Fluctuation and Noise Letters
Vol. 8, Nos. 3–4 (2008) L341–L348

FLUCTUATIONS INDUCE TRANSITIONS IN FRUSTRATED SPARSE NETWORKS

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Communicated by Gyorgy Korniss

With the aim of describing a general benchmark for several complex systems, we analyze, by means of statistical mechanics, a sparse network with random competitive interactions among dichotomic variables pasted on the nodes. The model is described by an infinite series of order parameters (the multi-overlaps) and has two tunable degrees of freedom: the noise level and the connectivity (the averaged number of links). We show that there are no multiple transition lines, one for every order parameter, as a naive approach would suggest, but just one corresponding to ergodicity breaking. We explain this scenario within a novel and simple mathematical technique via a driving mechanism such that, as the first order parameter (the two replica overlap) becomes different from zero due to a real second order phase transition (with properly associated diverging rescaled fluctuations), it enforces all the other multi-overlaps toward positive values thanks to the strong correlations which develop among themselves and the two replica overlap at the critical line.

Keywords: Sparse networks; complex systems; spin glasses.

1. Introduction

Among several different complex systems [8], [21] and a large amount of tools for their investigation [16], [19], statistical mechanics of disordered systems earned an always increasing weight in the last two decades [1], [14].

In this paper, the *complex networks* we analyze by statistical mechanics can be understood as follows: they are networks because we allow the variables to live on the node of a non trivial graph (a Poissonian Erdos-Renyi graph [8]), the links among the nodes being the interacting fields they exchange.

They are complex because, as opposite i.e. to the Ising model [3] (in which all the variables share the same coupling constants) here the variables interact with equal probability via a positive coupling or a negative one, giving rise to frustration [14] and forming what is often called, in the language of statistical mechanics, a *diluted* *spin glass* [11], [20], while, its zero temperature limit is known, in the language of the theoretical computer science counterpart, as a pairwise Random X-OR-SAT [15] (strictly speaking random satisfability problems deal with p-spin models where interactions happen in groups larger than couples [13]; this is not a minor point as criticality in these systems is related to the $p = 2$ case, while for $p \geq 3$ the phase transition is discontinuous [4], [7], even though not first order in the sense of Ehrenfest [12] as there is no latent heat [9]).

As these models are not Gaussian, they need not just a (functional) order parameter (i.e. q_2) as their fully connected counterpart (i.e. the SK model [14]) but the whole series of multi-overlaps (i.e. $q_2, q_4, ..., q_{2n}$ [11], [20]) and one may ask if there are several transition lines for these multi-overlaps (one for every of them) or they share the unique transition line at which ergodicity breaks (the critical line for q_2). In a previous recent work [6] we proved only mathematically, by bounds, the latter scenario to be the correct one, but the physics behind was still rather obscure and in particular no ideas concerning the nature of this transition were presented.

In this paper we show both mathematically (extending our previous results) and physically (offering a picture for the nature of the transition) a complete scenario as follows: At the boundaries of the ergodic region the fluctuations of the first order parameter (i.e. q_2) start diverging, accordingly to a well-defined second order phase transition, while the fluctuations of all the others do not (suggesting the validity of the several transition alternative); however, due to the strong correlations that develop at the critical point among all the order parameters, this growth to a non zero value for q_2 drives all the others toward its direction, acting as an 'ad hoc' field in the space of these parameters. So the transition for the multi-overlaps surprisingly is nor first order neither second order; it is a driven transition via a self-generated coupling field which raises on the broken ergodicity line.

2. Equilibrium Thermodynamics of the Sparse Frustrated Network

Consider N nodes, indexed by Latin letters i, j, etc., with an Ising spin $\sigma_i = \pm 1$ attached to each of them. Let $P_{\alpha N}$ be a Poisson random variable of mean αN, let $\{J_\nu\}$ be independent identically distributed copies of a random variable J with symmetric distribution. For the sake of simplicity (but without loss of generality) we will assume $J = \pm 1$. We consider randomly chosen points, we therefore introduce $\{i_\nu\}, \{j_\nu\}$ as independent identically distributed random variables, with uniform distribution over $1, \ldots, N$. The Hamiltonian of the model (a suitable version of the Viana-Bray [20] one) is the following symmetric random variable

$$H_N(\sigma, \alpha; \mathcal{J}) = -\sum_{\nu=1}^{P_{\alpha N}} J_\nu \sigma_{i_\nu} \sigma_{j_\nu}\,, \quad \alpha \in \mathbb{R}_+\,. \tag{1}$$

The non-negative parameter α is called *connectivity*.

The Gibbs measure ω and the partition function $Z_N(\beta)$ are defined by

$$\omega(\varphi) = \frac{1}{Z}\sum_\sigma \exp(-\beta H(\sigma))\varphi(\sigma),\ \ Z_N(\beta) = \sum_\sigma \exp(-\beta H_N(\sigma))\,,$$

where $\varphi : \{-1, +1\}^N \to \mathcal{R}$ and β is the noise level in the network.

When dealing with more than one configuration, the product Gibbs measure is denoted by Ω, and various configuration taken from each product space are called "replicas". $\mathbb{E}$ is the expectation with respect to all the (quenched) variables, i.e. all the random variables except the spins, collectively denoted by $\mathcal{J}$ and we preserve the symbol $\langle . \rangle$ for $\mathbb{E}\Omega(.)$. Sometimes we will deal with a perturbed Boltzmann measure, whose perturbation is triggered by a tunable parameter t and we stress the dependence on such a perturbation with a subscript t on the averages $\langle . \rangle \to \langle . \rangle_t$.

The (quenched) free energy density f_N is defined by

$$A_N(\beta, \alpha) = -\beta f_N(\beta, \alpha) = \frac{1}{N} \mathbb{E} \ln Z_N(\beta, \alpha) \, .$$

The whole physical behavior of the model is encoded by the even multi-overlaps $q_{1\cdots 2n}$ [6], which are functions of several configurations $\sigma^{(1)}, \sigma^{(2)}, \ldots$ and defined by

$$q_{1\cdots 2n} = \frac{1}{N} \sum_{i=1}^{N} \sigma_i^{(1)} \cdots \sigma_i^{(2n)} \, .$$

For the sake of simplicity, often we will denote by $\theta = \theta(\beta)$ the expression $\tanh(\beta J) = \tanh(\beta)$.

Looking for order parameter responses, in these networks, one usually perturbs the system with a random field so to have

$$\tilde{H}_N(\sigma, h) = H_N(\sigma) + \sum_{i=1}^{N} h_i(t) \sigma_i \, , \tag{2}$$

where the tilde stands for the perturbed Hamiltonian, h_i are the random fields acting on the spins and $t \in [0, 1]$ a tuning of the amplitude of the perturbation, eventually sent to zero afterwards (of course $h(0) = 0$).

In our approach, due to the randomness of the coupling J and the gauge invariance of the model (the transformation $\sigma \to \sigma\epsilon$, with $\epsilon \pm 1$ which leaves the Hamiltonian unaffected being $\epsilon^2 = 1$) we can think at the random perturbation as a term $h_i \sim \sum_{\nu}^{P_{2\bar{\alpha}t}} \tilde{J}_\nu \sigma_{i_\nu}$ then, by applying the gauge $\sigma_{i_\nu} \to \sigma_{i_\nu} \sigma_{N+1}, \forall i_\nu$, we can turn the perturbation into a cavity field, mirroring an unperturbed system made by $N+1$ spins (whose properties are the same of the N-spin system, for large N).

Notice that, thanks to the additivity property of the Poisson variables, we can also write, in distribution,

$$H_{N+1}(\sigma; \alpha) \sim H_N(\sigma; \bar{\alpha}) + h_\tau \sigma_1 \, , \qquad \bar{\alpha} = \alpha \frac{N}{N+1} \, , \qquad h_\tau = -\sum_{\nu=1}^{P_{2\bar{\alpha}}} \tilde{J}_\nu \sigma_{k_\nu} \, . \tag{3}$$

Let us define further a *cavity function* $\Psi_{N,t}(\alpha, \beta)$ as the following quantity:

$$\Psi_{N,t}(\alpha, \beta) = \mathbf{E} \ln \omega\big(e^{\beta \sum_{\nu=1}^{P_{2\bar{\alpha}t}} \tilde{J}_\nu \sigma_{i_\nu}}\big). \tag{4}$$

Note that the cavity function takes into account the perturbation applied to the original Hamiltonian; it plays a fundamental role in the expansion of the free energy as it is immediately clear by the next theorem [2], [6]:

Theorem 1. *The following relation among free energy, its connectivity increment and cavity function holds in the $N \to \infty$ limit*:

$$A_N(\alpha,\beta) + \alpha\partial_\alpha A_N(\alpha,\beta) = \ln 2 + \Psi_{N,t=1}(\alpha,\beta)\,. \tag{5}$$

The next two straightforward propositions express explicitly the two term by which the free energy can be decomposed thanks to eq. (5).

- The incremental contribution to the free energy by the connectivity is [6]

$$\alpha\partial_\alpha A(\alpha,\beta) = 2\alpha\sum_1^\infty \frac{1}{2n}\theta^{2n}(1-\langle q_{2n}^2\rangle)\,. \tag{6}$$

- The cavity function can be represented by the integral of the series of all the fillable multi-overlaps weighted by the powers of θ [6]:

$$\Psi_{N,t}(\beta,\alpha) = \int_0^t 2\bar{\alpha}\sum_{n=1}^\infty \frac{1}{2n}\theta^{2n}(\beta J)(1-\langle q_{2n}\rangle_t')dt'\,. \tag{7}$$

The next two propositions help us in understanding how to deal with these two expressions:

- *Robustness* states that all the multi-overlaps which are "filled" , i.e. they have each replica appearing an even number of times (like $\langle q_{12}^2\rangle$, $\langle q_{1234}^2\rangle$, $\langle q_{12}q_{34}q_{1234}\rangle$) are not affected by the perturbation.

  More sharply in the $N \to \infty$ limit, the average $\langle\cdot\rangle_t$ of filled monomials is not affected by the presence of the perturbation modulated by t, that is, for instance,
$$\int_{\bar{\alpha}_1}^{\bar{\alpha}_2}\langle q_{12}q_{23}q_{13}\rangle_t d\bar{\alpha} = \int_{\bar{\alpha}_1}^{\bar{\alpha}_2}\langle q_{12}q_{23}q_{13}\rangle d\bar{\alpha}\,,$$
  $\forall[\bar{\alpha}_1,\bar{\alpha}_2]$. We call this property of filled monomials "robustness" [5].

- *Saturability* states that, once called "fillable" the other multi-overlap monomials, in the $t \to 1$, $N \to \infty$ limits, fillable monomials become filled (i.e. $\lim_{N\to\infty}\lim_{t\to1}\langle q_2\rangle_t = \langle q_2^2\rangle$, $\lim_{N\to\infty}\lim_{t\to1}\langle q_{12}q_{34}\rangle_t = \langle q_{12}q_{34}q_{1234}\rangle$).
  More sharply, let $q_{1\cdots 2n}$ be a fillable monomial of the multi-overlaps, such that $q_{1\cdots 2n}Q_{1\cdots 2n}$ is filled. Then

$$\lim_{N\to\infty}\langle q_{1\cdots 2n}\rangle_{t=1} = \langle q_{1\cdots 2n}Q_{1\cdots 2n}\rangle\,.$$

  We refer to this property as "saturability" [5].

To obtain a stochastically stable and gauge invariant iterative expression for the free energy, we have to expand the cavity function via filled monomials: Neglecting orders higher than $(2\alpha\theta^2)^2$ we get

$$\Psi_{N,t}(\alpha,\beta) = \int_0^t dt' 2\alpha\left(\frac{\theta^2}{2}(1-\langle q_{12}\rangle_{t'}) + \frac{\theta^4}{4}(1-\langle q_{1234}\rangle_{t'}) + \cdots\right), \tag{8}$$

which can be filled by expanding its internal multi-overlap monomials (i.e. $\langle q_{12}\rangle_t = 2\alpha\theta^2 t\langle q_{12}^2\rangle + O(t^2)$, $\langle q_{1234}\rangle_t = 2\alpha\theta^4 t\langle q_{1234}^2\rangle + O(t^2)$) and than trivially integrated back thanks to robustness.

We can now use Eq. (5) to write down our free energy expansion of the model. Presenting just the first orders, and remembering that we call $\tau = 2\alpha\theta^2$, we have

$$A(\alpha,\beta) = \ln 2 + \left(\frac{1}{2\alpha}\right)^0 \left(\frac{\tau}{2} - \frac{\tau}{4}(1-\tau\theta^0)\langle q_{12}^2\rangle + \frac{\tau^3}{3}\langle q_{12}q_{23}q_{13}\rangle + \cdots\right) + \left(\frac{1}{2\alpha}\right)^2 \left(\frac{\tau}{4} - \frac{\tau}{8}(1-\tau\theta^2)\langle q_{1234}^2\rangle + \frac{3\tau^3}{4}\langle q_{1234}q_{12}q_{34}\rangle + \cdots\right) + \cdots . \quad (9)$$

Note that in the high connectivity limit [11] the expression (9) approaches the well known expression for the free energy of the SK model [2], [14].

3. Order Parameter Fluctuations and Uniqueness of Critical Line

The multi-overlaps among any $2n$ configurations is typically small in the ergodic region defined by $2\alpha\tanh^2(\beta) = 1$ and their fluctuation can be studied on the $\sqrt{N}$ scale by defining

$$\eta_{2n} = \sqrt{N}q_{2n} = \frac{1}{\sqrt{N}}\sum_i^N \sigma_i^1 ... \sigma_i^{2n} . \quad (10)$$

Then it is possible to show that these rescaled multi-overlaps behave, in this region, like independent centered Gaussian variables, in the infinite volume limit, and the following theorem holds [11]:

Theorem 2. *In the annealed region* $2\alpha\tanh^2(\beta) < 1$ *the variables* η_{2n} *converge to centered Gaussian process with covariances*

$$\langle \eta_{a_1,\dots,a_{2n}}\rangle = \frac{1}{(1-2\alpha\mathbb{E}\tanh^{2n}(\beta J))} , \quad (11)$$

$$\langle \eta_{a_1,\dots,a_{2n}}\eta_{b_1,\dots,b_{2n}}\rangle = 0 \quad if \quad \exists i : a_i \neq b_i \quad (12)$$

and, when the boundary of the annealed region is approached, *only* the variance of η_2 diverges.

This theorem for the fluctuations of q_2 and for finding its critical line is straightforward within our method so we sketch the proof:

Sketched Proof. At first we expand the 2-replica overlap

$$\langle q_{12}\rangle_t = 2\alpha\theta^2\langle q_{12}^2\rangle - 4\alpha^2\theta^4\langle q_{12}q_{23}\rangle_t + O(q^3) . \quad (13)$$

Then, by simple polynomial integrations, we can evaluate the overlap expansion in terms of filled monomials.

$$\langle q_{12}\rangle_t = 2\alpha\theta^2\langle q_{12}^2\rangle t - 4\alpha^2\theta^4 \int_0^t dt' \int_0^{t'} dt''\langle q_{12}q_{23}q_{13}\rangle + O(q^6) . \quad (14)$$

Now, by applying "saturability", we get $\langle q_{12}\rangle_t = \langle q_{12}^2\rangle$, consequently, forgetting $O(q^4)$ terms and multiplying by N, we have

$$\langle \eta_2^2\rangle = \frac{2(2\alpha\theta^2)^2}{(1-(2\alpha\theta^2))}\langle \eta_{12}\eta_{23}\eta_{13}\rangle\,. \tag{15}$$

We see that at the r.h.s. the overlap order is 3 while at the l.h.s. is 2: By a Central Limit Theorem argument we see that the only diverging point, for the rescaled overlap fluctuations is $2\alpha\theta^2 = 1$, where the r.h.s. denominator explodes. □

To try and show our physical picture, let us start by the following theorem:

Theorem 3. *Given two integer numbers c, d such that $cd = 2n$ and $m \in N$ the following families of bounds hold generically and also at finite N:*

$$\langle q_{2n}^m\rangle \geq \langle q_{1..c}^m q_{c+1..2c}^m \cdots q_{c(d-1)+1..2n}^m\rangle \geq \langle q_{1..c}^m\rangle^d\,. \tag{16}$$

Sketched Proof. Always using q_2 and q_4 as examples, we prove the theorem for $c = d = 2$ and $m = 1$. Its generalization is straightforward.

Exploiting the factorization of the Boltzmann state at fixed J one has

$$\langle q_{1234}\rangle = \mathbb{E}\frac{1}{N}\sum_i \omega^4(\sigma_i) \geq \mathbb{E}(\frac{1}{N}\sum_i \omega^2(\sigma_i))^2 = \mathbb{E}\omega^2(q_{12}) \geq (\mathbb{E}\omega(q_{12}))^2 = \langle q_{12}\rangle^2\,,$$

where we have used $\mathbb{E}[a^2] \geq \mathbb{E}^2[a]$ for any real-valued random variable, first for $a = \omega^4(\sigma_i)$ and with the expectation taken over the uniform distribution on $i = 1, ..., N$ and then for $a = \omega(q_{12})$ with the expectation over $P(J)$. □

The conclusion is that it is not possible to have several spin glass transitions in any model: as soon as $\langle q_{12}\rangle$ becomes nonzero, also $\langle q_{1234}\rangle$ must be, and so on.

The mechanism we provide is again ultimately based on saturability. In fact at the critical point the fillable multi-overlap $\langle q_{12}q_{34}\rangle$, applying saturability, gets

$$\lim_{N\to\infty}\lim_{t\to 1}\langle q_{12}q_{34}\rangle_t = \langle q_{12}q_{34}q_{1234}\rangle\,, \tag{17}$$

which couples the first multi-overlap q_2 and the second multi-overlap q_4 together, generating the correlation which drives the transition for $\langle q_{1234}\rangle$. Saturability can be applied as we are at the boundary of the ergodicity breaking (the last point in which it still holds due to a real second order phase transition of q_2).

So remembering once more that we are taking just the first two multi-overlaps but the scheme applies to all them and, for the sake of the clearness consequently forgetting all the higher order not necessary terms, we can write the free energy, that we call $f(q_2, q_4)$ stressing the dependence by the two multi-overlaps as

$$f(q_2,q_4) = \left(\theta - \left(\frac{1}{2\alpha}\right)^{\frac{1}{2}}\right)q_2^2 + \left(\theta - \left(\frac{1}{2\alpha}\right)^{\frac{1}{4}}\right)q_4^2 - \frac{3\tau^3}{4}q_2^2 q_4 \tag{18}$$

and we want to know how the minima of $f(q_2, q_4)$ evolve with θ (at fixed α, or viceversa). If a bifurcation analysis of the saddle point equations from the origin is performed, one would find two transition lines, $\theta_{q_2} = (1/2\alpha)^{1/2}$ and $\theta_{q_4} = (1/2\alpha)^{1/4}$. However, when looking at the actual minima it is possible to see just the first transition. After that the two minima are away from the origin and so the second "potential transition line" at $\theta_{q_4} = (\frac{1}{2\alpha})^{\frac{1}{4}}$ never appears: when approaching this line the system is already in a completely different part of its phase space. We stress that above $2\alpha\theta^2 = 1$, where the quadratic expansion of $f(q_2, q_4)$ around the origin determines the Gaussian fluctuations, q_2 and q_4 are uncorrelated, than, below this line, the third-order term produces an interaction $(q_{12}q_{34}q_{1234})$ and so, as soon as q_2 becomes non zero, it also drives q_4 to a non zero value. It is also straightforward to check that near $2\alpha\theta^2$ the minima scale as $q_2 \sim (2\alpha\theta^2 - 1)^{1/2}, q_4 \sim (2\alpha\theta^2) \sim q_2^2$ accordingly with the proved scaling for random spins at criticality [6].

4. Summary

In this paper we analyzed the genesis of the phase transition in frustrated sparse networks, by matching a rigorous approach (essentially based on modern cavity interpolation [2]) with a theoretical picture (essentially provided via replica trick [20]). Overall a clear scenario for the transition in these systems has been achieved: at the onset of ergodicity breaking the first order parameter (i.e. q_2) undergoes a second-order phase transition; due to the correlations among this parameter and all the others (i.e. q_4), it drives the latter to a positive value too. The positivity of the values assumed by these parameters (another prescription of Parisi theory [14]) is a straightforward application of the saturability property on themselves. This has interesting consequences, ranging from disordered statistical mechanics to computer science as well as random matrix theory. On the same line, we stress that in recent years, even on the last subject [17], an increasing formalization (avoiding replicas), from Girko's framework [10], has been achieved [18].

Acknowledgements

Peter Sollich is warmly acknowledged for illuminating discussions. This work is supported by Technological Voucher Contract n.11606 of Calabria Region.

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