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2011 J. Phys. A: Math. Theor. 44 245002

(http://iopscience.iop.org/1751-8121/44/24/245002)

IOP PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND THEORETICAL

J. Phys. A: Math. Theor. **44** (2011) 245002 (22pp) doi:10.1088/1751-8113/44/24/245002

Equilibrium statistical mechanics of bipartite spin systems

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Received 8 December 2010, in final form 18 April 2011
Published 11 May 2011
Online at stacks.iop.org/JPhysA/44/245002

Abstract

The aim of this paper is to give an extensive treatment of bipartite mean field spin systems, pure and disordered. At first, bipartite ferromagnets are investigated, and an explicit expression for the free energy is achieved through a new minimax variational principle. Then, via the Hamilton–Jacobi technique, the same structure of the free energy is obtained together with the existence of its thermodynamic limit and the minimax principle is connected to a standard max one. The same is investigated for bipartite spin-glasses. By the Borel–Cantelli lemma we obtain the control of the high temperature regime, while via the double stochastic stability technique we also obtain the explicit expression of the free energy in the replica symmetric approximation, uniquely defined by a minimax variational principle again. We also obtain a general result that states that the free energies of these systems are convex linear combinations of their independent one-party model counterparts. For the sake of completeness, we show further that at zero temperature the replica symmetric entropy becomes negative and, consequently, such a symmetry must be broken. The treatment of the fully broken replica symmetry case is deferred to a forthcoming paper. As a first step in this direction, we start deriving the linear and quadratic constraints to overlap fluctuations.

PACS numbers: 02.50.−r, 64.60.Cn, 64.60.A−

1. Introduction

The investigation by statistical mechanics of pure and random mean field spin systems is experiencing a huge increasing interest in the last few decades. The motivations are at least twofold: from one side, at the rigorous mathematical level, even though several contributions appeared along the years (see for instance [2, 20, 22, 31]), a full clear picture is still to be achieved (it is enough to think at the whole community dealing with ultrametricity in the case of random interactions as in glasses [3, 27, 28]); at the applied level, these toy models are starting to be used in several different contexts, ranging from quantitative sociology [7, 14, 15] to theoretical biology [6, 29].

The need for always stronger and simpler methods to analyze the enormous amount of all the possible 'variations on theme' is then obvious, the theme being the standard Curie–Weiss model (CW) [4, 21] for the pure systems, or the paradigmatic Sherrington–Kirkpatrick model (SK) [20, 27] for the random ones.

As a result, inspired by the recent attention raised for interactions between two different parties (i.e. decision-making processes in econometrics [16, 24] or metabolic networks in biology [25, 26]), we decided to focus, in this paper, on the equilibrium statistical mechanics of two bipartite spin systems, namely the bipartite CW and the bipartite SK.

Firstly, we approach the problem of the bipartite model by studying in section 2 the bipartite ferromagnet, obtaining, both via a standard approach and through our mechanical interpretation of the interpolation method [4, 8, 17, 19], a variational principle for the free energy in the thermodynamic limit.

In section 3, we open the investigation of the bipartite spin glass model, and following the path already outlined in [9, 10] we get the annealed free energy (more precisely, the pressure), with a characterization of the region of the phase diagram where it coincides with the true one in the thermodynamic limit, and the replica symmetric free energy, by the double stochastic stability technique, which stems from a minimax variational principle, whose properties are also discussed.

Finally, we calculate the zero temperature observable and, by noting that the entropy is negative, we conclude that replica symmetry must be broken. Despite a full replica symmetry breaking scheme deserves a separate paper, here we start introducing its typical linear and quadratic constraints, obtained by Landau self-averaging of the internal energy density [18].

The last section is left for conclusions and outlooks.

2. Bipartite ferromagnets

We are interested in considering a set of N Ising spin variables, separated in two subsets of size, respectively N_1 and N_2. We assume the variable's label of the first subset as σ_i, $i = 1, \dots, N_1$, while the spins of the second one are introduced by τ_j, $j = 1, \dots, N_2$. Of course we have $N_1 + N_2 = N$, and we name the relative size of the two subsets $N_1/N = \alpha_N$, $N_2/N = 1 - \alpha_N$.

For the sake of simplicity, in what follows we deal with both parties formed by binary variables, but we stress that the method works on a very general class of random variables with symmetric probability measure and compact support [17].

The spins interact via the Hamiltonian $H_{N_1,N_2}(\sigma, \tau, h_1, h_2)$, with $h_1 \geqslant 0$, $h_2 \geqslant 0$:

$$H_{N_1,N_2}(\sigma, \tau, h_1, h_2) = -\frac{2}{N_1 + N_2} \sum_{i=1}^{N_1} \sum_{j=1}^{N_2} \sigma_i \tau_j - h_1 \sum_{i=1}^{N_1} \sigma_i - h_2 \sum_{j=1}^{N_2} \tau_j.$$

We note that spins in each subsystem interact only with spins in the other one, but not among themselves; we have chosen to skip the self-interactions to tackle only genuine features stemming from the exchange ones. The reader, interested in a (different) treatment of bipartite ferromagnetic models with self-interactions, may refer to [16].

Partition function Z, pressure A and free energy per site f are defined naturally for the model:

$$Z_{N_1,N_2}(\beta, h_1, h_2) = \sum_\sigma \sum_\tau e^{-\beta H_{N_1,N_2}(\sigma,\tau,h_1,h_2)},$$
$$A_{N_1,N_2}(\beta, h_1, h_2) = \frac{1}{N_1 + N_2} \log Z_{N_1,N_2}(\beta, h_1, h_2),$$
$$f_{N_1,N_2}(\beta, h_1, h_2) = -\frac{1}{\beta} A_{N_1,N_2}(\beta, h_1, h_2).$$

The thermodynamic limit of A, f will be denoted via $A(\alpha, \beta, h_1 h_2) = \lim_N A_{N_1,N_2}(\beta, h_1, h_2)$ and $f(\alpha, \beta, h_1, h_2) = \lim_N f_{N_1,N_2}(\beta, h_1, h_2)$, where we stressed the prescription adopted in taking the infinite volume limit, performed in such a way that when $N, N_1, N_2 \to \infty$, $N_1/N \to \alpha \in (0, 1)$, and $N_2/N \to 1-\alpha \in (0, 1)$. For the sake of simplicity in the notations, we often write α in place of N_1/N, and $1-\alpha$ in place of N_2/N, by neglecting terms irrelevant in the infinite volume limit.

Assuming $z(\sigma, \tau)$ as a generic function of the spin variables, we can also specify the Boltzmann state of our system as

$$\langle z(\sigma, \tau)\rangle = \frac{\sum_\sigma \sum_\tau z(\sigma, \tau) \exp(-\beta H_{N_1,N_2}(\sigma, \tau, h_1, h_2))}{Z_{N_1,N_2}(\beta, h_1, h_2)}. \tag{1}$$

As usual the order parameters (the respective magnetizations of the two systems) are

$$m_{N_1} = \frac{1}{N_1} \sum_i^{N_1} \sigma_i, \tag{2}$$

$$n_{N_2} = \frac{1}{N_2} \sum_j^{N_2} \tau_j; \tag{3}$$

thus, the Hamiltonian reads off as

$$H_{N_1,N_2}(\sigma, \tau, h_1, h_2) = -N[2\alpha_N(1-\alpha_N)m_{N_1}n_{N_2} + h_1\alpha_N m_{N_1} + h_2(1-\alpha_N)n_{N_2}]. \tag{4}$$

Remark 1. As will be clear soon, the choice of the factor 2 in the Hamiltonian is made in such a way that the balanced bipartite model with $\alpha = 1/2$ has the same critical point of the single party model, i.e. $\beta = 1$.

2.1. The occurrence of a minimax principle for the free energy

Now we give the explicit form of the pressure of the model, together with some interesting properties. For the sake of convenience, we rename $\beta h_1 \to h_1$ and $\beta h_2 \to h_2$, as switching back to the original variables is straightforward in every moment, but this lightens the notation. The main result is as follows.

Theorem 1. *In the thermodynamic limit, the pressure of the bipartite ferromagnetic model is given by the following variational principle:*

$$A(\alpha, \beta, h_1, h_2) = \max_{\bar m} \min_{\bar n} A_{\text{trial}}(\bar m, \bar n), \tag{5}$$

where

$$A_{\text{trial}}(\bar m, \bar n) = \log 2 + \alpha \log\cosh(2\beta(1-\alpha)\bar n + h_1) + (1-\alpha)\log\cosh(2\beta\alpha\bar m + h_2) - 2\beta\alpha(1-\alpha)\bar m\bar n.$$

Furthermore, the solution is uniquely defined by the intersection of

$$\bar{m} = \tanh\left(2\beta(1-\alpha)\bar{n} + h_1\right), \tag{6}$$

$$\bar{n} = \tanh\left(2\beta\alpha\bar{m} + h_2\right) \tag{7}$$

for $\bar{m} \geqslant 0$ *and* $\bar{n} \geqslant 0$.

Proof. The proof can be achieved in several ways (i.e. a direct approach by marginalizing the free energy with respect to both the parties); we chose to follow the path outlined in [9] as this may act as a guide later, dealing with frustrated interactions.

To perform our task we need to introduce two trial parameters $\bar{m}, \bar{n}$ that mimic the magnetizations inside each party, one interpolating parameter $t \in [0, 1]$ and an interpolating function $A(t)$ for the free energy as follows:

$$\mathrm{A}(t) = \frac{1}{N}\log\sum_{\sigma}\sum_{\tau} \mathrm{e}^{t(2\beta\frac{N_1N_2}{N_1+N_2}m(\sigma)n(\tau))}\mathrm{e}^{(1-t)(2\beta(1-\alpha)\bar{n}\sum_i^{N_1}\sigma_i + 2\beta\alpha\bar{m}\sum_j^{N_2}\tau_j)}\mathrm{e}^{h_1\sum_i\sigma_i + h_2\sum_j\tau_j}, \tag{8}$$

such that for $t = 1$ our interpolating function reduces to the free energy of the model (the pressure strictly speaking), while for $t = 0$ reduces to a sum of one-body models whose solution is straightforward.

We can then apply the fundamental theorem of calculus to obtain the following sum rule:

$$A(1) = A(0) + \int_0^1 \frac{\mathrm{d}A(t)}{\mathrm{d}t}\mathrm{d}t. \tag{9}$$

To quantify the latter we need to sort out both the derivative of the interpolating function and its value at $t = 0$, namely

$$\partial_t A(t) = 2\beta\alpha(1-\alpha)[\langle mn\rangle - \bar{n}\langle m\rangle - \bar{m}\langle n\rangle + \bar{m}\bar{n}] - 2\beta\alpha(1-\alpha)\bar{m}\bar{n}, \tag{10}$$

$$A(t=0) = \log 2 + \alpha\log\cosh[2\beta(1-\alpha)\bar{n} + h_1] + (1-\alpha)\log\cosh[2\beta\alpha\bar{m} + h_2], \tag{11}$$

where in equation (10) we added and subtracted the term $\beta\alpha(1-\alpha)\bar{m}\bar{n}$ to write explicitly the sum rule in terms of a trial function $A_{\mathrm{trial}}(\alpha, \beta, h_1, h_2)$ and an error term $S(\bar{m}, \bar{n})$:

$$A(\alpha, \beta, h_1, h_2) = A_{\mathrm{trial}}(\alpha, \beta, h_1, h_2) + S(\bar{m}, \bar{n}), \tag{12}$$

where

$$A_{\mathrm{trial}}(\alpha, \beta, h_1, h_2) = \log 2 + \alpha\log\cosh[2\beta(1-\alpha)\bar{n} + h_1] + (1-\alpha)\log\cosh[2\beta\alpha\bar{m} + h_2] - 2\beta\alpha(1-\alpha)\bar{m}\bar{n}, \tag{13}$$

$$S(\bar{m}, \bar{n}) = 2\beta\alpha(1-\alpha)\int_0^1 \langle(m-\bar{m})(n-\bar{n})\rangle_t\,\mathrm{d}t, \tag{14}$$

for every trial parameters $\bar{m}, \bar{n}$.

Note that the averages $\langle\cdot\rangle_t$ take into account that the Boltzmannfaktor is no longer the standard one introduced in equation (1), but incorporates the interpolating structure tuned by the parameter t.

We stress that, at this stage, the error term $S(\bar{m}, \bar{n})$ is given in terms of the fluctuations of the order parameters, which are expected to reduce to zero in the thermodynamic limit. On the other hand, S is not positive at first sight, as in many other cases (i.e. monopartite spin glasses [20]). However, the idea of choosing properly $\bar{m}, \bar{n}$ to make it smaller and smaller (eventually zero) still holds obviously.

We study at fixed $\bar{n}$ the behavior of our trial function in $\bar{m}$ by looking at its derivative:

$$\partial_{\bar{m}} A_{\mathrm{trial}}(\alpha, \beta, h_1, h_2) = 2\beta\alpha(1-\alpha)[\tanh(2\beta\alpha\bar{m} + h_2) - \bar{n}]. \tag{15}$$

So, at given $\bar{n}$, this derivative is increasing in $\bar{m}$ and $A_{\text{trial}}(\alpha, \beta, h_1, h_2)$ is convex in $\bar{m}$.

By a direct calculation we obtain the same result inverting $\bar{n} \Leftrightarrow \bar{m}$:

$$\partial_{\bar{n}} A_{\text{trial}}(\alpha, \beta, h_1, h_2) = 2\beta\alpha(1-\alpha)[\tanh(2\beta(1-\alpha)\bar{n} + h_1) - \bar{m}]. \tag{16}$$

Due to the ferromagnetic character of the interaction, as it is crystal clear that the roles of $\bar{m}, \bar{n}$ are of the local magnetizations, we can allow ourselves in considering only values $\bar{m} \geqslant \tanh(\beta h_1)$ such that there exists a unique $\bar{n}(\bar{m}) \geqslant 0 : \tanh[2\beta(1-\alpha)\bar{n}(\bar{m}) + h_1] = \bar{m}$. Obviously, $\bar{n}(\bar{m})$ is monotone increasing in $\bar{m}$, and $\bar{n}(\bar{m})/\bar{m}$ is also increasing in $\bar{m}$.

From now on let us switch from $A_{\text{trial}}(\alpha, \beta, h_1, h_2)$ to a different labeling $\tilde{A}_{\text{trial}}(\bar{m}, \bar{n})$, which aims to stress the relevant dependence on its trial variables. For the value $\bar{n}(\bar{m})$ lastly obtained, the trial function has its minimum in $\bar{n}$ at fixed $\bar{m}$ such that we can substitute it, and obtain a new reduced trial function $\tilde{A}_{\text{trial}}(\bar{m}) = \tilde{A}_{\text{trial}}(\bar{m}, \bar{n}(\bar{m}))$.

Now, as

$$\partial_{\bar{m}} \tilde{A}_{\text{trial}}(\bar{m})/\bar{m} = 2\beta\alpha(1-\alpha)[\tanh(2\beta\alpha\bar{m} + h_2) - \bar{n}(\bar{m})]/\bar{m}, \tag{17}$$

we can consider $\tilde{A}(\bar{m})$ as a function of $\bar{m}^2$ to see easily that $\tilde{A}$ is concave in $\bar{m}^2$, so it has its unique maximum where its derivative vanishes.

Overall we can define the optimal trial $\tilde{A}(\alpha, \beta, h_1, h_2) = \sup_{\bar{m}} \inf_{\bar{n}} \tilde{A}(\bar{m}, \bar{n})$, whose stationary point is uniquely defined by the solution of the system of self-consistence relations

$$\bar{m} = \tanh\left(2\beta(1-\alpha)\bar{n} + h_1\right), \tag{18}$$

$$\bar{n} = \tanh\left(2\beta\alpha\bar{m} + h_2\right), \tag{19}$$

as stated in theorem 1. Furthermore, if we restrict ourselves in considering $\bar{n} = \bar{n}(\bar{m})$—as imposed by the variational principle—the error term (the fluctuation source) results positive: this statement can be understood by marginalizing with respect to the τ party the free energy (summing over all the τ-configurations) to substitute $n(\tau)$ by $\tanh[2\beta\alpha m(\sigma)t + 2\beta\alpha\bar{m}(1-t) + h_1]$ and noting that $m(\sigma) \geqslant \bar{m}$ implies $\tanh[2\beta\alpha m t + 2\beta\alpha\bar{m}(1-t) + h_1] \geqslant \bar{n}(\bar{m})$ such that

$$\langle (m - \bar{m})[\tanh(2\alpha\beta\bar{m} + h_2) - \bar{n}] \rangle \geqslant 0,$$

the error term is positive.

Now, in order to show that the error term is zero in the thermodynamic limit (such that the expression of the trial becomes correct) we proceed in a different way: the idea is to marginalize with respect to one party to remain with a single ferromagnetic party with a modified interaction, and then use the standard body of knowledge developed for this case.

So, at first, we marginalize the free energy with respect to the τ variables:

$$A_{N_1,N_2}(\beta, h_1, h_2) = \frac{1}{N_1 + N_2} \log Z_{N_1,N_2}(\beta, h_1, h_2) \tag{20}$$

$$= \frac{1}{N_1 + N_2} \log \sum_{\sigma} 2^{N_2} \cosh^{N_2}(2\alpha\beta m + h_2) \exp\left(h_1 \sum_i \sigma_i\right) \tag{21}$$

$$= (1-\alpha)\log 2 + \frac{1}{N_1 + N_2} \log \sum_{\sigma} \mathrm{e}^{N_2 \log\cosh(2\beta\alpha m + h_2) + h_1 \sum_i \sigma_i}. \tag{22}$$

We can now use the convexity of the logarithm of the hyperbolic cosine to obtain

$$\log\cosh(2\alpha\beta m + h_2)) \geqslant \log\cosh(2\alpha\beta\bar{m} + h_2)) + 2\alpha\beta(m - \bar{m})\tanh(2\alpha\beta\bar{m} + h_2).$$

Through a standard calculation we bound the pressure through a new trial function $\hat{A}_{\text{trial}}(\bar{m})$ as

$$A_{N_1,N_2}(\beta, h_1, h_2) \geqslant \hat{A}_{\text{trial}}(\bar{m}), \tag{23}$$

where

$$\begin{aligned}\hat{A}_{\text{trial}}(\bar{m}) = \log 2 + \log\cosh(2\alpha\beta\bar{m} + h_2)) - 2\alpha(1-\alpha)\beta\bar{m}\tanh(2\alpha\beta\bar{m} + h_2) \\ +\alpha\log\cosh[2\beta(1-\alpha)\tanh(2\alpha\beta\bar{m} + h_2) + h_1].\end{aligned} \tag{24}$$

We can look for the $\bar{m}$ derivative of the trial, namely

$$\partial_{\bar{m}}\hat{A}_{\text{trial}}(\bar{m}) = (2\beta\alpha)^2(1-\alpha)(1-\tilde{n}^2)[\tanh(2\beta(1-\alpha)\tilde{n} + h_1) - \bar{m}], \tag{25}$$

where we have defined $\tilde{n}(\bar{m}) = \tanh(2\alpha\beta\bar{m} + h_2)$, with the customary monotone properties. If we now consider the derivative with respect to $\tilde{n}$ of $\hat{A}_{\text{trial}}$ we obtain

$$\partial\tilde{n}\hat{A}_{\text{trial}}(\bar{m}) = (\partial_{\tilde{n}}\bar{m})\partial_{\bar{m}}\hat{A}_{\text{trial}}(\bar{m}) = 2\beta\alpha(1-\alpha)[\tanh(2\beta(1-\alpha)\tilde{n}) - \bar{m}(\tilde{n})], \tag{26}$$

where $\bar{m}(\tilde{n})$ is the inverse of $\tilde{n}(\bar{m})$. We see that $\partial\tilde{n}\hat{A}_{\text{trial}}(\bar{m})/\tilde{n}$ is decreasing in $\tilde{n}$, so that the trial is concave in $\tilde{n}^2$. There exists a unique maximum where the derivative vanishes. By properly choosing $\bar{m} \to \bar{m}$ and $\tilde{n} \to \bar{n}$ we get exactly the statement of the theorem, as far as the optimal parameters and the optimal trial are concerned. Up to now we have obtained only a lower bound, but the opposite bound can be achieved exactly as in the standard Curie–Weiss model, because what matters is only the convexity of the interaction in $m_N(\sigma)$. Therefore, the theorem is proven. □

As we are going to deepen the extremization procedure in these bipartite models, we stress that an important feature that seems to arise from our study is the occurrence of a min max principle for the pressure, usually given by a maximum principle in the ordered models, and a minimum principle in the frustrated ones.

We stress that these kinds of principles appeared early in statistical mechanics (in the context of separable many body problems), see for instance [11, 12].

We finally report an interesting result about the form of the pressure (or equivalently, the free energy) of the model. Indeed it turns out to be written as the convex combination of the pressures of two different monopartite CW models, at different inverse temperatures, stated as follows.

Proposition 1. *In the thermodynamic limit, the following decomposition of bipartite ferromagnetic model free energies into convex sums of monopartite ones is allowed:*

$$A(\alpha, \beta, h_1, h_2) = \alpha A^{\text{CW}}(\beta', h_1) + (1-\alpha)A^{\text{CW}}(\beta'', h_2),$$

with $\beta' = 2\beta(1-\alpha)\frac{\bar{n}}{\bar{m}}$ *and* $\beta'' = 2\beta\alpha\frac{\bar{m}}{\bar{n}}$.

Proof. We start setting the trial values for the inverse temperatures of the two monopartite models, as $\beta' = 2\beta(1-\alpha)a^2$ and $\beta'' = 2\beta\alpha/a^2$, with a a real parameter to be determined later, and we set the external fields to zero for the sake of clarity. It is

$$Z_{N_1,N_2}(\beta) = \sum_{\sigma}\sum_{\tau}\exp(2\beta N\alpha(1-\alpha)mn)\,,$$

and since $2mn \leqslant m^2a^2 + \frac{n^2}{a^2}$, $\forall a \neq 0$, we have

$$
\begin{aligned}
Z_{N_1,N_2}(\beta) &\leqslant \sum_\sigma \sum_\tau \exp\left(2\beta N\alpha(1-\alpha)\frac{m^2a^2}{2} + 2\beta N\alpha(1-\alpha)\frac{n^2}{2a^2}\right)\\
&= \sum_\sigma \exp\left(\beta' N_1 \frac{m^2}{2}\right)\sum_\tau \exp\left(\beta'' N_2\frac{n^2}{2}\right)\\
&= Z_{N_1}(\beta')Z_{N_2}(\beta'');
\end{aligned}
$$

thus, we conclude that when $N \to \infty$

$$A(\alpha,\beta) \leqslant \alpha A^{\mathrm{CW}}(\beta') + (1-\alpha)A^{\mathrm{CW}}(\beta''). \tag{27}$$

The inverse bound is proven noting that

$$
\begin{aligned}
\frac{Z_{N_1,N_2}(\beta)}{Z_{N_1}(\beta')Z_{N_2}(\beta'')} &= \frac{\sum_\sigma\sum_\tau \exp\left(2\beta N\alpha(1-\alpha)mn - \frac{\beta' m^2}{2} - \frac{\beta'' n^2}{2}\right)\exp\left(\frac{\beta' m^2}{2}+\frac{\beta'' n^2}{2}\right)}{Z_{N_1}(\beta')Z_{N_2}(\beta'')}\\
&= \Omega\left[\exp\left(2\beta N\alpha(1-\alpha)mn - \frac{\beta' m^2}{2} - \frac{\beta'' n^2}{2}\right)\right]\\
&\geqslant \exp\left(\Omega\left[2\beta N\alpha(1-\alpha)mn - \frac{\beta' m^2}{2} - \frac{\beta'' n^2}{2}\right]\right)\\
&= \exp\left(2\beta N\alpha(1-\alpha)\bar{m}\bar{n} - \frac{\beta' \bar{m}^2}{2} - \frac{\beta'' \bar{n}^2}{2}\right),
\end{aligned}
$$

where we have denoted with Ω the joint state of the two monopartite systems. Now, bearing in mind the definition of β' and β'' given at the beginning, we obtain

$$\frac{Z_{N_1,N_2}(\beta)}{Z_{N_1}(\beta')Z_{N_2}(\beta'')} \geqslant \mathrm{e}^{-2N\beta\alpha(1-\alpha)\left(a\bar{m}-\frac{\bar{n}}{a}\right)^2}$$

and then, in thermodynamic limit

$$A(\alpha,\beta) \geqslant \alpha A^{\mathrm{CW}}(\beta') + (1-\alpha)A^{\mathrm{CW}}(\beta'') - 2\beta\alpha(1-\alpha)\left(\left(a\bar{m} - \frac{\bar{n}}{a}\right)^2\right). \tag{28}$$

Now we note that the extrema in (27) and (28), respectively a minimum and a maximum, are obtained with the choice $a^2 = \frac{\bar{n}}{\bar{m}}$; this completes the proof. □

It is worthwhile to stress that the two monopartite models are independent from a physical point of view, and the order parameters are mathematically related by

$$
\begin{aligned}
\bar{m} &= \tanh(\beta'\bar{m}),\\
\bar{n} &= \tanh(\beta''\bar{n}),
\end{aligned}
$$

provided the temperatures are properly adjusted.

2.2. *The free energy again: a maximum principle*

Our aim is now to tackle the mathematical study of this model with the approach described in [4, 8, 17, 19], based on a mechanical interpretation of the interpolation method. The problem of finding the free energy in the thermodynamic limit is here translated in solving a Hamilton–Jacobi equation with certain suitable boundary conditions, and an associated Burgers transport equation for the order parameter of the model. In order to reproduce this scheme, with the freedom of interpretation of the label t for the time and x for the space, let us introduce now the (x,t)-dependent interpolating partition function

$$Z_N(x,t)=\sum_\sigma\sum_\tau \exp N\Bigg(t\alpha_N(1-\alpha_N)m_Nn_N + (2\beta-t)\left(a^2\alpha^2m_N^2+\left(\frac{1-\alpha}{a}\right)^2n_N^2\right)+x\left(a\alpha_Nm_N-\frac{1-\alpha_N}{a}n_N\right) + h_1\alpha_Nm_N+h_2(1-\alpha_N)n_N\Bigg),$$

such that the thermodynamical partition function of the model is recovered when $t=2\beta$ and $x=0$. At this level a is a free parameter to be determined later. We can go further and explicitly define the function

$$\varphi_N(x,t)=\frac{1}{N}\log Z_N(x,t) \tag{29}$$

that is just the pressure of the model for a suitable choice of (x,t). Now, computing derivatives of $\varphi_N(x,t)$, we note that putting $D_N=a\alpha_Nm_N-\frac{1-\alpha_N}{a}n_N$, it is

$$\begin{aligned}
\partial_t\varphi_N(x,t)&=-\frac{1}{2}\left\langle D_N^2\right\rangle(x,t),\\
\partial_x\varphi_N(x,t)&=\langle D_N\rangle(x,t),\\
\partial_x^2\varphi_N(x,t)&=\frac{N}{2}\left(\left\langle D_N^2\right\rangle-\langle D_N\rangle^2\right).
\end{aligned}$$

Thus, we can build our differential problems through a Hamilton–Jacobi equation for $\varphi_N(x,t)$:

$$\begin{cases}\partial_t\varphi_N(x,t)+\frac{1}{2}(\partial_x\varphi_N(x,t))^2+\frac{1}{2N}\partial_x^2\varphi_N(x,t)=0 & \text{in } \mathbb{R}\times(0,+\infty)\\ \varphi_N(x,0)=\alpha_NA_{N_1}^{\rm CW}(\beta',\alpha(h_1+x))+(1-\alpha_N)A_{N_2}^{\rm CW}(\beta'',(1-\alpha)(h_2-x)) & \text{on } \mathbb{R}\times\{t=0\},\end{cases} \tag{30}$$

where $A_{N_1}^{\rm CW}$ is the pressure of the Curie–Weiss model made of N_1 σ spins with inverse temperature β', and $A_{N_2}^{\rm CW}$ is the same referred to the N_2 τ spin with inverse temperature β'', with $\beta'=2\beta a^2(1-\alpha)$ and $\beta''=2\beta a^{-2}\alpha$, and through a Burgers equation for the velocity field $D_N(x,t)$:

$$\begin{cases}\partial_tD_N(x,t)+D_N(x,t)\partial_xD_N(x,t)+\frac{1}{2N_1}\partial_x^2D_N(x,t)=0 & \text{in } \mathbb{R}\times(0,+\infty)\\ D_N(x,0)=\alpha m(\beta,h_1+x)-(1-\alpha_N)n(\alpha_N\beta,h_2-x) & \text{on } \mathbb{R}\times\{t=0\}.\end{cases} \tag{31}$$

This is true of course for every choice of the parameter a that has the role of balancing the weights of the single party contributions. Since we have seen that the function φ_N is decreasing in time, if we put $x=0$, with no external fields, we gain

$$A_{N_1,N_2}(\beta)\leqslant\alpha_NA_{N_1}^{\rm CW}(\beta')+(1-\alpha_N)A_{N_2}^{\rm CW}(\beta'').$$

If we take $a^2=\sqrt{\frac{1-\alpha}{\alpha}}$, such that it is $\beta'=\beta''=\bar\beta=2\beta\sqrt{\alpha(1-\alpha)}$, we have, in the infinite volume limit,

$$A(\beta)\leqslant\alpha A^{\rm CW}(\bar\beta)+(1-\alpha)A^{\rm CW}(\bar\beta)$$

that easily give us the critical line of the bipartite model, $2\beta\sqrt{\alpha(1-\alpha)}=1$, obtained straightly by the critical point of the two Curie–Weiss models, $\bar\beta=1$. Anyway, for reasons that will become clear soon, hereafter it will adopt the different value $a=1$.

Remark 2. We stress that the boundary condition in equation (30) is always an upper bound for φ_N.

In order to work out an explicit solution for the thermodynamic limit of the pressure, *in primis* we note that the main difference with respect to the single party (namely the Curie Weiss [17]) is the more delicate form of the boundary conditions. In fact, we have that interactions do not factorize trivially (in a way independent by the size of the system). It is

$$\varphi_N(x,0) = \alpha_N A^{\mathrm{CW}}_{N_1}(\beta', h_1 + x) + (1-\alpha_N) A^{\mathrm{CW}}_{N_2}(\beta'', h_2 - x); \tag{32}$$

however, it is known [17] how to get a perfect control of the function on the rhs of (32), and we have

$$\varphi_N(x,0) = \alpha A^{\mathrm{CW}}(\beta', h_1 + x) + (1-\alpha) A^{\mathrm{CW}}(\beta'', h_2 - x) + O\left(\frac{1}{N}\right). \tag{33}$$

Evenly we have for the velocity field in $t = 0$

$$\alpha_N m_{N_1}(\beta', h_1 + x) - (1-\alpha_N) n_{N_2}(\beta'', h_2 - x) = \alpha m(\beta', h_1 + x) - (1-\alpha) n(\beta'', h_2 - x) + O\left(\frac{1}{\sqrt{N}}\right).$$

We obtained a Hamilton–Jacobi equation for the free energy with a vanishing dissipative term in the thermodynamic limit, while the velocity field, that is, the order parameter, satisfies a Burgers equation with a mollifier dissipative term.

We stress that our method introduces by itself the correct order parameter, without imposing it by hands.

Remark 3. As the next condition on $\varphi_N(x,t)$ is tacitly required by the following lemma, we stress that the function $D_N(x,t)$ is bounded uniformly in $N, \alpha, \beta, h_1, h_2$ that implies the function $\varphi_N(x,t)$ to be Lipschitz continuous.

We can replace the sequence of differential problems with boundary conditions dependent by N with the same sequence of equations but with obvious fixed boundary conditions, that is, the well-defined limiting value of φ_N and D_N in $t = 0$. To this purpose the following is useful.

Lemma 1. *The two differential problems*

$$\begin{cases} \partial_t \varphi_N(x,t) + \frac{1}{2}(\partial_x \varphi_N(x,t))^2 + \frac{1}{2N_1}\partial_x^2 \varphi_N(x,t) = 0 & \text{in } \mathbb{R}\times(0,+\infty) \\ \varphi_N(x,0) = \alpha_N A^{\mathrm{CW}}_{N_1}(\beta', h_1 + x) + (1-\alpha_N) A^{\mathrm{CW}}_{N_2}(\beta'', h_2 - x) = h_N(x) & \text{on } \mathbb{R}\times\{t=0\} \end{cases} \tag{34}$$

and

$$\begin{cases} \partial_t \bar\varphi_N(x,t) + \frac{1}{2}(\partial_x \bar\varphi_N(x,t))^2 + \frac{1}{2N_1}\partial_x^2 \bar\varphi_N(x,t) = 0 & \text{in } \mathbb{R}\times(0,+\infty) \\ \varphi_N(x,0) = \alpha A^{\mathrm{CW}}(\beta', h_1 + x) + (1-\alpha) A^{\mathrm{CW}}(\beta'', h_2 - x) = h(x) & \text{on } \mathbb{R}\times\{t=0\} \end{cases} \tag{35}$$

are completely equivalent, i.e. in thermodynamic limit they have the same solution $\varphi_N \to \varphi$ and $\bar\varphi_N \to \varphi$ and it is

$$|\varphi_N - \bar\varphi_N| \leqslant O\left(\frac{1}{N}\right).$$

Proof. By a Cole–Hopf transform, we can easily write the general form of $\delta_N(x,t) = |\varphi_N(x,t) - \bar\varphi_N(x,t)|$ as

$$\delta_N = \frac{1}{N}\left|\log \frac{\int_{-\infty}^{+\infty} \mathrm{d}y \Delta(y,(x,t)) \mathrm{e}^{-N R_N(y)}}{\int_{-\infty}^{+\infty} \mathrm{d}y \Delta(y,(x,t))}\right|,$$

where we introduced the modified heat kernel $\Delta(y,(x,t)) = \sqrt{\frac{N}{2\pi t}}\exp(-N[(x-y)^2/2t + h(y)])$, and $R_N(y) = |h(y) - h_N(y)|$ with $\lim_N NR_N < \infty, \forall\, y$. Now we note that as a y^* certainly exists such that

$$\sup_y R_N(y) = y^* \qquad \text{and} \qquad \lim_N NR_N(y^*) < \infty,$$

it is

$$\delta_N(x,t) \leqslant \frac{1}{N}|\log e^{-NR_N(y^*)}| = \frac{1}{N}[NR_N(y^*)] \leqslant O\left(\frac{1}{N}\right) \tag{36}$$

that completes the proof. □

Of course a similar result holds also for the Burgers' equation for the velocity field D_N.

Now the path to follow is clear: the problem of existence and uniqueness of the thermodynamic limit is translated here into the convergence of the viscous mechanical problem to the free one. We can readapt a theorem that resumes a certain amount of results due to Douglis, Hopf, Lax and Oleinik [17, 23] which assures the existence and uniqueness of the solution to the free problem.

Theorem 2. *The pressure of the generalized bipartite ferromagnet, in the thermodynamic limit, exists, is unique and is given by*

$$A(\beta,\alpha,h_1,h_2) = -2\alpha(1-\alpha)\beta\bar{n}\bar{m} + \alpha\log\cosh(h_1 + 2(1-\alpha)\beta\bar{n}) + (1-\alpha)\log\cosh(h_2 + 2\alpha\beta\bar{m}), \tag{37}$$

where, given the well-defined magnetization for the generalized CW model respectively for σ and τ, $m(\beta,h)$ and $n(\beta,h)$, it is

$$\bar{m}(\beta,\alpha,h_1,h_2) = m(2\beta, h_1 - D) \tag{38}$$

$$\bar{n}(\beta,\alpha,h_1,h_2) = n(2\beta, h_2 + D). \tag{39}$$

Furthermore, it is

$$|A_N(\beta,h_1,h_2) - A(\beta,\alpha,h_1,h_2)| \leqslant O\left(\frac{1}{N}\right). \tag{40}$$

Proof. Well-known results about the CW model (see for instance [4, 17]) give us the existence and the form of the free solution. We know [17] that the free Burgers equation can be solved along the characteristics

$$\begin{cases} t = s \\ x = x_0 + sD(x_0,0), \end{cases} \tag{41}$$

where

$$D(x_0,0) = \alpha m(2\beta(1-\alpha), \alpha(h_1+x_0)) + (1-\alpha)n(2\beta\alpha, (1-\alpha)(h_2-x_0)),$$

and it is

$$D(x,t) = D(x_0(x,t),0) = \alpha m(2\beta(1-\alpha), \alpha(h_1 + x - tD(x_0,0))) + (1-\alpha)n(2\beta\alpha, (1-\alpha)(h_2 - x + tD(x_0,0))). \tag{42}$$

Then we can note that

$$m(\beta', h_1 - x + tD(x_0,0)) = \tanh(h_1 - x + t(1-\alpha)n), \tag{43}$$

$$n(\beta'', h_2 + x - tD(x_0, 0)) = \tanh\,(h_2 + x + t\alpha m)\,, \tag{44}$$

which coincide with (38) and (39) when $x = 0$ and $t = 2\beta$. At this point we know [17] that the minimum is taken for $y = x - tD(x, t)$, such that we have

$$\begin{aligned}\varphi(x,t)_{(x=0,t=2\beta)} = \Big[&\frac{t}{2}D^2(x,t) - \frac{t}{2}\alpha^2 m^2(\beta', h_1 + x - tD(x_0,0)) - \frac{t}{2}(1-\alpha)^2 n^2(\beta'', h_2 - x\\ &+ tD(x_0,0)) + \alpha \log\cosh\,(h_1 - x + m(2\beta(1-\alpha) + t\alpha) - t(1-\alpha)n)\\ &+ (1-\alpha)\log\cosh\,(h_2 + x + t\alpha m - n((1-\alpha)t - 2\beta\alpha))\Big]_{(x=0,t=2\beta)}\\ = {}& A(\beta,\alpha,h_1,h_2),\end{aligned}$$

where $A(\beta, \alpha, h_1, h_2)$ is just given by (37), bearing in mind the right definition of $\tilde{M}$ and $\tilde{N}$. Now we must only prove the convergence of the true solution to the free one. But equation (40) follows by standard techniques, because of the uniform concavity of

$$\frac{(x-y)^2}{2t} + \alpha A^{\mathrm{CW}}(\beta, h_1 + y) + (1-\alpha)A^{\mathrm{CW}}(\beta, h_2 - y)$$

with respect to y. In fact, we have that by a Cole–Hopf transform, the unique bounded solution of the viscous problem is

$$\begin{aligned}\varphi_N(x,t) = \frac{1}{N}\log\sqrt{\frac{N}{t}}\int\frac{\mathrm{d}y}{\sqrt{2\pi}}\\ \times \exp\left[-N\left(\frac{(x-y)^2}{2t} + \alpha A^{\mathrm{CW}}(\beta, h_1 + y) + (1-\alpha)A^{\mathrm{CW}}(\beta, h_2 - y)\right)\right]\end{aligned}$$

and we have, by standard estimates of a Gaussian integral, that

$$|\varphi(x,t) - \varphi_N(x,t)| \leqslant O\left(\frac{1}{N}\right),$$

i.e. equation (40) is also proven. □

It is interesting to note that here the minimax principle discussed in the previous section has become a pure maximum principle for the free energy, because of the natural choice of the order parameter D, that is in our formalism the analog of the velocity field. Thus, we have outlined the framework for stating the next

Proposition 2. *As an alternative to theorem 1, the free energy of the bipartite ferromagnet can be obtained even within a classical extremization procedure as it is uniquely given by the following variational principle:*

$$\begin{aligned}f(\alpha,\beta,h_1,h_2) = \max_D\Big[&-D^2 + \alpha^2 m^2(D) + (1-\alpha)^2 n(D)^2\\ &- \frac{\alpha}{\beta}\log\cosh\,(h_1 + \beta(1-\alpha)n(D) - \frac{1-\alpha}{\beta}\log\cosh\,(h_2 + \beta\alpha m(D))\Big].\end{aligned}$$

3. Bipartite spin glasses

Let us consider a set of N_1 i.i.d. random spin variables σ_i, $i = 1, \ldots, N_1$, and also consider another set of i.i.d. random spin variable τ_j, $j = 1, \ldots, N_2 = N - N_1$. We will consider for the sake of simplicity only binary spin variables, although our scheme is easily extensible to other spin distributions, symmetric and with compact support. Therefore, we have two distinct sets (or parties hereafter) of different spin variables, and we let them interact through the following Hamiltonian:

$$H_{N_1,N_2}(\sigma,\tau) = -\sqrt{\frac{2}{N_1+N_2}}\sum_{i=1}^{N_1}\sum_{j=1}^{N_2}\xi_{ij}\sigma_i\tau_j, \tag{45}$$

where the ξ_{ij} are also i.i.d. r.v., with $\mathbb{E}[\xi]=0$ and $\mathbb{E}[\xi^2]=1$, i.e. the quenched noise ruling the mutual interaction between parties. In particular we deal with an $\mathcal{N}(0,1)$ quenched disorder. It is then defined a bipartite model of spin glass where emphasis is given on its bipartite nature by neglecting self-interactions, mirroring the strategy outlined in the first part of the work.

For the sake of simplicity, as external fields in complex systems must be considered much more carefully with respect to the simple counterparts, we are going to work out the theory neglecting them at this stage.

Of course, once the Hamiltonian is given, its results defined the partition function, the pressure and the free energy of the model as

$$Z_{N_1,N_2}(\beta) = \sum_{\sigma,\tau}\exp(-\beta H_{N_1,N_2}(\sigma,\tau)), \tag{46}$$

$$A_{N_1,N_2}(\beta) = \frac{1}{N_1+N_2}\mathbb{E}_J\log Z_{N_1,N_2}(\beta), \tag{47}$$

$$f_{N_1,N_2}(\beta) = -\frac{1}{\beta}A_{N_1,N_2}(\beta). \tag{48}$$

We can also define the Boltzmann state for an observable function of the spin variables $z(\sigma,\tau)$:

$$\omega_{N_1,N_2}(z) = Z_{N_1,N_2}^{-1}(\beta)\sum_{\sigma,\tau}[z(\sigma,\tau)\exp(-\beta H_{N_1,N_2}(\sigma,\tau))]$$

and, as in glasses we need to introduce replicas (equivalent copies of the system with the same identical quenched disorder), we can even define the Boltzmann product state as $\Omega_{N_1,N_2}=\omega_{N_1,N_2}\times\cdots\times\omega_{N_1,N_2}$, where the amount of replicas can be specified one at a time.

Remark 4. In analogy with the prescription introduced in the normalization of the bipartite ferromagnet, the factor $\sqrt{2}$ in the Hamiltonian is put *ad hoc* in order to obtain for the balanced bipartite spin glass ($\alpha=1/2$) the same critical point of the Sherrington–Kirkpatrick single party model, as will be clarified in the next section.

The main achievement of the theory would be a complete control of the free energy, or the pressure, in the thermodynamic limit, i.e. for $N_1,N_2\to\infty$. We stress that two different cases may arise: the first is that the size of one party grows faster than the other; the second is that the two sizes grows in the same way, such that the ratio $N_1/N\to\alpha\in(0,1)$ is well defined and $N_2/N\to(1-\alpha)\in(0,1)$ again for coherence with the strategy outlined in the first part of the work and for a general higher interest in this case. We will adopt this latter definition of the thermodynamic limit, and thus the thermodynamic functions depend on the additional parameter α ruling the relative ratio among the parties:

$$\lim_N A_{N_1,N_2}(\beta) = A(\alpha,\beta).$$

We must stress that at the moment no rigorous proof of the existence of such a limit is known and that there is a deep connection among this limit and the one of the Hopfield model for neural networks [9, 10]. Finally we must introduce an overlap, that is, correlation functions among replicas. It naturally arises how in this model we have two of these observables, one referred to each party; in fact we define immediately

$$q_{ab} = \frac{1}{N_1}\sum_i \sigma_i^a \sigma_i^b,$$
$$p_{ab} = \frac{1}{N_2}\sum_j \tau_j^a \tau_j^b.$$

3.1. High temperature behavior

We start our study of the model by characterizing the high temperature regime. It turns out that the system behaves like the annealed one in a wide region of the (α, β)-plane, stated as follows.

Theorem 3. *In the infinite volume limit the pressure of the model, without the quenched average, converges J-almost surely to its annealed value*

$$\lim_{N\to\infty} \frac{1}{N_1+N_2} \log Z_{N_1,N_2}(\beta) = A_A(\alpha,\beta) = \log 2 + \beta^2\alpha(1-\alpha), \tag{49}$$

in the region of the (α, β) *plane defined by* $2\beta^2\sqrt{\alpha(1-\alpha)} \leqslant 1$.

Proof. At first we calculate the annealed partition function

$$\mathbb{E}[Z_{N_1,N_2}(\beta)] = \exp(N(\log 2 + \beta^2\alpha(1-\alpha))); \tag{50}$$

thus,

$$A_A(\alpha,\beta) = \lim_{\substack{N_1,N_2\\ \frac{N_1}{N}\to\alpha}} (N)^{-1} \log \mathbb{E}[Z_{N_1,N_2}(\beta)] = \log 2 + \beta^2\alpha(1-\alpha).$$

Now, following a standard method [10, 30], we use the Borel–Cantelli lemma on $Z_{N_1,N_2}/\mathbb{E}[Z_{N_1,N_2}]$. We evaluate the second moment of the partition function:

$$\begin{aligned}
\mathbb{E}\big[Z_{N_1,N_2}^2\big] &= \mathbb{E}_\xi \sum_{\{\sigma,\tau\}} \exp\left(\beta\sqrt{\frac{2}{N}}\sum_{ij}\xi_{ij}(\sigma_i\tau_j + \sigma_i'\tau_j')\right)\\
&= \sum_{\{\sigma,\tau\}} \exp\left(\beta^2\frac{N_1N_2}{N}(2+2q_{12}p_{12})\right)\\
&= \mathrm{e}^{2N(\ln 2+\beta^2\alpha(1-\alpha))}\sum_{\{\sigma,\tau\}}\exp(2N\beta^2\alpha(1-\alpha)q_{12}p_{12}),
\end{aligned} \tag{51}$$

where we neglected terms leading to an error for the pressure $O(N^{-2})$, by replacing α_N with α. Now we perform the transformation $\sigma \to \sigma\sigma'$ and $\tau \to \tau\tau'$ in order to obtain $q_{12} \to m$ and $p_{12} \to n$. Thus, we have

$$\begin{aligned}
\frac{\mathbb{E}\big[Z_{N_1,N_2}^2\big]}{\mathbb{E}[Z_{N_1,N_2}]^2} &= \sum_{\{\sigma,\tau\}} \exp((N_1+N_2)2\beta^2\alpha(1-\alpha)mn)\\
&= \sum_{\{\sigma,\tau\}} \exp((N_1+N_2)\beta'\alpha(1-\alpha)mn),
\end{aligned}$$

with $\beta' = \beta^2$. The last term, as we have seen in the previous section about bipartite ferromagnetic models, is bounded for $2\beta'\sqrt{\alpha(1-\alpha)} \leqslant 1$, i.e. $2\beta^2\sqrt{\alpha(1-\alpha)} \leqslant 1$. A standard application of the Borel–Cantelli lemma completes the proof.

□

We note that this is a result slightly different with respect to the Hopfield model [10]. In fact we have that the annealed free energy and the true one are the same still at small temperatures, depending on the different weights assumed by the parties (that are of course ruled by α). This is reflected by the symmetry of the high temperature region with respect to the line $\alpha = 1/2$. The following argument will clarify this point.

Given respectively an $N_1 \times N_1$ and an $N_2 \times N_2$ random matrix J and J', with both J_{ij} and J'_{ij} normal distributed random variables for every i, j, we introduce the interpolating partition function

$$Z_{N_1,N_2}(\beta,t) = \sum_{\sigma,\tau} \mathrm{e}^{(\beta\sqrt{\frac{2t}{N}}\sum_{ij}^{N_1,N_2}\xi_{ij}\sigma_i\tau_j + a^2\beta\sqrt{\frac{2(1-t)}{N}}\sum_{ij}^{N_1}J_{ij}\sigma_i\sigma_j + \frac{\beta}{a^2}\sqrt{\frac{2(1-t)}{N}}\sum_{ij}^{N_2}J'_{ij}\tau_i\tau_j)},$$

where a is a parameter to be determined *a posteriori*. Putting $\beta' = \beta a^2\sqrt{2\alpha}$ and $\beta'' = \beta a^{-2}\sqrt{2(1-\alpha)}$, and neglecting terms vanishing when N grows to infinity, we can rewrite the latter expression as

$$Z_{N_1,N_2}(\beta,\beta',\beta'',t) = \sum_{\sigma,\tau} \mathrm{e}^{(\beta\sqrt{\frac{2t}{N}}\sum_{ij}^{N_1,N_2}\xi_{ij}\sigma_i\tau_j + \beta'\sqrt{\frac{(1-t)}{N_1}}\sum_{ij}^{N_1}J_{ij}\sigma_i\sigma_j + \beta''\sqrt{\frac{(1-t)}{N_2}}\sum_{ij}^{N_2}J'_{ij}\tau_i\tau_j)}.$$

Now we introduce the function

$$\phi_{N_1,N_2}(t,\beta,\beta',\beta'') = \frac{1}{N}\mathbb{E}\log Z_{N_1,N_2}(\beta,\beta',\beta'',t) + \frac{t}{4}\big(\beta'^2\alpha + \beta''^2(1-\alpha) + 4\beta^2\alpha(1-\alpha)\big).$$

It is easily verified that

$$\begin{cases} \phi_{N_1,N_2}(t=0) & = & \alpha A^{\mathrm{SK}}_{N_1}(\beta') + (1-\alpha)A^{\mathrm{SK}}_{N_2}(\beta'') \\ \phi_{N_1,N_2}(t=1) & = & A_{N_1,N_2}(\beta) + \frac{1}{4}(\beta'^2\alpha + \beta''^2(1-\alpha) - 4\beta^2\alpha(1-\alpha)) \end{cases} \tag{52}$$

where, of course, A^{SK} is the pressure for the Sherrington–Kirkpatrick model, that is, the only one-party model. Furthermore, we can take the derivative in t and obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\phi_{N_1+N_2} = \langle(\beta'\sqrt{\alpha}q_{12} - \beta''\sqrt{1-\alpha}\,p_{12})^2\rangle \geqslant 0,$$

since $2\beta'\beta''\sqrt{\alpha(1-\alpha)} = 4\beta\alpha(1-\alpha)$. Hence, we obtain the following bound for the pressure:

$$A_{N_1,N_2}(\beta) \geqslant \alpha A^{\mathrm{SK}}_{N_1}(\beta') + (1-\alpha)A^{\mathrm{SK}}_{N_2}(\beta'') - \tfrac{1}{4}(\beta'^2\alpha + \beta''^2(1-\alpha) + 4\beta^2\alpha(1-\alpha)). \tag{53}$$

Now we can fix a^2 in such a way that $\beta' = \beta'' = \bar{\beta}$. As a consequence, it results $a^4 = \sqrt{(1-\alpha)/\alpha}$ and $\bar{\beta}^2 = 2\beta^2\sqrt{\alpha(1-\alpha)}$, and the formula (53) becomes

$$A_{N_1,N_2}(\beta) \geqslant \alpha A^{\mathrm{SK}}_{N_1}(\bar{\beta}) + (1-\alpha)A^{\mathrm{SK}}_{N_2}(\bar{\beta}) - \frac{\bar{\beta}^2}{4} + \beta^2\alpha(1-\alpha). \tag{54}$$

Thus, the pressure is always greater than the convex sum of the single party Sherrington–Kirkpatrick pressure. The extra term is build in such a way that we get an equality in the annealed region. In fact we have, if $\bar{\beta} \leqslant 1$, i.e. $2\beta^2\sqrt{\alpha(1-\alpha)} \leqslant 1$, that in thermodynamic limit both $A^{\mathrm{SK}}_{N_1}$ and $A^{\mathrm{SK}}_{N_2}$ are $\log 2 + \bar{\beta}^2/4$, and therefore we get

$$\log 2 + \beta^2\alpha(1-\alpha) \geqslant A(\alpha,\beta) \geqslant \log 2 + \beta^2\alpha(1-\alpha),$$

where, as usual, the upper bound is given by the Jensen inequality.

3.2. Replica symmetric free energy

In order to obtain an explicit expression for the free energy density (or equivalently the pressure $A(\alpha, \beta)$), we apply the double stochastic stability technique recently developed in [9]. In a nutshell, the idea is to perturb stochastically both the parties via random perturbations; these are coupled with scalar parameters to be set *a fortiori* in order to get the desired level of approximation. With these perturbations the calculations can be reduced to a sum of one-body problems via a suitable sum rule for the free energy; by the latter, the replica symmetric approximation can be obtained straightforwardly by neglecting the fluctuations of the order parameters.

Concretely we introduce the following interpolating partition function, for $t \in [0, 1]$:

$$Z_{N_1,N_2}(t) = \sum_\sigma \sum_\tau \exp\left(\sqrt{t}\frac{\sqrt{2}\beta}{\sqrt{N}} \sum_{ij}^{N_1,N_2} \xi_{i,j}\sigma_i\tau_j\right) \cdot \exp\left(\sqrt{1-t}\left[\beta\sqrt{2(1-\alpha)\bar{p}}\sum_i^{N_1}\eta_i\sigma_i + \beta\sqrt{2\alpha\bar{q}}\sum_j^{N_2}\tilde{\eta}_j\tau_j\right]\right), \tag{55}$$

where $\eta, \tilde{\eta}$ are stochastic perturbations, namely i.i.d. random variables $\mathcal{N}[0, 1]$, whose averages are still encoded into $\mathbb{E}$, and, so far, $\bar{q}$, $\bar{p}$ are Lagrange multipliers to be determined later.

Now we introduce the interpolating function

$$A_{N_1,N_2}(t) = \frac{1}{N}\mathbb{E}\log Z_{N_1,N_2}(t) + (1-t)\alpha(1-\alpha)\beta^2(1-\bar{q})(1-\bar{p}).$$

It is easily seen that at $t = 1$ we recover the original pressure $A(\alpha, \beta)$, while for $t = 0$ we obtain a factorized one-body problem:

$$\begin{cases} \lim_N A_{N_1,N_2}(t=1) &= A_{N_1,N_2}(\beta), \\ \lim_N A_{N_1,N_2}(t=0) &= A_0(\alpha,\beta) + \alpha(1-\alpha)\beta^2(1-\bar{q})(1-\bar{p}), \end{cases} \tag{56}$$

with

$$\begin{aligned} A_0(\alpha,\beta) &= \frac{1}{N}\mathbb{E}\log\sum_\sigma \exp\left(\beta\sqrt{2(1-\alpha)\bar{p}}\sum_i \eta_i\sigma_i\right) \\ &\quad + \frac{1}{N}\mathbb{E}\log\sum_\tau \exp\left(\beta\sqrt{2\alpha\bar{q}}\sum_j \tilde{\eta}_j\tau_j\right) \\ &= \ln 2 + \alpha\mathbb{E}_g\log\cosh(g\beta\sqrt{2(1-\alpha)\bar{p}}) + (1-\alpha)\mathbb{E}_g\log\cosh(g\beta\sqrt{2\alpha\bar{q}}), \end{aligned} \tag{57}$$

where $\mathbb{E}_g$ indicates the expectation with respect to the $\mathcal{N}(0, 1)$ r.v. g.

Now we must evaluate the t-derivative of $A_{N_1,N_2}(t)$ in order to get a sum rule, namely

$$A(t=1) = A_0(\alpha,\beta) + \int_0^1 \mathrm{d}t\left(\frac{\mathrm{d}}{\mathrm{d}t}A(t)\right). \tag{58}$$

Denoting via $\langle\rangle_t$ the extended Boltzmann measure encoded into the structure (55)—that reduces to the standard one for $t = 1$ as it should—we obtain three terms by deriving the four contributions into the extended Maxwell–Boltzmann exponential, that we call $\mathcal{A}, \mathcal{B}, \mathcal{C}$ and follow

$$\mathcal{A} = \frac{1}{N}\mathbb{E}\frac{\beta}{2\sqrt{tN}}\sum_{ij}\xi_{ij}\omega(\sigma_i\tau_j) = \alpha(1-\alpha)\beta^2(1-\langle q_{12}p_{12}\rangle_t), \tag{59}$$

$$\mathcal{B} = \frac{-1}{N}\mathbb{E}\frac{\beta\sqrt{2(1-\alpha)\bar{p}}}{2\sqrt{1-t}}\sum_i \eta_i\omega(\sigma_i) = -\frac{\alpha(1-\alpha)}{2}\beta^2\bar{p}(1-\langle q_{12}\rangle_t), \tag{60}$$

$$\mathcal{C} = \frac{-1}{N}\mathbb{E}\frac{\beta\sqrt{2\alpha\bar{q}}}{2\sqrt{1-t}}\sum_j \tilde{\eta}_j\omega(\tau_j) = -\frac{\alpha(1-\alpha)}{2}\beta^2\bar{q}(1-\langle p_{12}\rangle_t). \tag{61}$$

So we can build the *t*-derivative of the interpolant $A_{N_1,N_2}(t)$ as

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}A_{N_1,N_2}(t) &= \mathcal{A}+\mathcal{B}+\mathcal{C}-\alpha(1-\alpha)\beta^2(1-\bar{q})(1-\bar{p})\\ &= \alpha(1-\alpha)\beta^2\,(1-\langle q_{12}p_{12}\rangle_t) - \frac{\alpha(1-\alpha)}{2}\beta^2\bar{p}\,(1-\langle q_{12}\rangle_t)\\ &\quad - \frac{\alpha(1-\alpha)}{2}\beta^2\bar{q}\,(1-\langle p_{12}\rangle_t) - \alpha(1-\alpha)\beta^2(1-\bar{q})(1-\bar{p})\\ &= -\alpha(1-\alpha)\beta^2\langle(\bar{q}-q_{12})(\bar{p}-p_{12})\rangle_t.\end{aligned} \tag{62}$$

Now we stress that the Lagrange multipliers $\bar{q}$ and $\bar{p}$ can be understood here as trial values for the order parameters. According to this point of view, the replica symmetric condition, i.e. the request that the overlaps do not fluctuate, is equivalent to impose $\langle(\bar{q}-q_{12})(\bar{p}-p_{12})\rangle_t = 0$, bringing us to conclude that in RS regime $A_{N_1,N_2}(t)$ is a steady function of *t*, and then

$$A_{N_1,N_2}(t=1) = A_{N_1,N_2}(\beta) = A_{N_1,N_2}(t=0) = A_0(\alpha,\beta)+\alpha(1-\alpha)\beta^2(1-\bar{q})(1-\bar{p}),$$

that is, in the thermodynamic limit

$$\begin{aligned}\bar{A}(\bar{p},\bar{q},\alpha,\beta) &= \ln 2+\alpha\mathbb{E}_g\log\cosh(g\beta\sqrt{2(1-\alpha)\bar{p}})\\ &\quad+(1-\alpha)\mathbb{E}_g\log\cosh(g\beta\sqrt{2\alpha\bar{q}})+\alpha(1-\alpha)\beta^2(1-\bar{q})(1-\bar{p}).\end{aligned} \tag{63}$$

Now we follow the same considerations exploited in [9]. Indeed, the last expression holds barely for every possible choice of the trial values $\bar{q}$ and $\bar{p}$ of the order parameters $\langle q_{12}\rangle$ and $\langle p_{12}\rangle$. Our purpose is then to fix the right value of $\bar{q}$ and $\bar{p}$, imposed by the RS condition $\langle(\bar{q}-q_{12})(\bar{p}-p_{12})\rangle = 0$. *In primis* we note that the trial function $\bar{A}$, as a function of the trial order parameters $\bar{q}$, $\bar{p}$, is uniformly concave with respect to $\bar{p}$. In fact it is easily seen that

$$\partial_{\bar{p}}\bar{A}(\bar{q},\bar{p},\alpha,\beta) = \alpha(1-\alpha)\beta^2(\bar{q}-\mathbb{E}_g\tanh^2(g\beta\sqrt{2(1-\alpha)\bar{p}})),$$

and since $\mathbb{E}_g\tanh^2(g\beta\sqrt{2(1-\alpha)\bar{p}})$ is increasing in $\bar{p}$ we have the assertion. Furthermore, for any fixed $\bar{q}$, the function $\bar{A}$ takes its maximum value where the derivative vanishes, which defines implicitly a special value for $\bar{p}(\bar{q})$:

$$\bar{q} = \mathbb{E}_g\tanh^2(g\beta\sqrt{2(1-\alpha)\bar{p}(\bar{q})}).$$

Of course we have that $\bar{p}$ is an increasing function of $\bar{q}$, with $\bar{p}(0) = 0$. Now we are concerned about $\bar{A}$ at a fixed level set, i.e. $A(\bar{p}(\bar{q}),\bar{q})$, and state that it is convex in $\bar{q}^2$. In fact it is easily seen from the last formula that $\bar{p}(\bar{q})/\bar{q}$ is an increasing function of $\bar{q}$, but of course $\mathbb{E}_g\tanh^2\left(g\beta\sqrt{2\alpha\bar{q}}\right)/\bar{q}$ is strictly decreasing; thus,

$$\partial_{\bar{q}}\bar{A}(\bar{q},\bar{p})/\bar{q} = \alpha(1-\alpha)\beta^2(\bar{p}(\bar{q})-\mathbb{E}_g\tanh^2(g\beta\sqrt{2\alpha\bar{q}}))/\bar{q}$$

is increasing in $\bar{q}$, that is, equivalent to convexity in $\bar{q}^2$. The last equation specifies the right value of the trial order parameter $\bar{q}$ in the RS approximation. Thus, the replica symmetric approximation of the pressure results uniquely defined by the minimax principle.

Theorem 4. *The replica symmetric free energy of the bipartite spin glass model is uniquely defined by the following variational principle:*

$$A^{\mathrm{RS}}(\bar{p}, \bar{q}, \alpha, \beta) = \min_{\bar{q}} \max_{\bar{p}} \bar{A}(\bar{p}, \bar{q}, \alpha, \beta), \tag{64}$$

where

$$\bar{A}(\bar{p}, \bar{q}, \alpha, \beta) = \ln 2 + \alpha \mathbb{E}_g \log\cosh(g\beta\sqrt{2(1-\alpha)\bar{p}}) + (1-\alpha)\mathbb{E}_g \log\cosh(g\beta\sqrt{2\alpha\bar{q}}) + \alpha(1-\alpha)\beta^2(1-\bar{q})(1-\bar{p}) \tag{65}$$

for which the saddle point is reached at the intersection of the following two curves in the (α, β) *plane*

$$\bar{q}(\alpha, \beta) = \mathbb{E}_g \tanh^2(g\beta\sqrt{2(1-\alpha)\bar{p}(\alpha, \beta)}) \tag{66}$$

$$\bar{p}(\alpha, \beta) = \mathbb{E}_g \tanh^2(g\beta\sqrt{2\alpha\bar{q}(\alpha, \beta)}). \tag{67}$$

We can go further and put some constraints, imposing the two curves to intersect away from $(\bar{p} = 0, \bar{q} = 0)$. Hence, we must have a precise relation among the slopes of such two curves near the origin, i.e. $\lim_{\bar{q}\to 0} \bar{p}(\bar{q})/\bar{q} \geqslant \lim_{\bar{p}\to 0} \bar{p}/\bar{q}(\bar{p})$; but since $\lim_{\bar{q}\to 0} \bar{p}(\bar{q})/\bar{q} = 2\alpha\beta^2$ and $\lim_{\bar{p}\to 0} \bar{p}/\bar{q}(\bar{p}) = 1/2\beta^2(1-\alpha)$, the latter inequality simply leads us to conclude that only the trivial intersection point is possible for $4\beta^4\alpha(1-\alpha) \leqslant 1$. Therefore, we can resume all these results in the following

Proposition 3. *In the thermodynamic limit, the replica symmetric pressure of the bipartite spin glass exists and is unique, given by (65). Furthermore, the region of the* (α, β) *plane such that the (66) and (67) have only the trivial solutions is characterized by* $4\beta^4\alpha(1-\alpha) \leqslant 1$.

Remark 5. In fact for $4\beta^4\alpha(1-\alpha) \leqslant 1$ the min max is obtained for $\bar{q}, \bar{p} = 0$, and, as it is easily seen, the pressure (65) reduces to the annealed one (49) that coincides with the true pressure of the model in the thermodynamic limit in such a region.

Furthermore, bearing in mind (65), together with (66) and (67), it is a remarkable result that the replica symmetric free energy of the bipartite model is given by the convex combination of two monopartite spin glasses, at different temperatures, exactly as happens in the ferromagnetic case. This is clarified by the following

Proposition 4. *Choosing* $\beta' = \beta\sqrt{2\alpha}\sqrt{\frac{1-\bar{q}}{1-\bar{p}}}$ *and* $\beta'' = \beta\sqrt{2(1-\alpha)}\sqrt{\frac{1-\bar{q}}{1-\bar{p}}}$, *we have*

$$A_{\mathrm{RS}}(\alpha, \beta) = \alpha A_{\mathrm{RS}}^{\mathrm{SK}}(\beta') + (1-\alpha) A_{\mathrm{RS}}^{\mathrm{SK}}(\beta''), \tag{68}$$

while, with the different scaling of the inverse temperatures $\beta' = \beta\sqrt{2\alpha}\sqrt{\frac{\bar{q}}{\bar{p}}}$ *and* $\beta'' = \beta\sqrt{2(1-\alpha)}\sqrt{\frac{\bar{q}}{\bar{p}}}$, *we have*

$$A_{\mathrm{RS}}(\alpha, \beta) - A_A(\alpha, \beta) = \alpha\left(A_{\mathrm{RS}}^{\mathrm{SK}}(\beta') - A_A^{\mathrm{SK}}(\beta')\right) + (1-\alpha)\left(A_{\mathrm{RS}}^{\mathrm{SK}}(\beta'') - A_A(\beta'')\right). \tag{69}$$

Proof. The proof follows by a straight calculation. If we take for the two monopartite model the two different inverse temperatures $\beta' = \beta\sqrt{2\alpha}a^2$ and $\beta'' = \beta\sqrt{2(1-\alpha)}a^2$, with a a free parameter to be determined at the end, we have

$$
\begin{aligned}
A_{\mathrm{RS}}(\alpha,\beta) &= \ln 2 + \alpha \mathbb{E}_g \log\cosh(g\beta\sqrt{2(1-\alpha)\bar{p}}) \\
&\quad + (1-\alpha)\mathbb{E}_g \log\cosh(g\beta\sqrt{2\alpha\bar{q}}) + \alpha(1-\alpha)\beta^2(1-\bar{q})(1-\bar{p}) \\
&= \ln 2 + \alpha\mathbb{E}_g \log\cosh(g\beta'' a^2\sqrt{\bar{p}}) \\
&\quad + (1-\alpha)\mathbb{E}_g \log\cosh\left(\frac{g\beta}{a^2}\sqrt{\bar{q}}\right) + \sqrt{\alpha(1-\alpha)}\beta'\beta''(1-\bar{q})(1-\bar{p}) \\
&\leqslant \alpha\left(\log 2 + \mathbb{E}_g \log\cosh(g\beta'' a^2\sqrt{\bar{p}}) + \frac{\beta''^2}{4}a^2(1-\bar{p})^2\right) \\
&\quad + (1-\alpha)\left(\log 2 + \mathbb{E}_g \log\cosh\left(g\frac{\beta'}{a^2}\sqrt{\bar{q}}\right) + \frac{\beta'^2}{4a^2}(1-\bar{q})^2\right).
\end{aligned}
\tag{70}
$$

Now it is easily seen that in the last formula, we obtain an equality with $a^4 = \frac{1-\bar{q}}{1-\bar{p}}$. Following exactly the same path of the previous part of the work (dealing with ferromagnetic models) we recover (69) with the choice $a^4 = \frac{\bar{q}}{\bar{p}}$. We stress that this last value of a is more meaningful, in the sense that in this case the two monopartite models are trivially independent and separated, with the order parameters given by usual self-consistency relations for the SK model:

$$
\begin{aligned}
\bar{q} &= \mathbb{E}_g \tanh^2(g\beta'\sqrt{\bar{q}}), \\
\bar{p} &= \mathbb{E}_g \tanh^2(g\beta''\sqrt{\bar{p}}),
\end{aligned}
$$

in perfect analogy with the ferromagnetic case. □

Lastly, as is well known, a theory with no overlap fluctuations allowed may not hold at low temperatures and we report about its properties in the limit $\beta \to \infty$ to check the stability of the replica symmetric ansatz. We will be concerned about the ground state energy $\hat{e}_{\mathrm{RS}}$ and its associated entropy $\hat{s}_{\mathrm{RS}}$, defined by

$$
\hat{e}_{\mathrm{RS}}(\alpha) = \lim_{\beta\to\infty} \partial_\beta \bar{A}_{\mathrm{RS}}(\alpha,\beta) = \lim_{\beta\to\infty} \bar{A}_{\mathrm{RS}}(\alpha,\beta)/\beta, \tag{71}
$$

$$
\hat{s}_{\mathrm{RS}}(\alpha) = \lim_{\beta\to\infty} (\bar{A}_{\mathrm{RS}}(\alpha,\beta) - \beta\partial_\beta \bar{A}_{\mathrm{RS}}(\alpha,\beta)). \tag{72}
$$

First of all, from the self-consistency equations (66), (67) through a long but straightforward calculation, we can compute the low temperature limit for the order parameters, $\bar{q}(\alpha,\beta), \bar{p}(\alpha,\beta) \to 1$, together with the rates they approach to their limit value, $\beta(1-\bar{q}(\alpha,\beta)) \to (\pi(1-\alpha))^{-1/2}$, $\beta(1-\bar{p}(\alpha,\beta)) \to (\pi\alpha)^{-1/2}$. Then, bearing in mind the explicit form of the pressure of the model in the replica symmetric regime (65), we derive the following expressions for the ground state energy and the entropy:

$$
\hat{e}_{\mathrm{RS}}(\alpha) = \sqrt{\frac{\alpha(1-\alpha)}{\pi}}, \tag{73}
$$

$$
\hat{s}_{\mathrm{RS}}(\alpha) = -\frac{2}{\pi}(1-\sqrt{\alpha(1-\alpha)}). \tag{74}
$$

Note that the entropy $\hat{s}_{\mathrm{RS}}(\alpha)$ is strictly less than zero for every $\alpha \in (0,1)$, that is, a typical feature of the replica symmetric *ansatz* for glassy systems. Therefore, the true solution of the model must involve replica symmetry breaking. Furthermore, it is a concave function of α, and assume its maximum value $\hat{s}_{\mathrm{RS}} = -1/\pi$ in $\alpha = 1/2$, i.e. the balanced bipartite, and its minimum value $\hat{s}_{\mathrm{RS}} = -2/\pi$ at the ending points $\alpha = 0, 1$, when the size of one party is negligible in the thermodynamic limit. Analogously, for the ground state energy, we find in the perfectly balanced case $\hat{e}_{RS}(1/2) = 1/2\sqrt{\pi}$, and at the extrema of the definition interval of α $\hat{e}_{RS}(0) = \hat{e}_{RS}(1) = 0$.

3.3. Constraints

While the order parameters for simple models (as the bipartite CW) are self-averaging, frustrated systems are expected to show the replica symmetry-breaking phenomenon [20, 27], which ultimately inhibits such a self-averaging properties for $\langle q_{12}\rangle$, $\langle p_{12}\rangle$. As a consequence, a certain interest for the constraints to free overlap fluctuations arose in the past [1, 5, 18, 19] (and recently has been deeply connected to ultrametricity [3, 28]) which motivates us to work out the same constraints even in bipartite models.

To fulfill this task the first step is obtaining an explicit expression for the internal energy density (which is self-averaging [13]).

Theorem 5. *The following expression for the quenched average of the internal energy density of the bipartite spin glass model holds in the thermodynamic limit:*

$$\lim_{N\to\infty}\frac{1}{N}\langle H_{N_1,N_2}(\sigma,\tau;\xi)\rangle = e(\alpha,\beta) = 2\alpha(1-\alpha)\beta^2(1-\langle q_{12}p_{12}\rangle). \tag{75}$$

As the proof can be achieved by direct evaluation, we skip it and turn to the constraints: starting with the linear identities we state the following

Proposition 5. *In the thermodynamic limit, and β almost everywhere, the following generalization of the linear overlap constraints holds for the bipartite spin glass:*

$$\langle q_{12}^2 p_{12}^2\rangle - 4\langle q_{12}p_{12}q_{23}p_{23}\rangle + 3\langle q_{12}p_{12}q_{34}p_{34}\rangle = 0. \tag{76}$$

Proof. Let us address our task by looking at the β derivative of the internal energy density, once expressed via $\langle q_{12}p_{12}\rangle$; in a nutshell, physically, we obtain these constraints by imposing that such a response cannot diverge, not even in the thermodynamic limit:

$$\partial_\beta\langle q_{12}p_{12}\rangle = \frac{1}{N_1N_2}\sum_{i,j}\mathbb{E}\partial_\beta\omega^2(\sigma_i\tau_j) = \frac{1}{N_1N_2}\sum_{i,j}\mathbb{E}2\omega(\sigma_i\tau_j)\partial_\beta\omega(\sigma_i\tau_j) \tag{77}$$

$$= \frac{2}{N_1N_2}\sum_{i,j}\mathbb{E}\omega(\sigma_i\tau_j)\xi_{i\nu}(\omega(\sigma_i\tau_j\sigma_j\tau_\nu) - \omega(\sigma_i\tau_j)\omega(\sigma_j\tau_\nu)); \tag{78}$$

now we use Wick's theorem on ξ and introducing the overlaps we have

$$\begin{aligned}\partial_\beta\langle q_{12}p_{12}\rangle = N_2\Big(&\langle p_{12}^2q_{12}^2\rangle - \langle p_{12}q_{12}p_{13}q_{13}\rangle\\ &- \langle p_{12}q_{12}p_{13}q_{13}\rangle + \langle p_{12}q_{12}p_{34}q_{34}\rangle + \langle\bar{p}q_{12}p_{12} - \langle p_{12}q_{12}p_{13}q_{13}\rangle\\ &- \langle p_{12}q_{12}p_{13}q_{13}\rangle + \langle p_{12}q_{12}p_{34}q_{34}\rangle - \langle\bar{p}q_{12}p_{12}\rangle + \langle p_{12}q_{12}p_{34}q_{34}\rangle.\end{aligned} \tag{79}$$

The several cancellations leave the following remaining terms:

$$\partial_\beta\langle q_{12}p_{12}\rangle = N_2\big(\langle q_{12}^2p_{12}^2\rangle - 4\langle q_{12}p_{12}q_{23}p_{23}\rangle + 3\langle q_{12}p_{12}q_{34}p_{34}\rangle\big) \tag{80}$$

and, again in the thermodynamic limit, the thesis is proved. □

Proposition 6. *In the thermodynamic limit, and in β-average, the following generalization of the quadratic Ghirlanda–Guerra relations holds for the bipartite spin glass:*

$$\langle q_{12}p_{12}q_{23}p_{23}\rangle = \frac{1}{2}\langle q_{12}^2 p_{12}^2\rangle + \frac{1}{2}\langle q_{12}p_{12}\rangle^2, \tag{81}$$

$$\langle q_{12}p_{12}q_{34}p_{34}\rangle = \frac{1}{3}\langle q_{12}^2 p_{12}^2\rangle + \frac{2}{3}\langle q_{12}p_{12}\rangle^2. \tag{82}$$

Proof. The idea is to impose, in the thermodynamic limit, the self-averaging of the internal energy (i.e. $\langle e^2(\alpha,\beta)\rangle - \langle e(\alpha,\beta)\rangle^2 = 0$); we obtain a rest that must be set to zero to give the quadratic control. Starting from

$$\mathbb{E}(e^2(\alpha,\beta)) = \frac{1}{(N_1+N_2)^3}\sum_{j,i}\sum_{\nu,j}\xi_{ij}\xi_{j\nu}\omega(\sigma_i\tau_j)\omega(\sigma_j\tau_\nu),$$

with a calculation perfectly analogous to the one performed in the proof of proposition 5 and comparing with the former relations, we obtain the linear system

$$0 = \langle q_{12}^2 p_{12}^2\rangle + 6\langle q_{12}p_{12}q_{34}p_{34}\rangle - 6\langle q_{12}p_{12}q_{23}p_{23}\rangle - \langle q_{12}p_{12}\rangle^2, \tag{83}$$

$$0 = \langle q_{12}^2 p_{12}^2\rangle - 4\langle q_{12}p_{12}q_{23}p_{23}\rangle + 3\langle q_{12}p_{12}q_{34}p_{34}\rangle, \tag{84}$$

whose solutions give exactly the expressions reported in proposition 6. □

4. Conclusion

In this paper, we analyzed the equilibrium behavior of bipartite spin systems (interacting both with ferromagnetic and with spin glass couplings) through statistical mechanics; these systems are made of two different subsets of spins (*a priori* of different nature [9, 17]), for the sake of clarity, each one interacting with the other, but with no self-interactions. For the former class through several techniques, among which our mechanical analogy of the interpolation method, early developed in [19] and successfully investigated in [4, 8, 17], we have seen that the thermodynamic limit of the pressure does exist and it is unique and we gave its explicit expression in a constructive way via a minimax principle. Furthermore, when introducing Burger's equation for the velocity field in our interpretation of the interpolating scheme, our method automatically 'chooses' the correct order parameter, which turns out to be a linear combination of the magnetizations of the two subsystems with different signs, to convert the minimax variational principle in a standard extremization procedure. Noting that the same structure can be recovered for many other models of greater interest, like bipartite spin glasses, we went over and analyzed even the latter.

For these models we have studied both the annealing and the replica symmetric approximation (the latter through the double stochastic stability technique recently developed in [9]) which allowed us to give an explicit expression for the free energy and to discover and discuss the same minimax principle of the bipartite ferromagnets.

Furthermore, we evaluated the replica symmetric observable in the low temperature limit confirming the classical vision about the need for a broken replica symmetry scheme: one step forward in this sense, by studying the properties of the internal energy, we derived all the classical constraints to the free overlap fluctuations (suitably obtained for these systems) and we worked out a picture of their criticality to conclude the investigation.

Future works on these subjects should be addressed toward a complete full replica symmetry-broken picture and to a systematic exploration of the multi-partite equilibria.

Acknowledgments

The authors are pleased to thank Jacques H H Perk and Yan V Fyodorov for useful discussions. They are further grateful to MiUR through the FIRB grant number RBFR08EKEV and to Sapienza University of Rome. AB acknowledges partial funding by GNFM (Gruppo Nazionale per la Fisica Matematica). FG also acknowlegdes partial funding by INFN (Istituto Nazionale di Fisica Nucleare).

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