_{\bar{\alpha}_1}^{\bar{\alpha}_2} \langle q_{12}q_{23}q_{13} \rangle d\bar{\alpha} \,,$$

for any interval $[\bar{\alpha}_1, \bar{\alpha}_2]$. We call this property of filled monomials "robustness."

Lemma 2. *Let $q_{1\cdots 2n}$ be a fillable monomial of the multioverlaps, such that $q_{1\cdots 2n}Q_{1\cdots 2n}$ is filled. Then*

$$\lim_{N\to\infty} \langle q_{1\cdots 2n} \rangle_{t=1} = \langle q_{1\cdots 2n} Q_{1\cdots 2n} \rangle \,,$$

where the right hand side is understood to be evaluated in the thermodynamic limit. We will refer to this property as "saturability."

As the whole paper strongly relies on the two lemmas above, we sketch their proofs in the following few lines:

Sketched Proof. Let us show how the fillable monomials turn out to be filled in the $N \to \infty$ limit. Then the robustness of the filled monomials will be a straightforward consequence: defining Q_{ab} as a fillable monomial and using Q_{ij} for the product of the filled replicas inside Q_{ab}, and leaving a and b as the nonfilled replicas, we have

$$\langle Q_{ab}\rangle_t = \left\langle \left(\sum_{ij} (\sigma_i^a \sigma_j^b)/N^2 \right) Q_{ij}(\sigma) \right\rangle_t .$$

Factorizing the state Ω we obtain

$$\langle Q_{ab}\rangle_t = \frac{\mathbf{E}}{N^2}\left(\sum_{ij} \Omega_t(\sigma_i^a \sigma_j^b Q_{ij}(\sigma)) \right) \tag{9}$$

$$= \frac{\mathbf{E}}{N^2}\left(\sum_{ij} \omega_t(\sigma_i^a)\omega_t(\sigma_j^b)\Omega_t(Q_{ij}) \right). \tag{10}$$

Now we rewrite the last expression for $t = 1$: by applying the symmetry $\sigma_i \to \sigma_i\sigma_{N+1}$, the states acting on the replicas a and b are $\omega_{t=1}(\sigma_i^a) \to \omega(\sigma_i^a\sigma_{N+1}^a) + O(N^{-1})$, while the remaining product state Ω_t continues to work on an even number of replicas and is not modified (giving rise to the saturability of the filled monomials). Putting all the replicas in a unique product state, we have

$$\omega(\sigma_i^a\sigma_{N+1}^a)\omega(\sigma_i^b\sigma_{N+1}^b)\Omega(Q_{ij}) = \Omega(\sigma_i^a\sigma_j^b\sigma_{N+1}^a\sigma_{N+1}^b Q_{ij}) . \tag{11}$$

By the gauge symmetry again, we can think of the index $N+1$ as a dumb hidden variable k, and multiplying by $1 = N^{-1}\sum_{k=1}^N$ in the thermodynamic limit we have the proof.

Now we introduce the instruments we need to deal with:

Proposition 3. *Let Φ be a function of s replicas. Then the following cavity streaming equation holds*:

$$\begin{aligned}\frac{d\langle\Phi\rangle_t}{dt} = -2\bar{\alpha}\langle\Phi\rangle_t + 2\bar{\alpha}\mathbb{E}\Bigg[\Omega_t\Phi\Bigg\{1 + J\sum_a^{1,s}\sigma_{i_1}^a\theta + \sum_{a<b}^{1,s}\sigma_{i_1}^a\sigma_{i_1}^b\theta^2 \\ + J\sum_{a<b<c}^{1,s}\sigma_{i_1}^a\sigma_{i_1}^b\sigma_{i_1}^c\theta^3 + \cdots\Bigg\}\Bigg\{1 - sJ\theta\omega_t + \frac{s(s+1)}{2!}\theta^2\omega_t^2 \\ - \frac{s(s+1)(s+2)}{3!}J\theta^3\omega_t^3 + \cdots\Bigg\}\Bigg]\forall\,\theta .\end{aligned} \tag{12}$$

Definition 4. Let us define a *cavity function* $\Psi_{N,t}(\alpha,\beta)$ as the following quantity:

$$\Psi_{N,t}(\alpha,\beta) = \mathbf{E}\ln\omega(e^{\beta\sum_{\nu=1}^{P_{2\bar{\alpha}t}}\tilde{J}_\nu\sigma_{i_\nu}}) . \tag{13}$$

Note that the cavity function takes into account the perturbation applied to the original Hamiltonian; it will play a fundamental role in the expansion of the free energy, as is immediately clear by the next theorem.

Theorem 5. *The following relation between free energy, its connectivity increment and cavity function holds*:

$$\lim_{N\to\infty}(A_N(\alpha,\beta)+\alpha\partial_\alpha A_N(\alpha,\beta)) = \ln 2 + \lim_{N\to\infty}\Psi_{N,t=1}(\alpha,\beta)\,. \tag{14}$$

Proof. Let us write down the partition function of a system built up by $N+1$ spins in terms of the one built only by the first N:

$$\begin{aligned} Z_{N+1}(\alpha,\beta) &= \sum_{\{\sigma_{N+1}\}}\exp\left(\beta\sum_{\nu=1}^{P_{\alpha N^2/(N+1)}} J_\nu\sigma_{i_\nu}\sigma_{j_\nu} + \beta\sum_{\nu=1}^{P_{2\alpha N/(N+1)}}\tilde{J}_\nu\sigma_{i_\nu}\sigma_{j_\nu}\right) \\ &= \sum_{\{\sigma_{N+1}\}}\exp\left(\beta\sum_{\nu=1}^{P_{\bar{\alpha}N}} J_\nu\sigma_{i_\nu}\sigma_{j_\nu} + \beta\sum_{\nu=1}^{P_{2\bar{\alpha}}}\tilde{J}_\nu\sigma_{i_\nu}\sigma_{j_\nu}\right) \\ &= 2\sum_{\{\sigma_N\}} e^{-\beta H_N\bar{\alpha}+\beta\sum_{\nu=1}^{P_{2\bar{\alpha}}}\tilde{J}_\nu\sigma_{i_\nu}}\,, \end{aligned}$$

where $\bar{\alpha}=\alpha N/(N+1)$ and in the last passage we have used gauge symmetry.

Multiplying and dividing by $Z_N(\beta,\bar{\alpha})$, we get

$$Z_{N+1}(\beta,\alpha) = 2Z_N(\beta,\bar{\alpha})\omega_{\bar{\alpha}}(e^{\beta\sum_{\nu=1}^{P_{\bar{\alpha}N}}\tilde{J}\sigma_{i_\nu}\sigma_{j_\nu}})\,, \tag{15}$$

where

$$\omega_{\bar{\alpha}}(\cdot) = \frac{\sum\{\sigma\}(\cdot)e^{-\beta H_N+\beta\sum_{\nu=1}^{P_{\bar{\alpha}N}}\tilde{J}_\nu\sigma_{i_\nu}}}{\sum\{\sigma\}e^{-\beta H_N+\beta\sum_{\nu=1}^{P_{\bar{\alpha}N}}\tilde{J}_\nu\sigma_{i_\nu}}}\,. \tag{16}$$

Now let us take the logarithm of $Z_{N+1}(\beta,\alpha)$

$$\ln Z_{N+1}(\beta,\alpha) = \ln 2 + \ln Z_N(\beta,\bar{\alpha}) + \ln\omega_{\bar{\alpha}}(e^{\beta\sum_{\nu=1}^{P_{2\bar{\alpha}}}\tilde{J}_\nu\sigma_{i_\nu}})\,, \tag{17}$$

summing and subtracting $\ln Z_{N+1}(\beta,\bar{\alpha})$ and expanding the logarithm around $\bar{\alpha}$ such that

$$\ln Z_{N+1}(\beta,\alpha) \sim \ln Z_N(\beta,\bar{\alpha}) + (\alpha-\bar{\alpha})\partial_\alpha\ln Z_{N+1}(\beta,\alpha)\mid_{\alpha=\bar{\alpha}} + O(d\alpha^2)\,,$$

where $\alpha=\bar{\alpha}(N+1)/N$ and $\alpha-\bar{\alpha}=\bar{\alpha}/N$; we have just to take the average $\mathbb{E}$ and remembering that $\lim_{N\to\infty}\bar{\alpha}=\alpha$ and that $\Psi_{N,t}(\alpha,\beta)=\mathbf{E}\ln\omega(\exp(\beta\sum_{\nu=1}^{P_{2\bar{\alpha}t}}\tilde{J}_\nu\sigma_{i_\nu}))$. In the thermodynamic limit we obtain the thesis of the theorem. $\square$

The next two propositions are due to express explicitly the two terms by which the free energy can be decomposed thanks to Theorem 5.

They are straightforward:

Proposition 6. *The incremental contribution to the free energy by the connectivity is*[30]

$$\alpha\partial_\alpha A(\alpha,\beta) = 2\alpha\sum_1^\infty\frac{1}{2n}\theta^{2n}(\beta J)(1-\langle q_{2n}^2\rangle)\,.$$

Proposition 7. *The cavity function can be represented by the integral of the series of all the fillable multioverlaps weighted by the powers of* θ[32]:

$$\Psi_{N,t}(\beta,\alpha) = \int_0^t 2\bar{\alpha}\sum_{n=1}^{\infty}\frac{1}{2n}\theta^{2n}(\beta J)(1-\langle q_{2n}\rangle_t')dt' \,. \tag{18}$$

So far we have exploited the machinery. Let us sketch how it works:

By Theorem 5 we know that we can express the free energy via its derivative with respect to the connectivity and that such a derivative is known and built up only by filled terms (Proposition 6); in other words, this part shows robustness.

Then we need also the cavity function contribution, which is expressed as an integral of fillable monomials (Proposition 7) which does not show robustness. Not to worry, because we can expand the linear multioverlaps appearing in Eq. (18) via the streaming equation (12) in terms of filled multioverlap and map the exact unfilled expression for the cavity function into an expansion in filled monomials.

Clearly, we lose the full solution for the free energy of the model but, as criticality neglects higher order terms, we can easily investigate its critical behavior.

4. Behavior of the Overlap in the Ergodic Regime

The multioverlaps among any $2n$ configurations are typically small in the ergodic region defined by $2\alpha\tanh^2(\beta)=1$ and their fluctuation can be studied on the $\sqrt{N}$ scale by defining

$$\eta_{2n} = \sqrt{N}q_{2n} = \frac{1}{\sqrt{N}}\sum_i^N \sigma_i^1\cdots\sigma_i^{2n}\,. \tag{19}$$

Then, as for the SK model, it is possible to show that these rescaled multioverlaps behave, in this region, like independent centered Gaussian variables in the infinite volume limit and the following theorem holds[5]:

Theorem 8. *In the annealed region* $2\alpha\tanh^2(\beta)<1$, *the variables* η_{2n} *converge to the centered Gaussian process with covariances*

$$\langle\eta_{a_1,\ldots,a_{2n}}\rangle = \frac{1}{1-2\alpha\mathbb{E}\tanh^{2n}(\beta J)}\,, \tag{20}$$

$$\langle\eta_{a_1,\ldots,a_{2n}}\eta_{b_1,\ldots,b_{2n}}\rangle = 0 \quad if \quad \exists i: a_i\neq b_i\,, \tag{21}$$

and when the boundary of the annealed region is approached only the variance of η_2 *diverges.*

Another way to look at the fluctuations of q_2 for finding its critical points is by using the interpolating cavity field method previously introduced. The idea is straightforward:

- By using the streaming equation (3) we can expand the two-replica overlap:

$$\langle q_{12}\rangle_t = 2\alpha\theta^2\langle q_{12}^2\rangle - 4\alpha^2\theta^4\langle q_{12}q_{23}\rangle_t + O(q_{ij}^3)\,. \tag{22}$$

- By simple polynomial integration we can evaluate the overlap expansion in terms of filled monomials:

$$\langle q_{12}\rangle_t = 2\alpha\theta^2\langle q_{12}^2\rangle t - 4\alpha^2\theta^4\int_0^t dt'\int_0^{t'} dt''\langle q_{12}q_{23}q_{13}\rangle + O(q_{ij}^6)\,, \tag{23}$$

$$\langle q_{12}\rangle_t = 2\alpha\theta^2\langle q_{12}^2\rangle t - \langle q_{12}q_{23}q_{13}\rangle 8\alpha^2\theta^4 t^2 + O(q_{ij}^6)\,. \tag{24}$$

- By applying "saturability" (Proposition 2) we get $\langle q_{12}\rangle_t = \langle q_{12}^2\rangle$.
- Consequently, forgetting the $O(q_{ij}^4)$ terms, we have

$$\langle q_2^2\rangle = \frac{2(2\alpha\theta^2)^2}{1-(2\alpha\theta^2)}\langle q_{12}q_{23}q_{13}\rangle\,. \tag{25}$$

- Multiplying by N Eq. (25) we obtain the same constraint for η_2^2 and we see that on the r.h.s. the overlap order is 3 while on the l.h.s. it is 2. By a central limit theorem argument we conclude that the only diverging point, for the rescaled overlap fluctuations is $2\alpha\theta^2 = 1$, where the denominator explodes, according to Theorem 8.

We stress once more that at the critical line only the rescaled fluctuations of q_2 diverge, while the others for higher order multioverlaps do not, suggesting in the sense a lack of a transition for them. Furthermore, applying the same reasoning to q_4, we are going to see that we will find another critical line given by $2\alpha\theta^4 = 1$: the two things coupled together strongly suggest the scenario with several transition temperatures as the right one.

5. Several Transitions by a Naive Calculation

In this section, we want to show the picture emerging by applying blindly the theory in Sec. 3 to the other multioverlaps. Not surprisingly, we are going to find a class of "hypothetical transition temperatures" in complete agreement with a second order expansion in the replica trick framework.[25]

- There are infinite transition temperatures, one for every (even) multioverlap q_{2n}, and they are obtainable via the relation

$$2\alpha\theta^{2n} = 1\,, \tag{26}$$

such that the critical temperature for q_2 is reached when $2\alpha\theta^2 = 1$, the one for q_4 when $2\alpha\theta^4 = 1$ and so on.
- The critical behavior of the multioverlaps is

$$\langle q_2^2\rangle \sim C(\tau-1)^2\,,\quad \tau = 2\alpha\theta^2\,, \tag{27}$$

$$\langle q_4^2\rangle \sim D(\tau'-1)^2\,,\quad \tau' = 2\alpha\theta^4\,, \tag{28}$$

with C and D real constants and so on.

Why is this?

Saturability is a powerful bridge between the expansion of the multioverlaps and the control of their fluctuations, in fact it played a key role in determining the behavior of the two-overlap at its critical point in the previous section. Let us show the saturability equations for the first two multioverlaps (which we are going to use as examples throughout the paper); starting by the streaming of $\langle q_{12}\rangle_t$ and $\langle q_{1234}\rangle_t$, thanks to the streaming equation (3), we know that

$$\langle q_{12}\rangle_t \sim 2\alpha\theta^2 t\langle q_{12}^2\rangle - 8\alpha^2\theta^4 t^2\langle q_{12}q_{23}q_{13}\rangle + \cdots\,, \tag{29}$$

$$\langle q_{1234}\rangle_t \sim 2\alpha\theta^4 t\langle q_{12}^2\rangle + 12\alpha^2\theta^4 t^2\langle q_{12}q_{34}q_{1234}\rangle + \cdots\,, \tag{30}$$

and the multioverlap monomials on the r.h.s. do not depend on t being all filled. Now let us evaluate both the equations for $t = 1$, we can apply saturability to get

$$\lim_{t\to 1}\langle q_{12}\rangle_t = \langle q_{12}^2\rangle_t = \langle q_{12}^2\rangle\,, \tag{31}$$

$$\lim_{t\to 1}\langle q_{1234}\rangle_t = \langle q_{1234}^2\rangle_t = \langle q_{1234}^2\rangle\,, \tag{32}$$

from which we obtain the expressions for the first two multioverlaps,

$$\langle q_{12}^2\rangle(2\alpha\theta^2 - 1) \sim 2(2\alpha\theta^2)^2\langle q_{12}q_{23}q_{13}\rangle + \text{h.o.}\,, \tag{33}$$

$$\langle q_{1234}^2\rangle(2\alpha\theta^4 - 1) \sim -3(2\alpha\theta^2)^2\langle q_{12}q_{34}q_{1234}\rangle + \text{h.o.}\,, \tag{34}$$

from which, using the same line of reasoning of Eqs. (22)–(25), we have the transition points $2\alpha\theta^2 = 1$ for the two-replica overlap and $2\alpha\theta^4 = 1$ for the four-replica overlap and so on.

The mistake is as follows: "Saturability" implicitly works thanks to the gauge invariance of the Viana–Bray Hamiltonian[18] and, as long as the Boltzmann state shares with the Hamiltonian this symmetry, saturability has to hold.

At the critical line of a second order transition, the symmetry of the system must contain all the symmetry elements of both the phases; they coincide only in that line. So, thanks to the "second order" nature of the two-replica overlap transition, at its critical line ($2\alpha\theta^2 = 1$) the gauge symmetry still holds and "saturability" effectively works as a key bridge between the critical regime and the ergodic phase.

As soon as the temperature (or the connectivity) is again decreased, the Boltzmann state no longer shares the gauge symmetry with the Hamiltonian because it is spontaneously broken by the added random field (negligible in the thermodynamic limit, so it is effectively a spontaneous change) and we cannot apply saturability any longer, and thus the key passage $\lim_{t\to 1}\langle q_{1234}\rangle_t = \langle q_{1234}^2\rangle$ does not work.

Remarkably, the transition lines suggested by this approach turn out to be the same ones suggested by the second order expansion within the replica trick framework.[25] The transitions discussed in Ref. 25 are "potential" transition temperatures: the authors refer to bifurcations from the replica symmetric state where all overlaps are zero. The bifurcation points are determined purely from the quadratic expansion of the free energy around zero overlaps; these temperatures for bifurcations from the origin then give the points where the curvature of the free energy along the various directions vanishes.

The actual transition happens at the highest of these bifurcation temperatures. Once this first bifurcation is achieved, the saddle point of the free energy is no longer at the origin (all overlaps equal zero) but at a different point.

6. Correct Scenario for the Critical Point

Before showing the scenario we propose, let us work out the expansion of the free energy, which will help immediately after.

Neglecting orders higher than $(2\alpha\theta^2)^2$, we have

$$\Psi_{N,t}(\alpha,\beta) = \int_0^t dt' 2\alpha \left(\frac{\theta^2}{2}(1 - \langle q_{12}\rangle_{t'}) + \frac{\theta^4}{4}(1 - \langle q_{1234}\rangle_{t'}) + \cdots \right), \tag{35}$$

so to evaluate the cavity function at the desired order we have to fulfill $\langle q_{12}\rangle_t$ at the order θ^4 and $\langle q_{1234}\rangle_t$ at the order θ^2 by the methodology explained in Sec. (3) and sketched throughout the last two sections.

Once Ψ has been decomposed in filled multioverlap fluctuations, we can use Theorem 5 to write down the free energy of the model. Presenting just the first orders and remembering that we call $\tau = 2\alpha\theta^2$, we have

$$\begin{aligned} A(\alpha,\beta) = \ln 2 &+ \left(\frac{1}{2\alpha}\right)^0 \left(\frac{\tau}{2} - \frac{\tau}{4}(1 - \tau\theta^0)\langle q_{12}^2\rangle + \frac{\tau^3}{3}\langle q_{12}q_{23}q_{13}\rangle + \cdots \right) \\ &+ \left(\frac{1}{2\alpha}\right)^2 \left(\frac{\tau}{4} - \frac{\tau}{8}(1 - \tau\theta^2)\langle q_{1234}^2\rangle + \frac{3\tau^3}{4}\langle q_{1234}q_{12}q_{34}\rangle + \cdots \right) + \cdots \end{aligned} \tag{36}$$

Note that in the high connectivity limit[5] the expression (36) approaches the well-known expression for the free energy of the SK model.[18]

In this section, we want to explain the correct scenario in which the only real transition line is the one where ergodicity breaks $(2\alpha\theta^2 = 1)$. Corresponding to this phenomenon, the two-replica overlap starts taking nonzero values. Thanks to the strong correlation between the two-replica overlap and all the other multioverlaps at the critical point, it drives the others to positive values too, and so neither in a standard second order way nor in a discontinuous first order way. All the multioverlaps start taking positive values continuously from zero but without diverging fluctuations at the critical point.

To try and show this picture, always using q_2 and q_4 as examples, let us start with the following:

Proposition 9. *As soon as* $\langle q_2\rangle > 0$, $\langle q_4\rangle$ *must also be* >0.

Proof. Exploiting the factorization of the Boltzmann state at fixed J, we have

$$\langle q_{1234}\rangle = \mathbb{E}\left[\frac{1}{N}\sum_i \omega^4(\sigma_i)\right] \geq \mathbb{E}\left[\left(\frac{1}{N}\sum_i \omega^2(\sigma_i)\right)^2\right]$$
$$= \mathbb{E}[\omega^2(q_{12})] \geq (\mathbb{E}[\omega(q_{12})])^2 = \langle q_{12}\rangle^2\,,$$

where we have used $\mathbb{E}[a^2] \geq \mathbb{E}^2[a]$ for any real-valued random variable, first for $a = \omega_4(\sigma_i)$ and with the expectation taken over the uniform distribution on $i = 1, \ldots, N$ and then for $a = \omega(q_{12})$ with the expectation over $P(J)$. □

We understand there is a huge family of such bounds, easily depicted by the following:

Theorem 10. *Given two integers c and d such that $cd = 2n$ and $m \in N$, the following families of bounds hold generically and also at finite N:*

$$\langle q_{2n}^m\rangle \geq \langle q_{1\cdots c}^m q_{c+1\cdots 2c}^m \cdots q_{c(d-1)+1\cdots 2n}^m\rangle \geq \langle q_{1\cdots c}^m\rangle^d\,. \tag{37}$$

Proposition 9 is a straightforward application of Theorem 10 with $c = d = 2$ and $m = 1$. We omit the proof of the theorem as it can be straightforwardly obtained along the lines of the previous proof.

The conclusion is that it is not possible to have several spin glass transitions in any model: as soon as $\langle q_{12}\rangle$ becomes nonzero, $\langle q_{1234}\rangle$ must also be nonzero, and so on.

Furthermore, we note that the bound works for finite N (and so it is not necessary to invoke pure states[14]); by making N large on both sides it must then also hold in the thermodynamic limit.

The last step missing is trying to understand in which way all the other multioverlaps start taking positive values at the critical line because, by standard statistical mechanics arguments, we do not see this behavior.

What is the genesis of the correlations among different multioverlaps?

The mechanism we provide is again ultimately based on saturability. In fact, at the critical point the fillable multioverlap $\langle q_{12}q_{34}\rangle$, applying saturability, gets

$$\lim_{N\to\infty}\lim_{t\to 1}\langle q_{12}q_{34}\rangle_t = \langle q_{12}q_{34}q_{1234}\rangle\,, \tag{38}$$

which couples the first multioverlap q_2 and the second multioverlap q_4 together, generating the correlation which drives the transition for $\langle q_{1234}\rangle$. Saturability can be applied as we are at the boundary of the ergodicity breaking (the last point at which it still holds).

So, remembering once more that we are taking just the first two multioverlaps but the scheme applies to all of them and, for the sake of clarify, forgetting all the higher order unnecessary terms, we can write the free energy, which we call $f(q_2, q_4)$

to stress the dependence by the two multioverlaps as

$$f(q_2, q_4) = \left(\theta - \left(\frac{1}{2\alpha}\right)^{\frac{1}{2}}\right) q_2^2 + \left(\theta - \left(\frac{1}{2\alpha}\right)^{\frac{1}{4}}\right) q_4^2 - \frac{3\tau^3}{4} q_2^2 q_4 \,, \tag{39}$$

and we want to know how the minima of $f(q_2, q_4)$ evolve with θ (at fixed α, or vice versa). If a bifurcation analysis of the saddle point equations from the origin is performed, one would find two transition temperatures, $\theta_{q_2} = (1/2\alpha)^{1/2}$ and $\theta_{q_4} = (1/2\alpha)^{1/4}$. However, when one is looking at the actual minima it is possible to see just the first transition. After that the two minima are away from the origin and so the second "potential transition temperature" at $\theta_{q_4} = (1/2\alpha)^{\frac{1}{4}}$ never becomes relevant: when one is approaching this temperature the system is already in a completely different part of its phase space. We stress that above $2\alpha\theta^2 = 1$, where

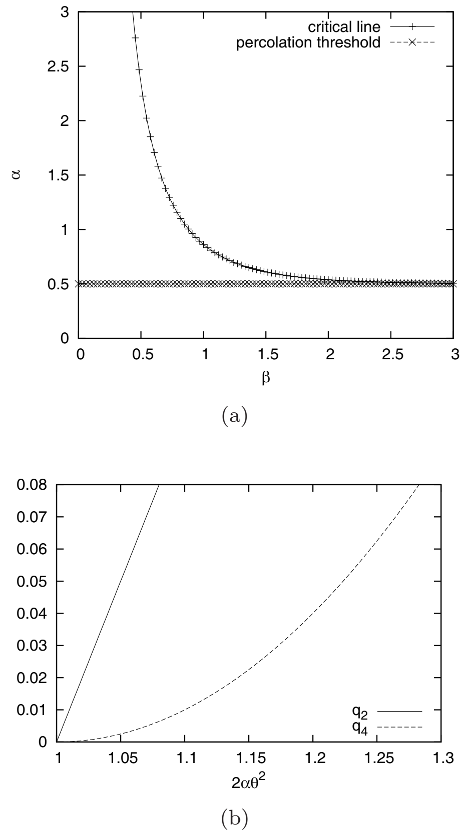

Fig. 1. (a) Phase diagram. For $\alpha < 1/2$ (percolation threshold) the dilution is too strong, there is no giant component in the graph and the spins are grouped in several isolated clusters which are thermodynamically averaged out. Above the percolation threshold the critical line defines two zones: at left the ergodic region, where the system behaves paramagnetically due to the high temperature; at right the broken ergodicity region where the system behaves as a spin glass. (b) Critical behavior of q_2 and q_4.

the quadratic expansion of $f(q_2, q_4)$ around the origin determines the Gaussian fluctuations, q_2 and q_4 are uncorrelated, then, below this point, the third order term produces an interaction $(q_{12}q_{34}q_{1234})$ and so, as soon as q_2 becomes nonzero, it also drives q_4 to a nonzero value (see Fig. 1). It is also straightforward to check that near $2\alpha\theta^2$ the minima scale as $q_2 \sim (2\alpha\theta^2 - 1)$ and $q_4 \sim (2\alpha\theta^2 - 1)^2 \sim q_2^2$, according to the proven scaling for random spins at criticality.[30]

7. Conclusion

In recent years, ever-increasing attention has been paid to the study of networks.[6,22] There are several questions which can be addressed when dealing with a random graph.[1] Beyond topology one can put variables (e.g., spins) on the nodes and impose rules for their interaction. When these rules are completely random with equal probability for the coupling to be positive (trying to align spins) or negative (opposite will) and the network is sparse (i.e., an Erdos–Renyi graph), we have a standard diluted spin glass.[25,32]

The ergodic region for these systems has been fully investigated and understood by Guerra and Toninelli,[5] while self-averaging properties have been obtained in Ref. 32 and a replica-like behavior in Ref. 19. Concerning the criticality, i.e. lines in the phase space where there is a macroscopic change in the system as the boundary of the ergodic region, it has recently been investigated in Ref. 30.

In this paper, we first showed how it is possible (and very intuitive) to think of a critical scenario for these systems with several critical lines, one for every multioverlap. Then, we proved this scenario to be wrong and we showed that there is just one critical line at which all the multioverlaps start to take nonzero values. The remarkable behavior of the multioverlaps is that they start to be positive without showing diverging (rescaled) fluctuations when approaching the boundary of the ergodic regime. In fact, we explained that these transitions are "induced" by a drive, which is ultimately due to the correlation among the multioverlaps and the two-replica overlaps, the latter undergoing a standard second order phase transition.

The whole discussion relied entirely on new methodologies such as robustness and saturability[18] of multioverlap monomials and did not require any kind of ansatz.

Further development should follow on diluted networks with different topologies (an attempt has been made in Ref. 10) and a detailed understanding of the broken replica phase (the whole nonergodic region).

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