Effective interactions in group competition with strategic diffusive dynamics

Elena Agliari,[1] Adriano Barra,[2, 3] Raffaella Burioni,[1] Federico Camboni,[2] and Pierluigi Contucci[3]

[1]*Dipartimento di Fisica and INFN, Università di Parma, Italy*
[2]*Dipartimento di Fisica, Sapienza Università di Roma, Italy*
[3]*Dipartimento di Matematica, Università di Bologna, Italy*

We analyze, on a random graph, a *diffusive strategic dynamics* with pairwise interactions, where nor Glauber prescription, neither detailed balance hold. We observe numerically that such a dynamics reaches a well defined steady state that fulfills a *shift* property: the critical temperature of the canonical ferromagnetic phase transition is higher with respect to the expected equilibrium one, known both numerically via Glauber relaxation or Monte Carlo simulations as well as analytically via cavity techniques or replica approaches.

We show how the relaxed states of this kind of dynamics can be described by statistical mechanics equilibria of a diluted p-spin model, for a suitable non-integer real p. Several implications from both theoretical physics and quantitative sociology points of view are discussed.

PACS numbers:

I. INTRODUCTION

Born as a theoretical background for thermodynamics, statistical mechanics provides nowadays a flexible approach to several scientific problems whose depth and wideness increases continuously. In the last decades in fact complex systems statistical mechanics has invaded fields as diverse as spin glasses [1], neural networks [2], protein folding [3], immunological memory [4], and also made some attempt to describe social networks [5], theoretical economy [6] and urban planning [7].

In this paper we study statistical mechanics of random diluted mean field systems, paying particular attention to its applications in social science: in a nutshell we show that the relaxed states of a particular non-Glauber two-body dynamics [8, 9] can be "effectively" described by a diluted p-spin model for a suitable non-integer real p.

After a review of the *diffusive strategic dynamics* previously introduced in [10, 11, 12], we implement it on an Erdös-Renyi mean field random model and work out the corresponding mean field equilibria: each agent (spin) is selected through a diffusive rule, and its flipping probability is not weighed "a la Glauber" [2]. The probabilities of its nearest neighbor flips are selected by favouring the flips which produce the maximum energy gain. We stress that this operation, although dynamically pairwise, effectively involves more than pairwise information evaluation, as the chosen agent interacts both with the first selected one as well as with its nearest neighbors, as a whole.

This dynamics is shown to relax to a well defined steady state, where all the properties of stationarity are recovered [13]. However a peculiarity happens, with respect to the standard relaxation investigated both by means of Glauber dynamics [2], and well known both by Monte-carlo simulation and analytical results [14]: the critical parameters (temperature, or equivalently the inverse of the strength of the interactions) are higher, of a few percent, than the expected.

The whole scenario suggests a "hidden" many body interaction, encoded into the particular rule for selecting the spins. To investigate this feature we perform further numerical analysis, which strongly supports a more-than-two body effective interaction. Then we work out analytically a theory for the randomly diluted p-spin model so to fit an effective $p \in \mathbb{R}$, which turns out to be $p = 2.15$, in order to match the numerical data available by the dynamics. This result has implication both in theoretical physics, as well as in quantitative sociology, where the effective interactions always play an important role in decision making [15, 16].

The paper is organized as follows:

In section II our dynamics is introduced and shortly discussed; then further numerical investigations toward a better understanding of a $p > 2$ behavior are presented.

In section III the randomly diluted p-spin model is defined and exploited in all details, both analytically within the cavity field techniques as well as numerically, within a Montecarlo approach. Full agreement is found among the two methods.

At the end, the last section is left for conclusions: the effective interaction is found and its implications analyzed. Furthermore, even though the paper is written within a theoretical physics approach, remarks concerning the application to quantitative sociology are sparse through all the work.

As a last remark, for a better reading of the manuscript, we decided to put all the long proofs of the various theorems in the appendix.

II. DIFFUSIVE STRATEGIC DYNAMICS

In this section we shortly discuss the general approach for the numerical investigation of an Ising-like system [9], then we introduce a particular dynamics and we explain the motivation behind our choice.

A. A strategy avoiding Glauber prescription

In order to simulate the dynamical evolution of a system described by a Curie-Weiss Hamiltonian $\hat{H}_N(\sigma)$, living on an Erdös-Renyi random graph [14, 17],

$$\hat{H}_N(\sigma) = \sum_{ij}^{N} A_{ij}\sigma_i\sigma_j,$$

where A_{ij} is a Poisson random adjacency matrix that, on average, connects α_i sites to the generic i^{th} spin (and thermodynamically $\alpha_i \to \alpha$, the connectivity, whenever $N \to \infty$), several different algorithms have been introduced. Among them a well established one is the so-called single-flip algorithm, which makes the system evolve by means of successive spin-flips, where we call "flip" on the node j the transformation $s_j \to -s_j$ [18].

More precisely, the generic single-flip algorithm is made up of two parts: first we need a rule according to which we select a spin to be updated, then we need a probability distribution which states how likely the spin-flip is. As for the latter, following the Glauber rule, given a configuration **s**, the probability for the spin-flip on the j-th node reads off as

$$p(\mathbf{s}, j, \mathbf{J}) = \frac{1}{1 + e^{\beta \Delta H(\mathbf{s}, j, \mathbf{J})}}, \tag{1}$$

where $\Delta H(\mathbf{s}, j, \mathbf{J}) = 2\sigma_i \sum_j A_{ij}\sigma_j$ is the variation in the cost function due to the flip $s_j \to -s_j$. Hence, for single-flip dynamics the cost variation ΔH, consequent to a flip, only depends on the spin of a few sites, viz. the j-th one undergoing the flipping process and its α_j nearest-neighbors (for the sake of clearness, we remember that the Erdös-Renyi, even if the amount of nearest neighbors is finite, still allows the model to be analyzed via mean-field techniques).

As for the selection rule according to which sites are extracted, there exist several different choices, ranging from purely random to deterministic. In several contexts (condensed-matter physics [10], sociology [8] etc.) unless no peculiar mechanisms or strategies are at work, the random updating seems to be the most plausible. In this case the probability that the current configuration **s** changes into $\mathbf{s}'_j$ due to the flip $s_j \to -s_j$, reads off as

$$\mathcal{P}^{\mathcal{R}}(\mathbf{s}, j; \mathbf{J}) = \frac{1}{N} p(\mathbf{s}, j, \mathbf{J}). \tag{2}$$

This algorithm mimics the coupling between the magnetic system with the thermal vibrations of the underlying structure, usually meant as heat-bath. The dynamics generated by $\mathcal{P}^{\mathcal{R}}$ has been intensively studied in the past (see e.g. [19]) and it has been shown to lead the system to the canonical equilibrium distribution, derived form the cost function $H_N(\sigma)$.

However, there exist several other different mechanisms yielding single flips [20]. For example, we can think

of a system endowed with a local thermostat where diffusing excitations affect the spin dynamics. Then it is reasonable to suppose that the spin-flips, and the related energy changes, have a diffusive character and that such a diffusion is biased towards those regions of the sample where energy variations are more likely to occur. Also in a social context a spin-flip can occur as a result of a direct interaction (phone call, mail exchange, etc.) between two neighbors and if agent i has just undergone an opinion-flip he will, in turn, prompt one out of his α_i neighbors to change opinion (opinion in social context plays the role of the spin orientation in material systems).

These aspects are neglected by traditional dynamics and can not be described by a random updating rule. In the past a different relaxation dynamics has been introduced and it is able to take into account these aspects, namely:

i. the selection rule exhibits a *diffusive character*: The sequence of sites selected for the updating can be thought of as the path of a random walk moving on the underlying structure.

ii. the diffusion is *biased*: The α_i neighbors are not equally likely to be chosen but, amongst the α_i neighbors, the most likely to be selected is also the most likely to undergo a spin-flip, namely the one which minimizes $\Delta H(\mathbf{s}, j, \mathbf{J})$.

Let us now formalize how the dynamics works. Our MC simulations are made up of successive steps [8]:

- Being i the newest updated spin/agent (at the very first step i is extracted randomly from the whole set of agents), we consider the corresponding set of nearest-neighbors defined as $\mathcal{N}_i = \{i_1, i_2, ..., i_{\alpha_i}\}$; we possibly consider also the subset $\tilde{\mathcal{N}}_i \subseteq \mathcal{N}_i$ whose elements are nearest-neighbors of i not sharing the same orientation/opinion: $j \in \tilde{\mathcal{N}}_i \Leftrightarrow j \in \mathcal{N}_i \wedge s_i s_j = -1$. Now, for any $j \in \mathcal{N}_i$ we compute the cost function variation $\Delta H(\mathbf{s}, j, \mathbf{J})$, which would result if the flip $s_j \to -s_j$ occurred; notice that $\Delta H(\mathbf{s}, j, \mathbf{J})$ involves not only the nearest-neighbors of i.

- We calculate the probability of opinion-flip for all the nodes in $\mathcal{N}_i$, hence obtaining $p(\mathbf{s}, i_1, \mathbf{J}), p(\mathbf{s}, i_2, \mathbf{J}), ..., p(\mathbf{s}, i_{\alpha_i}, \mathbf{J})$, where $p(\mathbf{s}, s'_j, \mathbf{J})$ (see Eq. 1), is the probability that the current configuration **s** changes due to a flip on the j-th site.

- We calculate the probability $\mathcal{P}^{\mathcal{S}}(\mathbf{s}; i, j; \mathbf{J})$ that among all possible α_i opinion-flips considered just the j-th one is realized; this is obtained by properly normalizing the $p(\mathbf{s}, j, \mathbf{J})$:

$$\mathcal{P}^{\mathcal{S}}(\mathbf{s}; i, j; \mathbf{J}) = \frac{p(\mathbf{s}, j, \mathbf{J})}{\sum_{k \in \mathcal{N}_i} p(\mathbf{s}, k, \mathbf{J})}. \tag{3}$$

We can possibly restrict the choice just to the set $\tilde{\mathcal{N}}_i$, hence defining $\tilde{\mathcal{P}}^{\mathcal{S}}(\mathbf{s}; i, j; \mathbf{J}) = p(\mathbf{s}, j, \mathbf{J})/\sum_{k \in \tilde{\mathcal{N}}_i} p(\mathbf{s}, k, \mathbf{J})$. Notice that, as we have verified, the use of $\tilde{\mathcal{P}}^{\mathcal{S}}$ instead of $\mathcal{P}^{\mathcal{S}}$ does not imply any qualitative change in the results.

- According to the normalized probability $\mathcal{P}^{\mathcal{S}}$ (see
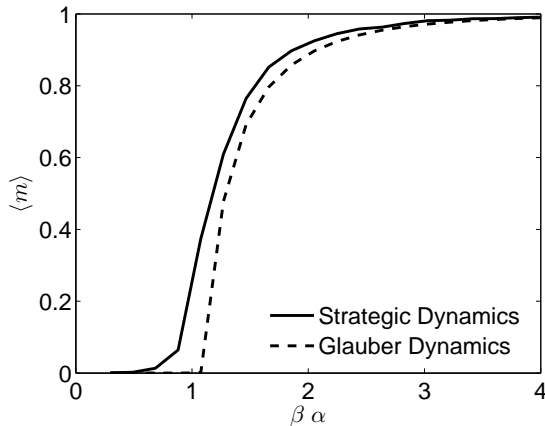
Figure 1: Critical behavior of the magnetization for the two dynamics (diffusive and standard Glauber) as a function of β and fixed $\alpha = 10$. The former dynamics gives rise to a critical point higher with respect to the latter.

Eq. 3), we extract randomly the node $\bar{i} \in \mathcal{N}_i$ and realize the opinion flip $s_{\bar{i}} \to -s_{\bar{i}}$.

- We set $\bar{i} \equiv i$ and we iterate the procedure.

Finally, it should be underlined that in this dynamics detailed balance is explicitly violated [9, 10]; indeed, its purpose is not to recover a canonical Bolzmann equilibrium but rather to model possible mechanism making the system evolve, and ultimately, to describe, at an effective level, the statics reached by a "socially plausible" dynamics for opinion spreading [8].

B. Equilibrium behavior

The diffusive dynamics was shown to be able to lead the system toward a well defined steady state and to recover the expected phase transition, although the critical temperature revealed was larger than the expected one [10]. Such results were also shown to be robust with respect to the the spin magnitude [9] and the underlying topology [8].

In particular, it was evidenced that there exists a critical value of the parameter $\beta_c^{\mathcal{S}}$ below which the system is spontaneously ordered. However, $\beta_c^{\mathcal{S}}$ was found to be appreciably smaller than the critical value $\beta_c(\alpha)$ expected for the canonical Ising model on a Erdos-Renyi random graph. Interestingly, it is not possible describe the system subjected to the diffusive dynamics by introducing an effective Hamiltonian obtained from Eq (II A) by a trivial rescaling. In fact, we consider the dependence on the magnetization displayed by the energies $E^{\mathcal{S}}(m)$ and $E(m)$, measured for system evolving according to the diffusive dynamics and to a traditional dynamics, respectively. As for the latter, from Eq. (II A) it is easy to see that $E = m^2$. As for $E^{\mathcal{S}}(m)$, we found that $E^{\mathcal{S}} < E$ for $0 < m < 1$, while $E^{\mathcal{S}} = E$ for $m = 0$ and $m = 1$. This is compatible with a power law behaviour $E^{\mathcal{S}} \sim m^{2+\epsilon}$. In

order to obtain an estimate for ϵ we measured the ratio $E^{\mathcal{S}}/E$ as a function of m; data are shown in the log-log scale plot of Fig. 2. Now, fitting procedures suggest that $\epsilon \approx 0.15$. Notice that the large deviations from the linear behaviour evidenced at small value of $\langle m \rangle$ are due to the fact that we are dividing two quantities close to zero.

So our idea is the following: as the critical temperature raises with p ranging from two to infinity, there can be a suitable real value of p that matches the critical temperature found numerically, and this is compatible with the plots of $E^{\mathcal{S}}(m)$ and $E(m)$. Notice that for $p > 2, p \in \mathbb{N}$, ferromagnetic transitions are no longer critical phenomena. At the critical line "jumps" in the magnetization and a latent heat do exist. However, if p is though of as real, for p slightly bigger than two, as suggested by our data, such a jump should be small (and it goes to zero whenever $p \to 2$), so it is difficult to check it just by looking at the magnetization as a function of α, β. To investigate this property, noting that the discontinuity of the entropy (latent heat) can be checked by looking at the compression rate of the phase space via the Shannon Theorem, we show the compression rate of the strategic dynamics R_s normalized by the compression rate of the Glauber one R, which offers another indication of the presence of a $p > 2$ behavior. In fig. 2, bottom panel, we plot $C = R_s/R$ as a function of the inverse temperature β; a minimum occurs just at the critical temperature of the system evolving according to the strategic dynamics.

III. STATICS OF MANY BODY INTERACTIONS

In this section we introduce a p-spin model to match the steady state recovered by the diffusive dynamics.

A. The diluted even mean field p-spin ferromagnet

In this section we exploit the properties of a diluted even p-spin ferromagnet: we restrict ourselves only to even values of p for mathematical convenience (the investigation with the cavities is much simpler), but, due to monotonicity of all the observables among p, there is no need to think at this as a real restriction (furthermore simulations on odd values of p confirm this statement). First of all, we define a suitable Hamiltonian acting on a Erdos-Renyi random graph, with connectivity α, made up by N agents $\sigma_i = \pm 1$, $i \in [1, N]$. Introducing p families $\{i_\nu^1\}, \{i_\nu^2\}, ..., \{i_\nu^p\}$ of i.i.d. random variables uniformly distributed on the previous interval, the Hamiltonian is given by the following expression

$$H_N(\sigma, \gamma(\alpha)) = - \sum_{\nu=1}^{k_{\gamma(\alpha)N}} \sigma_{i_\nu^1} \sigma_{i_\nu^2} ... \sigma_{i_\nu^p} \qquad (4)$$

where, reflecting the underlying network, k is a Poisson distributed random variable with mean value $\gamma(\alpha)N$.
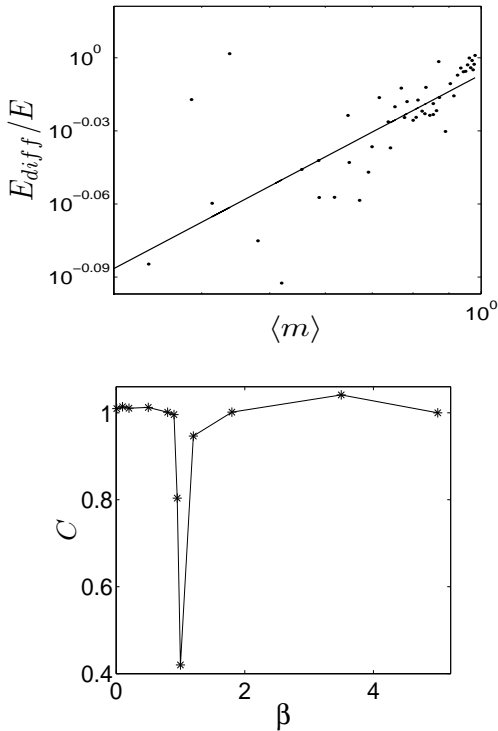
Figure 2: Up: We extrapolate from $m(\beta)$ and $E(\beta)$ the plot $E(m)$ for both the dynamics. It is worth noting that the diffusive dynamics deals to a curve living always "below" the one obtained by Glauber dynamics results. This strongly suggest the cooperation of $p > 2$ spins each interaction. Down: We show the phase space compression ratio $C = R_s/R$, where R_s is the phase space compression of the strategic dynamics and R stands for the standard Glauber one.

The relation among the coordination number α and γ is $\gamma \propto \alpha^{p-1}$: this will be easily understood a few line later by a normalization argument coupled with the high connectivity limit of this mean field model.

The quenched expectation of the model is given by the composition of the Poissonian average with the uniform one performed over the families $\{i_\nu\}$

$$\mathbf{E}[\cdot] = E_P E_i[\cdot] = \sum_{k=0}^{\infty} \frac{e^{-\gamma(\alpha)N}(\gamma(\alpha)N)^k}{k!N^p} \sum_{i_\nu^1,...,i_\nu^p}^{1,N} [\cdot]. \quad (5)$$

As they will be useful in our derivation, it is worth stressing the following properties of the Poisson distribution: Let us consider a function $g : \mathbb{N} \to \mathbb{R}$, and a Poisson variable k with mean γN, whose expectation is denoted by $\mathbb{E}$.
It is easy to verify that

$$\mathbb{E}[kg(k)] = \gamma N \mathbb{E}[g(k-1)] \quad (6)$$
$$\partial_{\gamma N}\mathbb{E}[g(k)] = \mathbb{E}[g(k+1) - g(k)] \quad (7)$$
$$\partial^2_{(\gamma N)^2}\mathbb{E}[g(k)] = \mathbb{E}[g(k+2) - 2g(k+1) + g(k)]. \quad (8)$$

The Hamiltonian (4), as written, has the advantage that it is the sum of (a random number of) i.i.d. terms. To see the connection to a more familiar Hamiltonian wrote in terms of adjacency tensor $A_{i_1,...,i_p}$, we note that the Poisson-distributed total number of bonds obeys $P_{\gamma N} = \gamma N + O(\sqrt{N})$ for large N. As there are N^p ordered spin p-plets $(i_1, ..., i_p)$, each gets a bond with probability $\sim \alpha/N$ for large N. The probabilities of getting two, three (and so on) bonds scale as $1/N^2, 1/N^3, \ldots$ so can be neglected. The probability of having a bond between any unordered p-plet of spins is $p!$ as large, i.e. $2\alpha/N$ for $p = 2$.
It is possible to show that our version of the Hamiltonian in fact is thermodynamically equivalent with the more familiar involving the explicit adjacency tensor $A_{i_1,...,i_p}$, by recall at first both the models

$$- H_N(\sigma; k) \sim -\hat{H}_N(\sigma) = \sum_{i_1,...,i_p}^{N} A_{i_1,...,i_p}\sigma_{i_1}...\sigma_{i_p}, \quad (9)$$

where k is a Poisson variable with mean $\gamma N \sim \alpha^{p-1}N$ and $A_{i_1,...,i_k}$ are all independent Poisson variables of mean $\gamma/N^{p-1} \sim (\alpha/N)^{p-1}$.
Then, it is enough to consider the streaming of the following interpolating free energy (whose structure proves the statement a priori by its thermodynamics meaning), depending on the real parameter $t \in [0, 1]$

$$\phi(t) = \frac{\mathbb{E}}{N} \ln \sum_\sigma e^{\beta(\sum_{\nu=1}^{k} \sigma_{i_\nu^1}...\sigma_{i_\nu^p} + \sum_{i_1,...,i_p}^{N} A_{i_1,...,i_p}\sigma_{i_1}...\sigma_{i_p})},$$

where k is a Poisson random variable with mean γNt and $A_{i_1,...,i_p}$ are random Poisson variables of mean $(1-t)\gamma/N^{p-1}$, so note that the two models separated are recovered in the two extremals of the interpolation (for $t = 0, 1$). By computing the t-derivative, we get

$$\frac{1}{\gamma} \frac{d\phi(t)}{dt} = \mathbb{E}\ln(1 + \Omega(\sigma_{i_0^1}...\sigma_{i_0^p})\tanh(\beta)) \quad (10)$$
$$- \frac{1}{N^p} \sum_{i_1,...,i_p}^{N} \ln(1 + \Omega(\sigma_{i_1}...\sigma_{i_p})\tanh(\beta)) = 0,$$

where the label 0 in i_0^k stands for a new spin, born in the derivative, accordingly to the Poisson property (7); as the i_0's are independent of the random site indices in the t-dependent Ω measure, the equivalence is proved.

Following a statistical mechanics approach, we know that the macroscopic behavior, versus the connectivity α and the inverse temperature β, is described by the following free energy density

$$A(\alpha,\beta) = \lim_{N\to\infty} A_N(\alpha,\beta) \quad (11)$$
$$= \lim_{N\to\infty} \frac{1}{N}\mathbf{E}\ln \sum_\sigma \exp(-\beta H_N(\sigma, \gamma(\alpha))).$$

The normalization constant can be checked by performing the expectation value of the cost function:

$$\mathbf{E}[H] = -\gamma N m^p$$

$$\mathbf{E}[H^2] - \mathbf{E}^2[H] = \gamma^2 N^2 \Big[(q_{12}^p - m^p) + O(\frac{1}{N}) \Big], \quad (12)$$

by which it is easy to see that the model is well defined, in particular it is linearly extensive in the volume. Then, in the high connectivity limit each agent interacts with all the others ($\alpha \sim N$) and, in the thermodynamic limit, $\alpha \to \infty$. Now, if $p = 2$ the amount of couples in the summation scales as $N(N-1)/2$ and, with $\gamma = 2\alpha$, a linear divergence of α (desired to get a finite ratio α/N for each coupling) provides the right scaling; if $p = 3$ the amount of triples scales as $N(N-1)(N-2)/3!$ and, with $\gamma = 3!\alpha^2$, again we find the right connectivity behavior. The generalization to every finite $p < N$ is straightforward.

B. Properties of the random diluted p-spin model

Let us now introduce the whole statistical mechanics machinery: we start by the partition function defined as

$$Z_N(\gamma, \beta) = \sum_{\{\sigma\}} e^{-\beta H_N(\sigma, \gamma)}, \quad (13)$$

the quenched pressure can be written as

$$A_N(\gamma, \beta) = \frac{1}{N} \mathbf{E} \ln Z_N(\gamma, \beta),$$

the Boltzmann state is given by

$$\omega(g(\sigma)) = \frac{1}{Z_N(\gamma, \beta)} \sum_{\{\sigma_N\}} g(\sigma) e^{-\beta H_N(\sigma, \gamma)}, \quad (14)$$

with its replicated form

$$\Omega(g(\sigma)) = \prod_s \omega^{(s)}(g(\sigma^{(s)})) \quad (15)$$

and the total average $\langle g(\sigma) \rangle$ is defined as

$$\langle g(\sigma) \rangle = \mathbf{E}[\Omega(g(\sigma))]. \quad (16)$$

Let us introduce further, as order parameters of the theory, the multi-overlaps

$$q_{1...n} = \frac{1}{N} \sum_{i=1}^{N} \sigma_i^{(1)} ... \sigma_i^{(n)}, \quad (17)$$

with a particular attention at the magnetization $m = q_1 = (1/N) \sum_{i=1}^{N} \sigma_i$ and to the two replica overlap $q_{12} = (1/N) \sum_{i=1}^{N} \sigma_i^1 \sigma_i^2$.

Before starting our free energy analysis, we want to point out also the connection among this diluted version and the fully connected counterpart.
Let us remember that the Hamiltonian of the fully connected p-spin model (FC) can be written as [21]

$$H_N^{FC}(\sigma) = \frac{p!}{2N^{p-1}} \sum_{1 \leq i_1 < ... < i_p \leq N} \sigma_{i_1} \sigma_{i_2} ... \sigma_{i_p}, \quad (18)$$

and let us consider the trial function $\hat{A}(t)$ defined as follows

$$\hat{A}(t) = \frac{1}{N} \mathbf{E} \ln \sum_{\sigma} \exp \Big[\beta \sum_{\nu}^{P_{\gamma N t}} \sigma_{i_\nu^1} \sigma_{i_\nu^2} ... \sigma_{i_\nu^p} + (1-t) \frac{\beta' N}{2} m^p \Big], \quad (19)$$

which interpolates between the fully connected p-spin model and the diluted one, such that for $t = 0$ only the fully connected survives, while the opposite happens for $t = 1$. Let us work out the derivative with respect to t to obtain

$$\partial_t \hat{A}(t) = (p-1)\alpha^{p-1} \ln \cosh(\beta) \quad (20)$$
$$- (p-1)\alpha^{p-1} \sum_n \frac{-1^n}{n} \theta^n \langle q_n^p \rangle - \frac{\beta'}{2} \langle m^p \rangle,$$

by which we see that the correct scaling, in order to recover the proper infinite connectivity model, is obtained when $\alpha \to \infty$, $\beta \to 0$ and $\beta' = 2(p-1)\alpha^{p-1} \tanh(\beta)$ is held constant.

Remark 1 *It is worth noting that in social modeling, usually, the role of the temperature is left, or at least coupled together, to the interaction strength J. As a consequence, in order to keep β' fixed, on different network dilution, the strength must be rescaled accordingly to*

$$J = \tanh^{-1} \Big(\frac{\beta'}{2(p-1)\alpha^{p-1}} \Big),$$

while, if present, an external field remains unchanged as it is a one-body term, like $h \sum_i^N \sigma_i$, unaffected by dilution.

Remark 2 *The dilute p-spin model reduces to the fully connected one, in the infinite connectivity limit, uniformly in the size of the system.*

C. The smooth cavity approach

In this section we want to look for an iterative expression of the free energy density by using a version of the cavity strategy [22, 23] that we briefly recall: the idea behind the cavity techniques [1, 24], which, for our purposes, resembles the stochastic stability approach [25, 26], is that information concerning the free energy density can be extrapolated when looking at the incremental extensive free energy given by the addition of a

spin.

In diluted models, this pasted spin changes also (infinitesimally in the high N limit) the connectivity and in evaluating how the free energy density varies conformingly with this, we are going to prove that it can be written in terms of a cavity function and such a connectivity shift. So the behavior of the system is encoded into these two terms. The latter is simpler as it is made only by stochastically stable terms (a proper definition of these terms will follow through the section). The former, instead, must be expressed via these terms and this will be achieved by iterative expansions.

At first we show how the free energy density can be decomposed via these two terms (the cavity function and the connectivity shift). Then, we analyze each term separately. We will see that they can be expressed by the momenta of the magnetization and of the multi-overlaps, weighted in a perturbed Boltzmann state, which recovers the standard one in the thermodynamic limit.

Theorem 3 *In the thermodynamic limit, the quenched pressure of the even p-spin diluted ferromagnetic model is given by the following expression*

$$A(\alpha, \beta) = \ln 2 - \frac{\alpha}{p-1}\frac{d}{d\alpha}A(\alpha, \beta) + \Psi(\alpha, \beta, t=1), \quad (21)$$

where the cavity function $\Psi(t, \alpha, \beta)$ is introduced as

$$\mathbf{E}\left[\ln \frac{\sum_{\{\sigma\}} e^{\beta \sum_{\nu=1}^{k_{\tilde{\gamma}N}} \sigma_{i_\nu^1}\sigma_{i_\nu^2}...\sigma_{i_\nu^p}} e^{\beta \sum_{\nu=1}^{k_{2\tilde{\gamma}t}} \sigma_{i_\nu^1}\sigma_{i_\nu^2}...\sigma_{i_\nu^{p-1}}}}{\sum_{\{\sigma\}} e^{\beta \sum_{\nu=1}^{k_{\tilde{\gamma}N}} \sigma_{i_\nu^1}\sigma_{i_\nu^2}...\sigma_{i_\nu^p}}}\right] =$$

$$\mathbf{E}\left[\ln \frac{Z_{N,t}(\tilde{\gamma}, \beta)}{Z_N(\tilde{\gamma}, \beta)}\right] = \Psi_N(\tilde{\gamma}, \beta, t), \quad (22)$$

with

$$\Psi(\gamma, \beta, t) = \lim_{N\to\infty} \Psi_N(\tilde{\gamma}, \beta, t). \quad (23)$$

For the sake of clearness, to avoid interrupting the paper with a long technical calculation, the proof of the Theorem is reported in the Appendix.

Thanks to the previous theorem, it is possible to figure out an expression for the pressure by studying the properties of the cavity function $\Psi(\alpha, \beta)$ and the connectivity shift $\partial_\alpha A(\alpha, \beta)$.

Using the properties of the Poisson distribution (6, 7), we can write

$$\frac{d}{d\alpha}A(\alpha, \beta) = \frac{(p-1)}{N}\alpha^{p-2}\frac{d}{d\gamma}\mathbf{E}\left[\ln Z_N(\gamma, \beta)\right] =$$

$$= (p-1)\alpha^{p-2}\mathbf{E}\left[\ln \sum_{\{\sigma\}} e^{\beta \sum_{\nu=1}^{k+1}\sigma_{i_\nu^1}...\sigma_{i_\nu^p}} - \right.$$

$$\left. - \ln \sum_{\{\sigma\}} e^{\beta \sum_{\nu=1}^{k}\sigma_{i_\nu^1}...\sigma_{i_\nu^p}}\right].$$

Now considering the relation (and definition)

$$e^{\beta \sigma_{i_0^1}...\sigma_{i_0^p}} = \cosh\beta + \sigma_{i_0^1}...\sigma_{i_0^p}\sinh\beta, \quad (24)$$

$$\theta = \tanh\beta, \quad (25)$$

we can write

$$\frac{d}{d\alpha}A(\alpha, \beta) = \quad (26)$$

$$(p-1)\alpha^{p-2}\left[\ln\cosh\beta + \mathbf{E}[\ln(1 + \omega(\sigma_{i_\nu^1}...\sigma_{i_\nu^p})\theta)]\right].$$

At the end, expanding the logarithm, we obtain

$$\frac{d}{d\alpha}A(\alpha, \beta) = (p-1)\alpha^{p-2}\ln\cosh\beta - \quad (27)$$

$$- (p-1)\alpha^{p-2}\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\theta^n\langle q_{1,...,n}^p\rangle.$$

With the same procedure it is possible to show that

$$\frac{d}{dt}\Psi(\tilde{\alpha}, \beta, t) = 2\tilde{\alpha}^{p-1}\ln\cosh\beta - \quad (28)$$

$$- 2\tilde{\alpha}^{p-1}\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\theta^n\langle q_{1,...,n}^{p-1}\rangle_{\tilde{\alpha},t},$$

by which,we see that even the cavity function, once integrated back against t the r.h.s. of eq.(28), can be expressed via all the order parameters of the model.

$$\Psi(\tilde{\alpha}, \beta, t) = 2\tilde{\alpha}^{p-1}\left(\ln\cosh(\beta) - \sum_{n=1}^{\infty}\frac{(-\theta)^n}{n}\int_0^t\langle q_{1,...,n}^{p-1}\rangle_{\tilde{\alpha},t}\right).$$

So we can understand the properties of the free energy by analyzing the properties of the order parameters: magnetization and overlaps, weighted in their extended Boltzmann state $\tilde{\omega}_t$.

Further, as we expect the order parameters being able to describe thermodynamics even in the true Boltzmann states ω, Ω [27], accordingly to the following definitions, we are going to show that *filled* order parameters (the ones involving even numbers of replicas) are stochastically stable, or in other words, are independent by the t-perturbation in the thermodynamic limit, while the others, not filled, become filled, again in this limit (such that for them $\omega_t \to \omega$ in the high N limit and thermodynamics is recovered). The whole is explained in the following definitions and theorems of this section.

Definition 4 *We define the t-dependent Boltzmann state $\tilde{\omega}_t$ as*

$$\tilde{\omega}_t(g(\sigma)) = \quad (29)$$

$$\frac{1}{Z_{N,t}(\gamma, \beta)}\sum_{\{\sigma\}}g(\sigma)e^{\beta \sum_{\nu=1}^{k_{\tilde{\gamma}N}}\sigma_{i_\nu^1}...\sigma_{i_\nu^p}+\beta\sum_{\nu=1}^{k_{2\tilde{\gamma}t}}\sigma_{i_\nu^1}...\sigma_{i_\nu^{p-1}}},$$

where $Z_{N,t}(\gamma, \beta)$ extends the classical partition function in the same spirit of the numerator of eq.(29) itself.

We see that the original Boltzmann state of a N-spin system is recovered by sending $t \to 0$, while, sending $t \to 1$ and gauging the spins, it is possible to build a Boltzmann state of a $N + 1$ spins, with a little shift both in α, β, which vanishes in the $N \to \infty$ limit, as prescribed in $(22, 23)$.

Coherently with the implication of thermodynamic limit (by which $A_{N+1}(\alpha, \beta) - A_N(\alpha, \beta) = 0$ for $N \to \infty$), we are going to define the *filled* overlap monomials and show their independence (stochastic stability) with respect to the perturbation encoded by the interpolating parameter t.

These parameters are already "good" order parameters describing the theory, while the others (the not-filled ones) must be expressed via the formers, and this will be achieved by expanding them with a suitably introduced streaming equation.

Definition 5 *We can split the class of monomials of the order parameters in two families:*

- *We define filled or equivalently stochastically stable those overlap monomials with all the replicas appearing an even number of times (i.e. q_{12}^2, m^2, $q_{12}q_{34}q_{1234}$).*

- *We define non-filled those overlap monomials with at least one replica appearing an odd number of times (i.e. q_{12}, m, $q_{12}q_{34}$).*

We are going to show three theorems that will play a guiding role for our expansion: as this approach has been deeply developed in similar contexts (as fully connected Ising and p-spin models [21, 23], fully connected spin glasses [22] or diluted ferromagnetic models [14, 28], which are the "boundaries" of the model of this paper) we will not show all the details of the proof, but we sketch them in the appendix as they are really intuitive. The interested reader will found a clear derivation in the appendix but can deepen this point by looking at the original works.

Theorem 6 *In the thermodynamic limit and setting $t = 1$ we have*

$$\tilde{\omega}_{N,t}(\sigma_{i_1}\sigma_{i_2}...\sigma_{i_n}) = \tilde{\omega}_{N+1}(\sigma_{i_1}\sigma_{i_2}...\sigma_{i_n}\sigma_{N+1}^n). \quad (30)$$

Theorem 7 *Let Q_{ab} be a not-filled monomial of the overlaps (this means that $q_{ab}Q_{ab}$ is filled). We have*

$$\lim_{N \to \infty} \lim_{t \to 1} \langle Q_{ab} \rangle_t = \langle q_{ab}Q_{ab} \rangle, \quad (31)$$

(examples:
for $N \to \infty$ we get $\langle m_1 \rangle_t \to \langle m_1^2 \rangle$, $\langle q_{12} \rangle_t \to \langle q_{12}^2 \rangle$).

Theorem 8 *In the $N \to \infty$ limit, the averages $\langle \cdot \rangle$ of the filled polynomials are t-independent in β average.*

D. Properties of the free energy

In this section we are going to address various points: at first we work out the constraints that the model must fulfil, which are in agreement both with a self-averaging behavior of the magnetization as with the replica-symmetric behavior of the multi-overlaps [29]; then we write an iterative expression for the free energy density and its links with known models as diluted ferromagnets ($p \to 2$ limit) and fully connected p-spin models ($\alpha \to \infty$ limit).

With the following definition

$$\tilde{\beta} = 2(p-1)\tilde{\alpha}^{p-1}\theta \quad (32)$$
$$= 2(p-1)\alpha^{p-1}\frac{N}{N+1}\theta \stackrel{N \to \infty}{\longrightarrow} 2(p-1)\alpha^{p-1}\theta = \beta',$$

we show (and prove in the appendix) the streaming of replica functions, by which not filled multi-overlaps can be expressed via filled ones.

Proposition 9 *Let F_s be a function of s replicas. Then the following streaming equation holds*

$$\frac{\partial \langle F_s \rangle_{t,\tilde{\alpha}}}{\partial t} = \tilde{\beta}\Big[\sum_{a=1}^s \langle F_s m_a^{p-1} \rangle_{t,\tilde{\alpha}} - s\langle F_s m_{s+1}^{p-1} \rangle_{t,\tilde{\alpha}}\Big] \quad (33)$$
$$+ \tilde{\beta}\theta\Big[\sum_{a<b}^{1,s} \langle F_s q_{a,b}^{p-1} \rangle_{t,\tilde{\alpha}} - s\sum_{a=1}^s \langle F_s q_{a,s+1}^{p-1} \rangle_{t,\tilde{\alpha}}$$
$$+ \frac{s(s+1)}{2!}\langle F_s q_{s+1,s+2}^{p-1} \rangle_{t,\tilde{\alpha}}\Big] + O(\theta^3).$$

Remark 10 *We stress that, at the first two level of approximation presented here, the streaming has has the structure of a θ-weighted linear sum of the Curie-Weiss streaming (θ^0 term) [23] and the Sherrington-Kirkpatrick streaming (θ^1 term) [22], conferring a certain degree of independence by the kind of quenched noise (frustration or dilution) to mathematical structures of disordered systems.*

It is now immediate to obtain the linear order parameter constraints (often known as Aizenman-Contucci polynomials [23, 25, 28]) of the theory: in fact, the generator of such a constraint is the streaming equation when applied on each filled overlap monomial (or equivalently it is possible to apply the streaming on a not-filled one and then gauge the obtained expression; for the sake of clearness both the methods will be exploited, the former for q_2 and the latter for m).

As examples, dealing with the terms m^{p-1} and q_2^{p-1}, it is straightforward to check that

$$0 = \lim_{N \to \infty} \frac{\partial \langle m_N^{p-1} \rangle_{t,\tilde{\alpha}}}{\partial t} = \tilde{\beta}\Big(\langle m_1^{2(p-1)} \rangle - \langle m_1^{p-1} \rangle^2\Big)$$
$$+ \tilde{\beta}\theta\Big(\langle m_1^{p-1}q_2^{p-1} \rangle - \langle m_1^{p-1} \rangle\langle q_2^{p-1} \rangle\Big) + O(\theta^3),$$

then, by gauging the above expression, in the thermodynamic limit, (as $\lim_{N\to\infty}\langle m_N^{p-1}\rangle_t \to \langle m^p\rangle$), we get

$$\left((\langle m_1^{2p}\rangle - \langle m_1^p\rangle^2) + \theta(\langle q_2^{2p}\rangle - \langle q_2^p\rangle^2)\right) = 0 \quad \forall\theta \in \mathcal{R}^+,$$

which, as holds for every θ suggests both self-averaging for the energy (by which all the linear constraints can be derived[28]) due to the first term, as well as replica symmetric behavior of the two replica overlap due to the last one.

In the same way, the contribution of the $\langle q_2^2\rangle$ generator is

$$0 = \left((\langle q_{12}^{p-1}m_1^{p-1}\rangle + \langle q_{12}^{p-1}m_2^{p-1}\rangle - 2\langle q_{12}^{p-1}m_3^{p-1}\rangle) + \right.$$
$$\left. + \theta(\langle q_{12}^{p-1}q_{12}^{p-1}\rangle - 4\langle q_{12}^{p-1}q_{23}^{p-1}\rangle + 3\langle q_{12}^{p-1}q_{34}^{p-1}\rangle)\right),$$

which shows replica symmetric behavior of the magnetization by the first term and the classical Aizenman-Contucci relations [25, 28] by the latter.

Furthermore, turning now our attention to the free energy, it is easy to see that the streaming equation allows to generate all the desired overlap functions coupled to every well behaved F_s. In this way, if F_s is a not filled overlap, we can always expand recursively it into a filled one, with the only price to pay given by the θ order that has to be reached or, which is equivalent, the number of derivatives that have to be performed.

Let us now remember the t-derivative of the cavity function (28), showing explicitly the first two terms of its expansion

$$\frac{d}{dt}\Psi(\tilde{\alpha},\beta,t) = 2\tilde{\alpha}^{p-1}\ln\cosh\beta + \tilde{\beta}\langle m_1^{p-1}\rangle_{\tilde{\alpha},t} - \qquad (34)$$
$$- \frac{\tilde{\beta}}{2}\theta\langle q_{12}^{p-1}\rangle_{\tilde{\alpha},t} - 2\tilde{\beta}^{p-1}\sum_{n=3}^{\infty}\frac{-1^n\theta^n}{n}\langle q_{1,...,n}^{p-1}\rangle_{\tilde{\alpha},t}.$$

As derivative of fillable terms involve filled ones, we can arrive to an analytical form of $\Psi(\alpha,\beta)$ if we calculate it as the t-integral of its t-derivative, together with the obvious relation $\Psi(t=0) = 0$. So, if we apply the streaming equation machinery to the overlaps constituting equation (34), we are able to fill them and to get them free from the t-dependence in the thermodynamic limit. In this way we are allowed to bring them out from the final t-integral.

In fact, without gauging (so, not only in the ergodic regime, where symmetries are preserved), we can expand the streaming of $\langle m^{p-1}\rangle_t$:

$$\frac{d\langle m_1^{p-1}\rangle_t}{dt} = \tilde{\beta}\left[\langle m_1^{2(p-1)}\rangle - \langle m_1^{p-1}m_2^{p-1}\rangle_t\right] +$$
$$- \tilde{\beta}\theta\left[\langle m_1^{p-1}q_{12}^{p-1}\rangle_t - \langle m_1^{p-1}q_{23}^{p-1}\rangle_t\right] + O(\theta^2).$$

We can note the presence of the filled monomial $\langle m_1^{2(p-1)}\rangle$, whose t-dependence has been omitted explicitly to underly its stochastic stability, while the overlaps

$\langle m_1^{p-1}m_2^{p-1}\rangle_t$ and $\langle m_1^{p-1}q_{12}^{p-1}\rangle_t$ can be saturated in two steps of streaming. This will be sufficient, wishing to have a fourth order expansion for the cavity function. We now derive these two functions and apply the same scheme to all the overlaps that appear and that have to be necessary filled in order to obtain the desired result.

$$\frac{d\langle m_1^{p-1}m_2^{p-1}\rangle_t}{dt} =$$
$$2\tilde{\beta}\left[\langle m_1^{2(p-1)}m_2^{p-1}\rangle_t - \langle m_1^{p-1}m_2^{p-1}m_3^{p-1}\rangle_t\right] +$$
$$\theta\tilde{\beta}\left[\langle m_1^{p-1}m_2^{p-1}q_{12}^{p-1}\rangle - 4\langle m_1^{p-1}m_2^{p-1}q_{13}^{p-1}\rangle_t + \right.$$
$$\left. 3\langle m_1^{p-1}m_2^{p-1}q_{34}^{p-1}\rangle_t\right], \qquad (35)$$

$$\frac{d\langle m_1^{2(p-1)}m_2^{p-1}\rangle_t}{dt} = \qquad (36)$$
$$2\tilde{\beta}\left[\langle m_1^{2(p-1)}m_2^{2(p-1)}\rangle_t\right] + \tilde{\beta}\left[\text{unfilled terms}\right] + O(\theta^2).$$

Integrating back in t and neglecting higher order terms we have

$$\langle m_1^{2(p-1)}m_2^{p-1}\rangle_t = \tilde{\beta}\left[\langle m_1^{2(p-1)}m_2^{2(p-1)}\rangle\right]t, \qquad (37)$$

and we can write

$$\langle m_1^{p-1}m_2^{p-1}\rangle_t = \qquad (38)$$
$$\tilde{\beta}\theta\langle m_1^{p-1}m_2^{p-1}q_{12}^{p-1}\rangle t + \tilde{\beta}^2\langle m_1^{2(p-1)}m_2^{2(p-1)}\rangle t^2.$$

Let us take a look now at the other overlap $\langle m_1^{p-1}q_{12}^{p-1}\rangle_t$:

$$\frac{d\langle m_1^{p-1}q_{12}^{p-1}\rangle_t}{dt} = \tilde{\beta}\left[\langle m_1^{2(p-1)}q_{12}^{p-1}\rangle_t - \langle m_1^{p-1}m_2^{p-1}q_{12}^{p-1}\rangle_t \right.$$
$$\left. - 2\langle m_1^{p-1}m_2^{p-1}m_3^{p-1}q_{12}^{p-1}\rangle_t\right] + O(\theta^2)(39)$$

that gives

$$\langle m_1^{p-1}q_{12}^{p-1}\rangle_t = \tilde{\beta}\langle m_1^{p-1}m_2^{p-1}q_{12}^{p-1}\rangle t + O(\theta^2). \qquad (40)$$

At this point we can write for $\langle m_1^{p-1}\rangle_{t,\tilde{\alpha}}$ (and consequently for $\langle q_{12}^{p-1}\rangle_{t,\tilde{\alpha}}$)

$$\langle m_1^{p-1}\rangle_{t,\tilde{\alpha}} = \tilde{\beta}\langle m_1^{2(p-1)}\rangle t - \frac{\tilde{\beta}^3}{3}\langle m_1^{2(p-1)}m_2^{2(p-1)}\rangle t^3$$
$$- \tilde{\beta}^2\theta\langle m_1^{p-1}m_2^{p-1}q_{12}^{p-1}\rangle t^2 + O(\theta^3),$$
$$\langle q_{12}^{p-1}\rangle_{t,\tilde{\alpha}} = \tilde{\beta}\theta\langle q_{12}^{2(p-1)}\rangle t + \tilde{\beta}^2\langle m_1^{p-1}m_2^{p-1}q_{12}^{p-1}\rangle t^2 + O(\theta^3).$$

With these relations, eq. (34) becomes

$$\frac{d}{dt}\Psi_N(\alpha,\beta,t) = 2\alpha^{p-1}\ln\cosh\beta + \tilde{\beta}^2\langle m_1^{2(p-1)}\rangle t$$
$$- \frac{\tilde{\beta}^2\theta^2}{2}\langle q_{12}^{2(p-1)}\rangle t - \frac{3\tilde{\beta}^3\theta}{2}\langle m_1^{p-1}m_2^{p-1}q_{12}^{p-1}\rangle t^2$$
$$- \frac{\tilde{\beta}^4}{3}\langle m_1^{2(p-1)}m_2^{2(p-1)}\rangle t^3 + O(\theta^5),$$

which ultimately allows us to write an iterated expressions for Ψ evaluated at $t = 1$

$$\Psi_N(\alpha, \beta, 1) = \tag{41}$$

$$2\alpha^{p-1}\ln\cosh\beta + \frac{\tilde{\beta}^2}{2}\langle m_1^{2(p-1)}\rangle - \frac{\tilde{\beta}^2\theta^2}{4}\langle q_{12}^{2(p-1)}\rangle -$$

$$\frac{\tilde{\beta}^3\theta}{2}\langle m_1^{p-1}m_2^{p-1}q_{12}^{p-1}\rangle - \frac{\tilde{\beta}^4}{12}\langle m_1^{2(p-1)}m_2^{2(p-1)}\rangle t^3 + O(\theta^5).$$

Overall the result we were looking for, a polynomial form of the free energy, reads off as

$$A(\alpha, \beta) = \ln 2 + \alpha^{p-1}\ln\cosh\beta + \tag{42}$$

$$+ \frac{\beta'}{2}\Big(\beta'\langle m^{2(p-1)}\rangle - \langle m^p\rangle\Big) + \tag{43}$$

$$+ \frac{\beta'\theta}{4}\Big(\beta'\theta\langle q_{12}^{2(p-1)}\rangle - \langle q_{12}^p\rangle\Big) + O(\theta^5).$$

Now, several conclusions can be addressed from the expression (42):

Remark 11 *At first let us note that, by constraining the interaction to be pairwise, critical behavior should arise [27]. Coherently, we see that for $p = 2$ we can write the free energy expansion as*

$$A(\alpha, \beta)_{p=2} = \ln 2 + \alpha\ln\cosh(\beta) - \frac{\beta'}{2}(1-\beta')\langle m^2\rangle - \frac{\beta'\theta}{4}\langle q_2^2\rangle,$$

which coincides with the one of the diluted ferromagnet [14] and displays criticality at $2\alpha\theta = 1$, where the coefficient of the second order term vanishes, in agreement with previous results[14].

Remark 12 *The free energy density of the fully connected p-spin model is [21] $A(\beta') = \ln 2 + \ln\cosh(\beta m^{p-1}) - (\beta/2)m^p$, which coincides with the expansion (42) in the limit of $\alpha \to \infty$ and $\beta \to 0$ with $\beta' = 2(p-1)\alpha^{p-1}\theta$.*

Remark 13 *It is worth noting that the connectivity no longer plays a linear role in contributing to the free energy density, as it does happen for the diluted two body models [14, 17], but, in complete generality as $p-1$. This is interesting in social networks, where, for high values of coordination number it may be interesting developing strategies with more than one exchange [19].*

Remark 14 *As from the numerics discussed in Section 2, the effective p turns out to be 2.15; to check for consistency with the analytic results, we note that close to criticality ($p = 2$), the temperature for the phase transition is given by $\beta_c = \tanh^{-1}(1/2\alpha^{p-1}) = \tanh^{-1}(1/2\alpha)$. Now, for $p = 2.15$ this expression becomes $\beta_c \sim \tanh^{-1}(1/2\alpha^{p-1}) = \tanh^{-1}(1/2\alpha^{1.15})$: The ratio among the two expressions, when evaluated for $\alpha = 10$ gets approximately 1.4, in agreement with data depicted in Fig. 1.*

E. Numerics

We now analyze the system described in this section, from the numerical point of view by performing extensive Monte Carlo simulations. Within this approach it is more convenient to use the second Hamiltonian introduced (see eq.(9)):

$$\hat{H}_N(\sigma) = -\sum_{i_i}^{N}\sigma_{i_1}\sum_{i_2<i_3<...<i_p=1}^{N}A_{i_1,...,i_p}\sigma_{i_2}\sigma_{i_3}...\sigma_{i_p}. \tag{44}$$

The product between the elements of the adjacency tensor ensures that the $p - 1$ spins considered in the second sum are joined by a link with i_1.

The evolution of the magnetic system is realized by means of a single spin flip dynamics based on the Metropolis algorithm [19]. At each time step a spin is randomly extracted and updated whenever its coordination number is larger than $p - 1$. For α large enough (at least above the percolation threshold, as obviously it is the case for the results found previously) and $p = 3, 4$ this condition is generally verified. The updating procedure for a spin σ_i works as follows: Firstly we calculate the energy variation ΔE_i due to a possible spin flip, which for $p = 3$ and $p = 4$ reads respectively

$$\Delta E_i = 2\sigma_i\sum_{j<k=1}^{N}A_{i,j}A_{i,k}\sigma_j\sigma_k, \tag{45}$$

$$\Delta E_i = 2\sigma_i\sum_{j<k<w=1}^{N}A_{i,j}A_{i,k}A_{i,w}\sigma_j\sigma_k\sigma_w. \tag{46}$$

Now, if $\Delta E_i < 0$, the spin-flip $\sigma_i \to -\sigma_i$ is realized with probability 1, otherwise it is realized with probability $e^{-\beta\Delta E}$.

The case $p = 3$ has been studied in details and some insight is provided also for the case $p = 4$, while for $p = 2$ we refer to [14]. Our investigations concern two main issues:
- the existence of a phase transition and its nature
- the existence of a proper scaling for the temperature as the parameter α is tuned.

As for the first point, we measured the so-called Binder cumulants defined as follows:

$$G_N(T) \equiv 1 - \frac{\langle m^4\rangle_N}{3\langle m^2\rangle_N^2}, \tag{47}$$

where $\langle\cdot\rangle_N$ indicates the average obtained for a system of size N [30]. The study of Binder cumulants is particularly useful to locate and catalogue the phase transition. In fact, in the case of continuous phase transitions, $G_N(T)$ takes a universal positive value at the critical point T_c, namely all the curves obtained for different system sizes N cross each other. On the other hand, for a first-order transition $G_N(T)$ exhibits a minimum at T_{min}, whose magnitude diverges as N. Moreover, a crossing point at
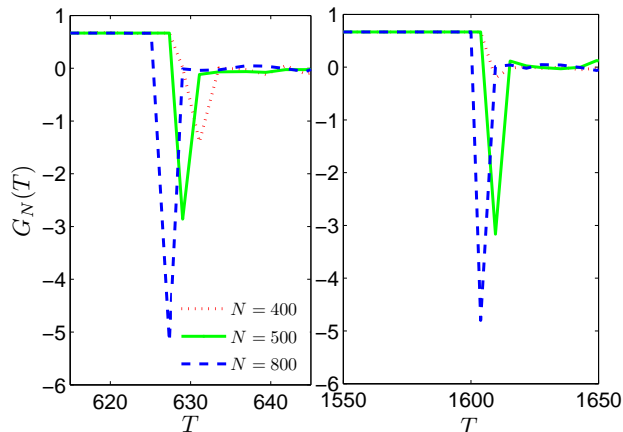
Figure 3: Binder cumulants $G_L(T)$ for systems of different size N, as shown in the legend, and connectivity $\bar{\alpha} = 50$ (left panel) and $\bar{\alpha} = 80$ (right panel).
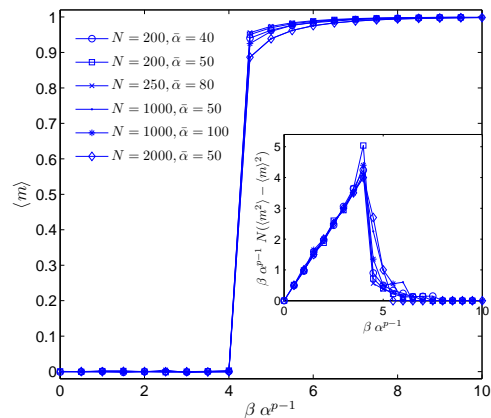

Figure 4: Magnetization (main figure) and its normalized fluctuations (inset) for systems of different sizes and different dilution as a function of $\beta\,\alpha^{p-1}$. The collapse of all the curves provides a strong evidence for the scaling of the temperature.

T_{cross} can be as well detected when curves pertaining to different sizes N are considered [31]. Now, T_{min} and T_{cross} scale as $T_{min} - T_c \propto N^{-1}$ and $T_{cross} - T_c \propto N^{-2}$, respectively.

In Fig. 3 we show data for $G_N(T)$ obtained for systems of different sizes ($N = 400$, $N = 500$, and $N = 800$) but equal connectivity ($\alpha = 50$ and $\alpha = 80$, respectively) as a function of the temperature T. The existence of a minimum is clear and it occurs for $T \approx 625$ and $T \approx 1600$. Similar results are found also for $p = 4$ and they all highlight the existence of a first-order phase transition at a temperature which depends on the connectivity α.

In order to deepen the role of connectivity in the evolution of the system we measure the macroscopic observable $\langle m \rangle$ and its (normalized) fluctuations $\langle m^2 \rangle - \langle m \rangle^2$, studying their dependence on the temperature β and on the dilution α. Data for different choices of size and dilution are shown in Figure 4.

The profile of the magnetization, with an abrupt jump, and the correspondent peak found for its fluctuations confirm the existence of a first order phase transition at a well defined temperature T_c whose value depends on the dilution α. More precisely, by properly normalizing the temperature in agreement with analytical results, namely $\tilde{\beta} \equiv \beta\,\bar{\alpha}^{p-1}$ we found a very good collapse of all the curves considered. Hence, we can confirm that the temperature scales like α^{p-1}. Moreover our data provide a very clear hint suggesting that the critical temperature can be written as $T = \alpha^{p-1}/4$.

IV. CONCLUSIONS

We have seen in this work a numerical and analytical study of a a diffusive strategy modeling group competition. The study is performed on a standard random graph in a ferromagnetic mean field model. The steady state equilibria show a *shifted-temperature* phenomena: the Hamiltonian equilibria differ from the stationary one. We propose an *effective* Hamiltonian description of the last by means of an analytically continued random p-spin model which turns out to provide a good approximation for the steady state for $p = 2.15$. The analysis of the free energy functional suggests moreover that the connectivity gives a non linear contribution to the equilibrium state. This open the possibility to consider extended models in which the connectivity is a dynamical variable for the system and is chosen thermodynamically to maximize stability. We plan to return on that topic in future works.

V. APPENDIX: ANALYTICAL PROOFS

In this section the proofs of al the Theorems and the Proposition 1 are reported.

Proof of Theorem 3
Bridging a system made of by $N+1$ spins with one made of by N spins implies the definition of rescaled γ, α parameters, accordingly to [14, 28]

$$\tilde{\gamma} = \gamma \frac{N}{N+1} \qquad \overset{N\to\infty}{\longrightarrow} \quad \gamma \qquad (48)$$

$$\tilde{\alpha} = \alpha \left[\frac{N}{N+1} \right]^{\frac{1}{p-1}} \qquad \overset{N\to\infty}{\longrightarrow} \quad \alpha. \qquad (49)$$

We have, in distribution, the Hamiltonian of a system

made of $N+1$ particles writable as

$$H_{N+1}(\sigma,\gamma) = -\sum_{\nu=1}^{k_{\gamma(N+1)}} \sigma_{i_\nu^1}\sigma_{i_\nu^2}...\sigma_{i_\nu^p} \sim \qquad (50)$$

$$-\sum_{\nu=1}^{k_{\tilde\gamma N}} \sigma_{i_\nu^1}\sigma_{i_\nu^2}...\sigma_{i_\nu^p} - \sum_{\nu=1}^{k_{2\tilde\gamma}} \sigma_{i_\nu^1}\sigma_{i_\nu^2}...\sigma_{i_\nu^{p-1}}\sigma_{N+1},$$

that we may rewrite as

$$H_{N+1}(\sigma,\gamma) = H_N(\sigma,\tilde\gamma) + \hat H_N(\sigma,2\tilde\gamma). \qquad (51)$$

Following the above decomposition, let us consider the partition function of the same $N+1$ spin model and let us introduce the gauge transformation $\sigma_i \rightarrow \sigma_i\sigma_{N+1}$ which is a symmetry of the Hamiltonian known as *spin-flip*.

$$Z_{N+1}(\gamma,\beta) \sim \sum_{\{\sigma_{N+1}\}} e^{-\beta H_N(\sigma,\tilde\gamma)-\beta\hat H_N(\sigma,\tilde\gamma)\sigma_{N+1}} = \qquad (52)$$

$$= \sum_{\{\sigma_{N+1}\}} e^{\beta H_N(\sigma,\tilde\gamma)+\beta\sum_{\nu=1}^{k_{2\tilde\gamma}}\sigma_{i_\nu^1}...\sigma_{i_\nu^{p-1}}\sigma_{N+1}} =$$

$$= 2\sum_{\{\sigma_N\}} e^{\beta\sum_{\nu=1}^{k_{\tilde\gamma N}}\sigma_{i_\nu^1}...\sigma_{i_\nu^p}+\beta\sum_{\nu=1}^{k_{2\tilde\gamma}}\sigma_{i_\nu^1}...\sigma_{i_\nu^{p-1}}} =$$

$$= 2Z_N(\tilde\gamma,\beta)\tilde\omega(e^{-\beta\hat H_N}),$$

where the new Boltzmann state $\tilde\omega$, and its replicated $\tilde\Omega$, are introduced as

$$\tilde\omega(g(\sigma)) = \frac{\sum_{\{\sigma_N\}} g(\sigma)e^{-\beta H_N(\tilde\gamma,\sigma)}}{\sum_{\{\sigma_N\}} e^{-\beta H_N(\tilde\gamma,\sigma)}}, \qquad (53)$$

$$\tilde\Omega(g(\sigma)) = \prod_i \tilde\omega^{(i)}(g(\sigma^{(i)})). \qquad (54)$$

To continue the proof we now take the logarithm of both sides of the last expression in eq. (52), apply the expectation $\mathbf{E}$ and subtract the quantity $\mathbf{E}[\ln Z_{N+1}(\tilde\gamma,\beta)]$. We obtain

$$\mathbf{E}[\ln Z_{N+1}(\gamma,\beta)] - \mathbf{E}[\ln Z_{N+1}(\tilde\gamma,\beta)] =$$

$$\ln 2 - \mathbf{E}[\ln \frac{Z_{N+1}(\gamma,\beta)}{Z_N(\tilde\gamma,\beta)}] + \Psi_N(\tilde\gamma,\beta,1), \qquad (55)$$

The left hand side gives

$$\mathbf{E}[\ln Z_{N+1}(\gamma,\beta)] - \mathbf{E}[\ln Z_{N+1}(\tilde\gamma,\beta)] = \qquad (56)$$

$$= (\gamma-\tilde\gamma)\frac{d}{d\gamma}\mathbf{E}[\ln Z_{N+1}(\gamma,\beta)]|_{\gamma=\tilde\gamma} =$$

$$= \gamma\frac{1}{N+1}\frac{d}{d\gamma}\mathbf{E}[\ln Z_{N+1}(\gamma,\beta)]|_{\gamma=\tilde\gamma} =$$

$$= \gamma\frac{d}{d\gamma}A_{N+1}(\gamma,\beta). \qquad (57)$$

Considering the α dependence of γ, we have

$$\partial_\gamma \propto \frac{1}{(p-1)\alpha^{p-2}}\partial_\alpha \quad \Rightarrow \quad \gamma\frac{d}{d\gamma}A \propto \frac{\alpha}{p-1}\frac{d}{d\alpha}A,$$

where the symbol $\propto$ instead of $=$ reflects the arbitrariness by which we include the $p!$ term, multiplying α, inside the definition of γ, or directly in α.

Performing now the thermodynamic limit, we see that at the right hand side we have

$$\lim_{N\rightarrow\infty} \mathbf{E}[\ln \frac{Z_{N+1}(\alpha,\beta)}{Z_N(\tilde\alpha,\beta)}] \longrightarrow A(\alpha,\beta) \qquad (58)$$

and the theorem is proved $\square$.

Proofs of Theorems 6,7,8

In this sketch we are going to show how to get Theorem (6) in some details; It automatically has as a corollary Theorem (7) which ultimately gives, as a simple consequence when applied on filled monomials, Theorem(8). Let us assume for a generic overlap correlation function Q, of s replicas, the following representation

$$Q = \prod_{a=1}^s \sum_{i_l^a} \prod_{l=1}^{n^a} \sigma_{i_l^a}^a I(\{i_l^a\})$$

where a labels the replicas, the internal product takes into account the spins (labeled by l) which contribute to the a-part of the overlap $q_{a,a'}$ and runs to the number of time that the replica a appears in Q. The external product takes into account all the contributions of the internal one and the I factor fixes the constraints among different replicas in Q; so, for example, $Q = q_{12}q_{23}$ can be decomposed in this form noting that $s = 3$, $n^1 = 1, n^2 = 2$, $I = N^{-2}\delta_{i_1^1,i_1^3}\delta_{i_1^2,i_2^3}$, where the δ functions fixes the links between replicas $1,3 \rightarrow q_{1,3}$ and $2,3 \rightarrow q_{2,3}$. The averaged overlap correlation function is

$$\langle Q\rangle_t = \mathbf{E}\sum_{i_l^a} I(\{i_l^a\}) \prod_{a=1}^s \omega_t(\prod_{l=1}^{n^a} \sigma_{i_l^a}^a).$$

Now if Q is a fillable polynomial, and we evaluate it at $t = 1$, let us decompose it, using the factorization of the ω state on different replica, as

$$\langle Q\rangle_t = \mathbf{E}\sum_{i_l^a,i_l^b} I(\{i_l^a\},\{i_l^b\}) \prod_{a=1}^u \omega_a(\prod_{l=1}^{n^a} \sigma_{i_l^a}^a) \prod_{b=u}^s \omega_b(\prod_{l=1}^{n^b} \sigma_{i_l^b}^b),$$

where u stands for the number of the unfilled replicas inside the expression of Q. So we split the measure Ω into two different subset ω_a and ω_b: in this way the replica belonging to the b subset are always in even number, while the ones in the a subset are always odds. Applying the gauge $\sigma_i^a \rightarrow \sigma_i^a\sigma_{N+1}^a, \forall i \in (1,N)$ the even measure is unaffected by this transformation ($\sigma_{N+1}^{2n} \equiv 1$) while the odd measure takes a σ_{N+1} inside the Boltzmann measure.

$$\langle Q\rangle = \qquad (59)$$

$$\sum_{i_l^a,i_l^b} I(\{i_l^a\},\{i_l^b\}) \prod_{a=1}^u \omega(\sigma_{N+1}^a\prod_{l=1}^{n^a} \sigma_{i_l^a}^a) \prod_{b=u}^s \omega(\sigma_{N+1}^b\prod_{l=1}^{n^b} \sigma_{i_l^b}^b).$$

At the end we can replace in the last expression the index $N + 1$ of σ_{N+1} by k for any $k \neq \{i_l^a\}$ and multiply by one as $1 = N^{-1} \sum_{k=0}^{N}$. Up to orders $O(1/N)$, which go to zero in the thermodynamic limit, we have the proof.

It is now immediate to understand that Theorem (6) on a fillable overlap monomial has the effect of multiplying it by its missing part to be filled (Theorem 7), while it has no effect if the overlap monomial is already filled (Theorem 8). $\square$

Proof of Proposition 9
The proof works by direct calculation:

$$\frac{\partial \langle F_s \rangle_{t,\tilde{\alpha}}}{\partial t} = \tag{60}$$

$$\frac{\partial \mathbf{E}}{\partial t} \left[\frac{\sum_{\{\sigma\}} F_s e^{\sum_{a=1}^{s}(\beta \sum_{\nu=1}^{k_{\tilde{\gamma}}N} \sigma_{i_\nu^1}^a...\sigma_{i_\nu^p}^a + \beta \sum_{\nu=1}^{k_2 \tilde{\gamma} t} \sigma_{i_\nu^1}^a...\sigma_{i_\nu^{p-1}}^a)}}{\sum_{\{\sigma\}} e^{\sum_{a=1}^{s}(\beta \sum_{\nu=1}^{k_{\tilde{\gamma}}N} \sigma_{i_\nu^1}^a...\sigma_{i_\nu^p}^a + \beta \sum_{\nu=1}^{k_2 \tilde{\gamma} t} \sigma_{i_\nu^1}^a...\sigma_{i_\nu^{p-1}}^a)}} \right] =$$

$$2\tilde{\alpha}^{p-1} \mathbf{E} \left[\frac{\tilde{\Omega}_t(F_s e^{\sum_{a=1}^{s}(\beta \sigma_{i_0^1}^a...\sigma_{i_0^{p-1}}^a)})}{\tilde{\Omega}_t(e^{\sum_{a=1}^{s}(\beta \sigma_{i_0^1}^a...\sigma_{i_0^{p-1}}^a)})} \right] - 2\tilde{\alpha}^{p-1} \langle F_s \rangle_{t,\tilde{\alpha}} =$$

$$2\tilde{\alpha} \mathbf{E} \left[\frac{\tilde{\Omega}_t(F_s \Pi_{a=1}^{s}(\cosh\beta + \sigma_{i_0^1}^a...\sigma_{i_0^{p-1}}^a \sinh\beta))}{\tilde{\Omega}_t(\Pi_{a=1}^{s}(\cosh\beta + \sigma_{i_0^1}^a...\sigma_{i_0^{p-1}}^a \sinh\beta))} \right] -$$

$$2\tilde{\alpha}^{p-1} \langle F_s \rangle_{t,\tilde{\alpha}} =$$

$$2\tilde{\alpha}^{p-1} (\mathbf{E} \left[\frac{\tilde{\Omega}_t(F_s \Pi_{a=1}^{s}(1 + \sigma_{i_0^1}^a...\sigma_{i_0^{p-1}}^a \theta))}{(1 + \tilde{\omega}_t(\sigma_{i_0^1}^a...\sigma_{i_0^{p-1}}^a)\theta)^s} \right] - \langle F_s \rangle_{t,\tilde{\alpha}}),$$

Now noting that

$$\Pi_{a=1}^{s}(1 + \sigma_{i_0^1}^a...\sigma_{i_0^{p-1}}^a \theta) = 1 + \sum_{a=1}^{s} \sigma_{i_0^1}^a...\sigma_{i_0^{p-1}}^a \theta$$

$$+ \sum_{a<b}^{1,s} \sigma_{i_0^1}^a...\sigma_{i_0^{p-1}}^a \sigma_{i_0^1}^b...\sigma_{i_0^{p-1}}^b \theta^2 + ...$$

$$\frac{1}{(1 + \tilde{\omega}_t \theta)^s} = 1 - s\tilde{\omega}_t \theta + \frac{s(s+1)}{2!} \tilde{\omega}_t^2 \theta^2 + ...$$

we obtain

$$\frac{\partial \langle F_s \rangle_{t,\tilde{\alpha}}}{\partial t} = 2\tilde{\alpha}^{p-1} \Big(\mathbf{E} \Big[\tilde{\Omega}_t \Big(F_s (1 + \sum_{a=1}^{s} \sigma_{i_0^1}^a...\sigma_{i_0^{p-1}}^a \theta + \tag{61}$$

$$+ \sum_{a<b}^{1,s} \sigma_{i_0^1}^a...\sigma_{i_0^{p-1}}^a \sigma_{i_0^1}^b...\sigma_{i_0^{p-1}}^b \theta^2 + ...) \Big) \times$$

$$\times \left(1 - s\tilde{\omega}_t \theta + \frac{s(s+1)}{2!} \tilde{\omega}_t^2 \theta^2 + ... \right) \Big] - \langle F_s \rangle_{t,\tilde{\alpha}} \Big),$$

from which our thesis follows. $\square$

[1] M. Mézard, G. Parisi and M. A. Virasoro, *Spin glass theory and beyond*, World Scientific, Singapore (1987).
[2] D.J. Amit, *Modeling brain function: The world of attractor neural network* Cambridge Univerisity Press, Cambridge (1992)
[3] K. Huang, *Lectures on Statistical Physics and Protein Folding*, World Scientific Publishing, London (2007).
[4] G. Parisi, PNAS **87**, 429 (1990).
[5] P. Contucci, S. Graffi (Ed.s), *How Can Mathematics Contribute to Social Sciences*, Special Issue of Quality and Quantity, **41**, 531 (2007).
[6] A.C.C. Coolen, *The Mathematical Theory of Minority Games - Statistical Mechanics of Interacting Agents*, Oxford University Press, Oxford (2005).
[7] D. Chowdhury, L. Santen, A. Schadschneider, Phys. Rep. **199**, 329 (2000).
[8] E. Agliari, R. Burioni, P. Contucci, submitted
[9] E. Agliari, R. Burioni, D. Cassi and A. Vezzani, Eur. Phys. J. B **46**, 109 (2005).
[10] P. Buonsante, R. Burioni, D. Cassi and A. Vezzani, Phys. Rev. E **66**, 036121 (2002).
[11] E. Agliari, R. Burioni, D. Cassi and A. Vezzani, Eur. Phys. J. B **49**, 119 (2005).
[12] E. Agliari, R. Burioni, D. Cassi and A. Vezzani, Eur. Phys. J. B **48**, 529 (2006).
[13] D. Evans, G. Morris, *Statistical mechanics of non equilibrium liquids*, Oxford Press, Oxford (1990).
[14] E. Agliari, A. Barra, F. Camboni, J. Stat. Mech. P10003, (2008)
[15] Durlauf, S. N.: 1999, *How can statistical mechanics contribute to social science?*. Proceedings of the National Academy of Sciences of the U.S.A. **96**, 10582-10584, (1999).
[16] D. Mc Fadden, The American Economic Review, **91**, 35 (2001).
[17] F. Guerra, F. L. Toninelli, J. Stat. Phys. **115**, 531 (2004).
[18] Liggett TM, *Stochastic Interacting Systems: Contact, Voter, and Exclusion Processes*, Springer-Verlag, New York (1999).
[19] M. E. J. Newman and G. T. Barkema , *Monte Carlo methods in Statistical Physics*, Oxford University Press, Oxford (2001).
[20] P. Sollich, F. Ritort, Adv. in Phys. **52**, 219 (2003).
[21] A. Barra, Math. Meth. Appl. Sc. **32**, 783 (2008).
[22] A. Barra, J. Stat. Phys. **123**, 601 (2006).
[23] A. Barra, J. Stat. Phys. **132**, 787 (2008).
[24] F. Guerra, Int. J. Mod. Phys. B **10**, 1675 (1996).
[25] M. Aizenman, P. Contucci, J. Stat. Phys. **92**, 765 (1998).
[26] G. Parisi, *Stochastic Stability*, Proceedings of the Conference Disordered and Complex Systems, London (2000).
[27] L. D. Landau and E.M. Lifshitz, Statistical Physics, Course of Theoretical Physics, Volume 5, Elsevier But-

terworth Heinemann, Amsterdam (1951).

[28] P.Contucci, A. Barra, F.Camboni, J. Stat. Mech. P03028, (2008).

[29] G. Semerjian, M. Weigt, J. Phys. A **37**, 5525 (2004).

[30] K. Binder, Rep. Prog. Phys. **60**, 487 (1997).

[31] K. Vollmayr, J. D. Reger, M. Scheucher and K. Binder, Zeitschrift für Physik B Condensed Matter, **91**, 113 (1993).