A Solvable Mean Field Model of a Gaussian Spin Glass

Adriano Barra*, Giuseppe Genovese†, Francesco Guerra‡, Daniele Tantari§

May 18, 2012

Abstract

We introduce a mean field spin glass model with gaussian distribuited spins and pairwise interactions, whose couplings are drawn randomly from a gaussian distribution $\mathcal{N}(0,1)$ too. We completely control the main thermodynamical properties of the model (free energy, phase diagram, fluctuations theory) in the whole phase space. In particular we prove that in thermodynamic limit the free energy equals its replica symmetric expression.

Introduction

Recently, some work has been done studying the properties of bipartite spin glasses [4][2][1]. The main interest in these models is related to the peculiarity of the Hopfield Model, a well known model of very hard solution from a mathematical point of view (see [21] and references therein), can be seen as a special bipartite model, with a party of *usual* dichotomic spin, and another party of *special* gaussian soft spin variables.

In particular, from the investigation of dichotomic bipartite spin glasses, it has been shown that, at least to the Replica Symmetric approssimation (with zero external field), the model can be written as a convex combination of two different Sherrington-Kirkpatrick models, at different temperatures [1]. This seems to be more than a hint that a similar structure should be conserved in the Hopfield Model, and infact we have recently shown that this is the case [3].

*Dipartimento di Fisica, Sapienza Università di Roma and GNFM, Gruppo di Roma1.
†Dipartimento di Matematica, Sapienza Università di Roma.
‡Dipartimento di Fisica, Sapienza Università di Roma and INFN, Sezione di Roma.
§Dipartimento di Fisica, Sapienza Università di Roma.

As a consequence, while the dichotomic spin glass has been intensively studied, the need for a clear picture of its gaussian counterpart is the main interest for this paper.
The Gaussian Spin Model has been originally introduced together with the Spherical Model in [6]. Then, due to the natural divergences arising in such a model, the main interest has been concentrated in the Spherical one [17][8][20][18][21], tought in some recent papers are discussed interesting properties of gaussian models similar to the one we introduce here for the first time [5][7][10].
Therefore in our work we extend here techniques previously developed for pairwise dichotomic spin glasses to their gaussian counterparts.
In Section 1 we introduce the model with all its related statistical mechanics package and regularize it so to avoid divergencies due to coupled fat tails of the soft (unbounded) spin.
In Section 2, we show how to get a rigorous control of the thermodynamic limit of the free energy.
Section 3 is left for the investigation of the high temperature limit (the ergodic behavior).
In Section 4 we develop a generalization of a sum rule for the free energy in terms of its replica symmetric approximation and an error term. The breaking of ergodicity is expected to be a critical phenomenon.
Section 5 is dedicated to a fluctuation theory for the order parameter.
In Section 6 we develop the broken replica symmetric bound, coupled with the Parisi-like equation, and we show that it is equivalent to the bound given by replica symmetric solution.
In Section 7 we finally prove a lower bound for the free energy, stating that the correct solution is the RS expression.

1 Definition of the Model

We introduce a system on N sites, whose generic configuration is defined by spin variables $z_i \in \mathbb{R}$, $i = 1, 2, ..., N$ attached on each site. We call the external quenched disorder a set of N^2 indipendent and identical distributed random variables J_{ij}, defined for each couple of sites (i, j). We assume each J_{ij} to be a centered unit Gaussian $\mathcal{N}(0, 1)$ i.e.

$$\mathbb{E}(J_{ij}) = 0, \quad \mathbb{E}(J_{ij}^2) = 1.$$

The interaction among spins is given by defining the Hamiltonian

$$H_N(z,J) = -\frac{1}{\sqrt{2N}}\sum_{i,j}^{N} J_{ij} z_i z_j - h\sum_{i=1}^{N} z_i.$$

The first sum, extending to all spin couples, with the factor $1/\sqrt{N}$, is the typical long range spin-spin interaction of the mean field spin glass model. The second sum, extending to all sites, is the one-body interaction with a scalar external field $h \in \mathbb{R}$. All the thermodynamic properties of the model are codified in the partition function that we write symbolically as

$$Z_N(\beta, J) = \sum_{configurations} e^{-\beta H_N(z,J)},$$

for a given inverse temperature β. In our model we state that there are a number of identical z-type configurations proportional to $d\mu(z)$, where $d\mu(z) = d\mu(z_1)...d\mu(z_N)$, $d\mu(z_i) = (2\pi)^{-\frac{1}{2}}\exp(-z_i^2/2)$, so to justify the following definition

$$Z_N(\beta, J) = \int d\mu(z) e^{-\beta H_N(z,J)} = \mathbb{E}_z e^{-\beta H_N(z,J)}. \tag{1}$$

Substantially, we called this kind of model "fully gaussian spin glass" because the external quenched disorder as well as the value of the soft-spin variables are drawn from a Gaussian distribution $\mathcal{N}(0,1)$. As early pointed out for instance in [6], unfortunately these kind of models need to be regularized; in fact, the right side of (1) is not always well defined as the pairwise interaction bridges soft spins which are both Gaussian distributed.
It will be clear soon that a good definition is

$$Z_N(\beta, J, \lambda) = \mathbb{E}_z \exp\big[-\beta H_N(z,J) - \frac{\beta^2}{4N}(\sum_{i=1}^{N} z_i^2)^2 + \frac{\lambda}{2}\sum_{i=1}^{N} z_i^2\big], \tag{2}$$

where the first additional term is needed for convergence of the integral over the Gaussian measure $\mu(z)$ as it essentially flattens the Gaussian tails of the variables z_i. The new parameter λ, within the last term of (2), instead is inserted just to modify the variance of the soft spins, as in several applications this can sensibly vary.
For a given inverse temperature (or noise level) β, we introduce the (quenched average of the) free energy per site $f_N(\beta)$, the Boltzmann state ω_J and the auxiliary function $A_N(\beta)$ (namely the pressure), according to the definition

$$-\beta f_N(\beta) = A_N(\beta) = N^{-1}\mathbb{E}\log Z_N(\beta, J), \tag{3}$$

$$\omega_J(O) = Z_N^{-1}\mathbb{E}_z O(z) \exp\big[-\beta H_N(z,J) - \frac{\beta^2}{4N}(\sum_{i=1}^N z_i^2)^2 + \frac{\lambda}{2}\sum_{i=1}^N z_i^2\big], \quad (4)$$

where O is a generic function of the z's. In the notation ω_J, we have stressed the dependence of the Boltzmann state on the external noise J, but, of course, there is also a dependence on β,h and N.
Let us now introduce the important concept of replicas. Consider a generic number s of independent copies of the system, characterized by the spin variables $z_i^{(1)}, \dots z_i^{(s)}$ distributed according to the product state

$$\Omega_J = \omega_J^{(1)} \dots \omega_J^{(s)}, \quad (5)$$

where all $\omega_J^{(\alpha)}$ act on each one $z_i^{(\alpha)}$'s, and are subject to the same sample J of the external noise. Finally, for a generic smooth function $F(z_i^{(1)}, \dots z_i^{(s)})$ of the replicated spin variables, we define the $\langle . \rangle$ average as

$$\left\langle F(z_i^{(1)}, \dots z_i^{(s)}) \right\rangle = \mathbb{E}\Omega_J(F(z_i^{(1)}, \dots z_i^{(s)})). \quad (6)$$

Correlation functions are also well defined as overlap q between replicas:

$$q_{ab,N} = \frac{1}{N}\sum_{i=1}^N z_i^a z_i^b.$$

Note that, once defined the overlap among replicas we can write

$$Z_N(\beta,\lambda,J) = \mathbb{E}_z \exp\left(-\beta\sqrt{\frac{N}{2}}\mathcal{K}(z) - \frac{\beta^2}{4N}(\sum_{i=1}^N z_i^2)^2 + \frac{\lambda}{2}\sum_{i=1}^N z_i^2\right), \quad (7)$$

where $\mathcal{K}(z)$ is a family of centered gaussian random variables with covariances $S_{zz'} = \mathbb{E}[\mathcal{K}(z)\mathcal{K}(z')] = q_{zz'}^2$ and the regularization term is just $\frac{1}{2}\frac{\beta^2 N}{2} q_{zz}^2 = \frac{1}{2}\frac{\beta^2 N}{2} S_{zz}$.

2 Thermodynamic Limit

The aim of this section is to show how to get a rigorous control of the infinite volume limit of the free energy f_N (or similarly A_N). The main idea, inspired by [14], is to compare A_N, A_{N_1} and A_{N_2}, with $N = N_1 + N_2$. For this purpose we consider both the original N site system and two independent subsystems made of by N_1 and N_2 soft spins respectively, so to define

$$\begin{aligned} Z_N(t) = \quad \mathbb{E}_z \quad & \exp\left(\beta\sqrt{\frac{t}{2N}}\sum_{i,j=1}^{N} J_{ij}z_iz_j - t\frac{\beta^2}{4N}(\sum_{i=1}^{N} z_i^2)^2\right) \\ & \exp\left(\beta\sqrt{\frac{1-t}{2N_1}}\sum_{i,j=1}^{N_1} J'_{ij}z_iz_j - (1-t)\frac{\beta^2}{4N_1}(\sum_{i=1}^{N_1} z_i^2)^2\right) \\ & \exp\left(\beta\sqrt{\frac{1-t}{2N_2}}\sum_{i,j=N_1+1}^{N} J''_{ij}z_iz_j - (1-t)\frac{\beta^2}{4N_2}(\sum_{i=N_1+1}^{N} z_i^2)^2\right) \\ & \exp\left(\beta h\sum_{i=1}^{N} z_i\right)\exp\left(\frac{\lambda}{2}\sum_{i=1}^{N} z_i^2\right), \end{aligned} \tag{8}$$

with $0 \leq t \leq 1$. The partition function $Z_N(t)$ interpolates between the original N-spin model (obtained for $t = 1$) and the two subsystems (of sizes N_1 and N_2, obtained for $t = 0$) equipped with independent noises J' and J'', both independent of J, *i.e.*

$$Z_N(1) = Z_N(\beta, J, h) \tag{9}$$
$$Z_N(0) = Z_{N_1}(\beta, J', h)Z_{N_2}(\beta, J'', h). \tag{10}$$

As a consequence, if we define the interpolating function

$$\varphi(t) = \frac{1}{N}\mathbb{E}\log Z_N(t), \tag{11}$$

taking into account the definition (3), we have

$$\begin{aligned} \varphi(1) &= A_N(\beta, h), \\ \varphi(0) &= \frac{N_1}{N}A_{N_1}(\beta, h) + \frac{N_2}{N}A_{N_2}(\beta, h). \end{aligned} \tag{12}$$

If we derive $\varphi(t)$ we obtain

$$\begin{aligned}
\frac{d}{dt}\varphi(t) \quad &= \frac{1}{N}\mathbb{E}\left(\frac{\beta}{2\sqrt{2tN}}\sum_{i,j=1}^{N} J_{ij}\omega_t(z_i z_j)\right) - \left\langle \frac{\beta^2}{4N^2}(\sum_{i=1}^{N} z_i^2)^2 \right\rangle \\
&-\frac{1}{N}\mathbb{E}\left(\frac{\beta}{2\sqrt{2(1-t)N_1}}\sum_{i,j=1}^{N_1} J'_{ij}\omega_t(z_i z_j)\right) \\
&-\frac{1}{N}\mathbb{E}\left(\frac{\beta}{2\sqrt{2(1-t)N_2}}\sum_{i,j=N_1+1}^{N} J''_{ij}\omega_t(z_i z_j)\right) \\
&+\left\langle \frac{\beta^2}{4NN_1}(\sum_{i=1}^{N_1} z_i^2)^2 \right\rangle + \left\langle \frac{\beta^2}{4NN_2}(\sum_{i=N_1+1}^{N} z_i^2)^2 \right\rangle . \qquad (13)
\end{aligned}$$

with $\omega_t(.)$ and the $\langle . \rangle$ average as defined in (3)(6) but corresponding to the *Boltzmannfaktor* coupled to $Z_N(t)$.
Let us now evaluate for example the first term: using a standard integration by parts on the external noise (Wick's theorem) we obtain

$$\frac{1}{N}\mathbb{E}\left(\frac{\beta}{2\sqrt{2tN}}\sum_{i,j=1}^{N} J_{ij}\omega_t(z_i z_j)\right) = \frac{\beta}{2N\sqrt{2tN}}\sum_{i,j=1}^{N}\mathbb{E}\left(\frac{\partial}{\partial_{J_{ij}}}\omega_t(z_i z_j)\right),$$

and then we get

$$\begin{aligned}
&\frac{\beta}{2N\sqrt{2tN}}\sum_{i,j=1}^{N}\mathbb{E}\left(\frac{\partial}{\partial_{J_{ij}}}\omega_t(z_i z_j)\right) = \\
= \quad &\frac{\beta^2}{4N^2}\sum_{i,j=1}^{N}\left\langle z_i^2 z_j^2\right\rangle - \frac{\beta^2}{4N^2}\sum_{i,j=1}^{N}\left\langle z_i^{(1)} z_j^{(1)} z_i^{(2)} z_j^{(2)}\right\rangle \\
= \quad &\left\langle \frac{\beta^2}{4N^2}(\sum_{i=1}^{N} z_i^2)^2 \right\rangle - \frac{\beta^2}{4}\left\langle q_{12,N}^2\right\rangle . \qquad (14)
\end{aligned}$$

The other terms of (13) can be evaluated in the same way, therefore

$$\frac{d}{dt}\varphi(t) = -\frac{\beta^2}{4}\left(\left\langle q_{12,N}^2\right\rangle - \frac{N_1}{N}\left\langle q_{12,N_1}^2\right\rangle - \frac{N_2}{N}\left\langle q_{12,N_2}^2\right\rangle\right), \qquad (15)$$

where we defined the overlaps

$$
\begin{aligned}
q_{12,N_1} &= \frac{1}{N_1}\sum_{i=1}^{N_1} z_i^{(1)} z_i^{(2)}, \\
q_{12,N_2} &= \frac{1}{N_2}\sum_{i=N_1+1}^{N_1} z_i^{(1)} z_i^{(2)}.
\end{aligned} \tag{16}
$$

Since $q_{12,N}$ is a convex linear combination of q_{12,N_1} and q_{12,N_2},

$$
q_{12,N} = \frac{N_1}{N} q_{12,N_1} + \frac{N_2}{N} q_{12,N_2}, \tag{17}
$$

and due to the convexity of the function $x \to x^2$, we have the inequality

$$
\left\langle q_{12,N}^2 - \frac{N_1}{N} q_{12,N_1}^2 - \frac{N_2}{N} q_{12,N_2}^2 \right\rangle \leq 0. \tag{18}
$$

A combination of the informations in (15) and (18) allows us to state the following result.

Lemma 1. *The interpolating function is increasing in t i.e.* $\frac{d}{dt}\varphi(t) \geq 0$.

By integrating in t we get

$$
\varphi(1) = \varphi(0) + \int_0^1 \frac{d}{dt}\varphi(t)dt \geq \varphi(0) \tag{19}
$$

and recalling the boundary conditions (12) we obtain the main result

Theorem 1. *The following superadditivity property holds*

$$
NA_N(\beta,h) \geq N_1 A_{N_1}(\beta,h) + N_2 A_{N_2}(\beta,h). \tag{20}
$$

The superadditivity property gives an immediate control of the thermodynamic limit [19], and we can state the next

Theorem 2. *The thermodynamic limit for $A_N(\beta,h)$ exists and equals its* sup *i.e.*

$$
\lim_{N\to\infty} A_N(\beta,h) = A(\beta,h) = \sup_N A_N(\beta,h). \tag{21}
$$

3 High Temperature behavior

We start to analyze our model characterizing the high temperature regime at zero external field. First we define the annealed free energy of the model

$$-\beta f_N^A(\beta,\lambda) = A_N^A(\beta,\lambda) = \frac{1}{N}\log \mathbb{E}Z_N(\beta,\lambda,J), \tag{22}$$

that can be easily computed as in the following

Proposition 1. *For $\lambda < 1$ the annealed free energy of the model in the thermodynamic limit is well defined and coincides with*

$$-\beta f^A(\beta,\lambda) = \lim_{N\to\infty} A_N^A(\beta,\lambda) = -\frac{1}{2}\log(1-\lambda). \tag{23}$$

Proof.

$$\begin{aligned}
\mathbb{E}_J Z_N &= \mathbb{E}_J\mathbb{E}_z \exp\left(\frac{\beta}{\sqrt{2N}}\sum_{i,j=1}^N J_{ij}z_iz_j - \frac{\beta^2}{4N}(\sum_{i=1}^N z_i^2)^2 + \frac{\lambda}{2}\sum_{i=1}^N z_i^2\right)\\
&= \mathbb{E}_z \exp\left(-\frac{\beta^2}{4N}(\sum_{i=1}^N z_i^2)^2 + \frac{\lambda}{2}\sum_{i=1}^N z_i^2\right)\mathbb{E}_J \exp\left(\frac{\beta}{\sqrt{2N}}\sum_{i,j=1}^N J_{ij}z_iz_j\right)\\
&= \mathbb{E}_z \exp\left(-\frac{\beta^2}{4N}(\sum_{i=1}^N z_i^2)^2 + \frac{\lambda}{2}\sum_{i=1}^N z_i^2\right)\exp\left(\frac{\beta^2}{4N}\sum_{i,j=1}^N z_i^2z_j^2\right)\\
&= \mathbb{E}_z \exp\left(\frac{\lambda}{2}\sum_{i=1}^N z_i^2\right)\\
&= (1-\lambda)^{-\frac{N}{2}}
\end{aligned} \tag{24}$$

Thus (23) follows from (22) and the proposition is proven. □

We define the high temperature regime as the region in the (β,λ) plane where the quenched free energy is equal to the annealed one. We already know that the annealed approximation is an upper bound for the pressure, infact a simple application of the Jensen inequality shows that

$$\frac{1}{N}\mathbb{E}\log Z_N(\beta,\lambda;J) \le \frac{1}{N}\log \mathbb{E}Z_N(\beta,\lambda,J) = A^A(\beta,\lambda). \tag{25}$$

On the other side we have that

$$\frac{1}{N}\mathbb{E}\log Z_N(\beta,\lambda;J) \ge \frac{1}{N}\mathbb{E}\log Z'_N(\beta,\lambda;J), \tag{26}$$

where $Z'_N(\beta,\lambda;J)$ is an auxiliary partition function in which diagonal terms of the spin-spin interaction are neglected, i.e.

$$\begin{aligned} Z'_N(\beta,\lambda;J) &= \mathbb{E}_z \exp\left(-\frac{1}{\sqrt{2N}}\sum_{i\neq j}^N J_{ij}z_iz_j - \frac{\beta^2}{4N}(\sum_{i=1}^N z_i^2)^2 + \frac{\lambda}{2}\sum_{i=1}^N z_i^2\right) \\ &= \mathbb{E}_z \exp\left(-\frac{1}{\sqrt{N}}\sum_{i<j}^N J_{ij}z_iz_j - \frac{\beta^2}{4N}(\sum_{i=1}^N z_i^2)^2 + \frac{\lambda}{2}\sum_{i=1}^N z_i^2\right), \end{aligned}$$

where we have noted that $\frac{1}{\sqrt{2}}(J_{ij}+J_{ji})$ is a centered gaussian random variable $\mathcal{N}(0,1)$ that we have simply denoted by J_{ij}. Inequality (26) follows by an other application of the Jensen inequality on the J_{ii} noises:

$$\begin{aligned} \mathbb{E}\log Z_N(\beta,\lambda;J) &= \mathbb{E}_{J_{ij}}\mathbb{E}_{J_{ii}}\log Z_N(\beta,\lambda;J_{ij},J_{ii}) \\ &\geq \mathbb{E}_{J_{ij}}\log Z_N(\beta,\lambda;J_{ij},\mathbb{E}_{J_{ii}}[J_{ii}]) \\ &= \mathbb{E}_{J_{ij}}\log Z_N(\beta,\lambda;J_{ij},0) = \mathbb{E}\log Z'_N(\beta,\lambda;J), \end{aligned}$$

Note that the auxiliary partition function Z'_N gives the same annealed approximation of Z_N; infact we have the following

Proposition 2. *For $\lambda < 1$,*

$$\lim_{N\to\infty}\frac{1}{N}\log\mathbb{E}_J Z'_N(\beta,\lambda;J) = -\frac{1}{2}\log(1-\lambda) = A^A(\beta,\lambda) \tag{27}$$

Proof.

$$\begin{aligned} \mathbb{E}_J Z'_N &= \mathbb{E}_J\mathbb{E}_z \exp\left(\frac{\beta}{\sqrt{N}}\sum_{i<j}^N J_{ij}z_iz_j - \frac{\beta^2}{4N}(\sum_{i=1}^N z_i^2)^2 + \frac{\lambda}{2}\sum_{i=1}^N z_i^2\right) \\ &= \mathbb{E}_z \exp\left(-\frac{\beta^2}{4N}(\sum_{i=1}^N z_i^2)^2 + \frac{\lambda}{2}\sum_{i=1}^N z_i^2\right)\mathbb{E}_J \exp\left(\frac{\beta}{\sqrt{N}}\sum_{i<j}^N J_{ij}z_iz_j\right) \\ &= \mathbb{E}_z \exp\left(-\frac{\beta^2}{4N}(\sum_{i=1}^N z_i^2)^2 + \frac{\lambda}{2}\sum_{i=1}^N z_i^2\right)\exp\left(\frac{\beta^2}{2N}\sum_{i<j}^N z_i^2z_j^2\right) \\ &= \mathbb{E}_z \exp\left(\frac{\lambda}{2}\sum_{i=1}^N z_i^2 - \frac{\beta^2}{4N}\sum_{i=1}^N z_i^4\right) \\ &= (1-\lambda)^{-\frac{N}{2}}\left(\int\frac{dz}{\sqrt{2\pi}}e^{-\frac{1}{2}z^2-\frac{\beta^2}{4N(1-\lambda)^2}z^4}\right)^N \end{aligned} \tag{28}$$

Now, putting $\beta_\lambda = \frac{\beta}{1-\lambda}$, we notice that the function in the integral

$$\int \frac{dz}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2 - \frac{\beta_\lambda^2}{4N}z^4} \tag{29}$$

approaches to 1 uniformly for $0 \le \lambda < 1$ when N grows to infinity, and so the integral, that completes the proof. $\square$

Now, we can control the high temperature region of Z'_N studying the fluctuations of the random variable $Z'_N/\mathbb{E}Z'_N$ in according to the Borel-Cantelli lemma approach [4][21]. The following lemma holds:

Lemma 2. *For $\beta_\lambda = \frac{\beta}{1-\lambda} \le 1$ we have*

$$\limsup_{N\to\infty} \frac{\mathbb{E}_J(Z'^2_N)}{\mathbb{E}_J^2(Z'_N)} \le \frac{1}{\sqrt{1-\beta_\lambda^2}}. \tag{30}$$

Before proving lemma 2 we note that it is a sufficient condition to state the following

Lemma 3. *In the region of the (β, λ) plane defined by $\beta_\lambda < 1$, i.e. $\beta < 1-\lambda$.*

$$\lim_{N\to\infty} \frac{1}{N}\mathbb{E}\log Z'_N(\beta,\lambda;J) = \lim_{N\to\infty} \frac{1}{N}\log \mathbb{E}Z'_N(\beta,\lambda;J) = A^A(\beta,\lambda) \tag{31}$$

Thanks to inequalities (25) and (26), we have proven the following main

Theorem 3. *The quenched free energy of the Gaussian spin glass model at zero external field does coincide with the annealed one*

$$-\beta f(\beta,\lambda) = \lim_{N\to\infty} \frac{1}{N}\mathbb{E}\log Z_N(\beta,\lambda;J) = -\frac{1}{2}\log(1-\lambda) \tag{32}$$

in the region of the (β, λ) plane defined by $\beta < 1-\lambda$.

Now we attack Lemma 2.

Proof. At first we evaluate $\mathbb{E}(Z'^2_N)$. By a straightforward calculation we have

$$\mathbb{E}(Z'^2_N) \le \mathbb{E}_{1,2}\exp\left(-\frac{\beta^2}{4N}\sum_{i=1}^N {z_i^{(1)}}^4 + {z_i^{(2)}}^4 + \frac{\lambda}{2}\sum_{i=1}^N {z_i^{(1)}}^2 + {z_i^{(2)}}^2 + \frac{\beta^2 N}{2}q_{12}^2\right),$$

where we neglected a term $e^{-\frac{\beta^2}{2N}\sum_i {z_i^{(1)}}^2 {z_i^{(2)}}^2} < 1$. We can linearize the overlap term introducing an auxiliary $\mathcal{N}(0,1)$ gaussian random variable g so

$e^{\frac{\beta^2 N}{2}q_{12}^2} = \mathbb{E}_g e^{\beta\sqrt{N}q_{12}g}$. In this way the average factorizes on i so, bearing in mind (28) and the overlap's definition, we can write

$$\frac{\mathbb{E}(Z_N'^2)}{\mathbb{E}^2(Z_N')} \leq \mathbb{E}_g \left(\frac{\mathbb{E}_{x,y} e^{-\frac{\beta^2}{4N}(x^4+y^4)+\frac{\lambda}{2}(x^2+y^2)+\frac{\beta}{\sqrt{N}}xyg}}{\mathbb{E}_{x,y} e^{-\frac{\beta^2}{4N}(x^4+y^4)+\frac{\lambda}{2}(x^2+y^2)}} \right)^N$$

$$= \mathbb{E}_g \left(\frac{\mathbb{E}_{x,y} e^{-\frac{\beta_\lambda^2}{4N}(x^4+y^4)+\frac{\beta_\lambda}{\sqrt{N}}xyg}}{\mathbb{E}_{x,y} e^{-\frac{\beta_\lambda^2}{4N}(x^4+y^4)}} \right)^N, \quad (33)$$

where we used the simpler notation $x = z_1^{(1)}$, $y = z_1^{(2)}$. Now it is sufficient to note that

$$\mathbb{E}_{x,y} e^{-\frac{\beta_\lambda^2}{4N}(x^4+y^4)+\frac{\beta_\lambda}{\sqrt{N}}xyg} = \mathbb{E}_x e^{-\frac{\beta_\lambda^2}{4N}x^4+\frac{\beta_\lambda^2}{2N}g^2x^2} \int \frac{dy}{\sqrt{2\pi}} e^{-\frac{1}{2}(y-\frac{\beta_\lambda}{\sqrt{N}}xg)^2-\frac{\beta_\lambda^2}{4N}y^4}$$

$$= \mathbb{E}_x e^{-\frac{\beta_\lambda^2}{4N}x^4+\frac{\beta_\lambda^2}{2N}g^2x^2} \mathbb{E}_y e^{-\frac{\beta_\lambda^2}{4N}(y+\frac{\beta_\lambda}{\sqrt{N}}xg)^4}$$

$$\leq \mathbb{E}_x e^{-\frac{\beta_\lambda^2}{4N}x^4+\frac{\beta_\lambda^2}{2N}g^2x^2} \mathbb{E}_y e^{-\frac{\beta_\lambda^2}{4N}y^4},$$

since the function $\mathbb{E}_y[e^{-(y+a)^4}]$ is concave in a, and it exhibits a unique maximum in $a = 0$, as it can be easily verified. Hence we finally get

$$\frac{\mathbb{E}(Z_N'^2)}{\mathbb{E}^2(Z_N')} \leq \mathbb{E}_g \left[\frac{\int \frac{dx}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} e^{\frac{\beta_\lambda^2 x^2 g^2}{2N}} e^{-\frac{\beta_\lambda^2 x^4}{4N}}}{\int \frac{dx}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} e^{-\frac{\beta_\lambda^2 x^4}{4N}}} \right]^N, \quad (34)$$

and we can split this integral as a sum over the complementary regions

$\{g^2 < N/\beta_\lambda^2\}$ and $\{g^2 \geq N/\beta_\lambda^2\}$. For the first one, we see that

$$\begin{aligned}
& \lim_{N\to\infty} \int_{g^2<N/\beta_\lambda^2} \frac{dg}{\sqrt{2\pi}} e^{-\frac{g^2}{2}} \left[\frac{\int \frac{dx}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} e^{\frac{\beta_\lambda^2 x^2 g^2}{2N}} e^{-\frac{\beta_\lambda^2 x^4}{4N}}}{\int \frac{dx}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} e^{-\frac{\beta_\lambda^2 x^4}{4N}}} \right]^N \\
= & \lim_{N\to\infty} \int_{g^2<N/\beta_\lambda^2} \frac{dg}{\sqrt{2\pi}} e^{-\frac{g^2}{2}} (1 - \frac{\beta_\lambda^2}{N} g^2)^{-\frac{N}{2}} \left(\frac{\int \frac{dx}{2\pi} e^{-\frac{1}{2}x^2 - \frac{\beta_\lambda^2}{4N(1-\frac{\beta_\lambda^2}{2N}g^2)^2} x^4}}{\int \frac{dx}{2\pi} e^{-\frac{1}{2}x^2 - \frac{\beta_\lambda^2}{4N} x^4}} \right)^N \\
\leq & \lim_{N\to\infty} \int_{g^2<N/\beta_\lambda^2} \frac{dg}{\sqrt{2\pi}} e^{-\frac{g^2}{2}} \left(1 - \frac{\beta_\lambda^2 g^2}{N} \right)^{-\frac{N}{2}} \\
\leq & \frac{1}{\sqrt{1-\beta_\lambda^2}}. \qquad (35)
\end{aligned}$$

For the second one, we cannot perform the change of variable for the variance of the gaussian, and we have to estimate the integrals. As we have seen, the integral in the denominator is given by (29). Thus we can rewrite the second term as

$$\int_{g^2 \geq N/\beta_\lambda^2} \frac{dg}{\sqrt{2\pi}} e^{-\frac{g^2}{4}} \frac{e^{-g^2/4} \left(\int \frac{dx}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} e^{\frac{\beta_\lambda^2 x^2 g^2}{2N}} e^{-\frac{\beta_\lambda^2 x^4}{4N}} \right)^N}{\left(\int \frac{dx}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} e^{-\frac{\beta_\lambda^2 x^4}{4N}} \right)^N},$$

and notice that the function at the numerator in the integral is concave in g, and it assumes the unique maximum point at $g = 0$, where it attains the same value of the denominator. Therefore we get the super-exponential decay of the gaussian tails:

$$\begin{aligned}
& \int_{g^2 \geq N/\beta_\lambda^2} \frac{dg}{\sqrt{2\pi}} e^{-\frac{g^2}{4}} \frac{e^{-g^2/4} F_N^N(g, \beta_\lambda)}{\left(\int \frac{dx}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} e^{-\frac{\beta_\lambda^2 x^4}{4N}} \right)^N} \\
\leq & \sqrt{2} P(g^2 \geq N/\beta_\lambda^2) \simeq C \sqrt{\frac{\beta_\lambda^2}{N}} e^{-\frac{N}{4\beta_\lambda^2}}, \qquad (36)
\end{aligned}$$

for a certain constant C, and the lemma is proven. □

4 Sum Rules for the Free Energy

In this section we introduce the replica symmetric approximation for the free energy density. In particular, we obtain it as an upper bound for $-\beta f$ togheter with the error, in the form of a sum rule. For this purpose, we apply a well known interpolation scheme [11][9][1] [2] to compare the original two-body interaction with a one-body interaction system. Concretely, we define, for $t \in [0,1]$ and $\bar{q} \geq 0$[1], the interpolating partition function

$$\begin{aligned} Z_N(t,J,J') &= \mathbb{E}_z \exp\left(\beta\sqrt{\frac{t}{2N}}\sum_{i,j=1}^N J_{ij}z_iz_j - t\frac{\beta^2}{4N}(\sum_{i=1}^N z_i^2)^2\right. \\ &\left. +\beta\sqrt{1-t}\sqrt{\bar{q}}\sum_{i=1}^N J_i'z_i + (1-t)\frac{c}{2}\sum_{i=1}^N z_i^2\right) \\ &\exp\left(\frac{\lambda}{2}\sum_{i=1}^N z_i^2\right), \end{aligned} \tag{37}$$

where the external noise J_i' are i.i.d Gaussian random variable $\mathcal{N}(0,1)$ and they are also independent from all J_{ij}. Here c is an additional lagrangian multiplier to be fixed later. Now we introduce the interpolating function

$$\varphi_N(t) = \frac{1}{N}\mathbb{E}\log Z_N(t,J,J'), \tag{38}$$

where we encode in $\mathbb{E}$ the averages respect to both J and J'. At $t=1$ the interpolating function (38) recovers the original system, while at $t=0$ it accounts for a simpler factorized one-body model and we can easily get

$$\begin{aligned} \varphi_N(0) &= \frac{1}{N}\mathbb{E}\log\prod_{i=1}^N \mathbb{E}_{z_i}\exp\left(\beta\sqrt{\bar{q}}J_i'z_i + \frac{(c+\lambda)}{2}z_i^2\right) \\ &= \mathbb{E}_{J'}\log\mathbb{E}_g\exp\left(\beta\sqrt{\bar{q}}J'g + \frac{(c+\lambda)}{2}g^2\right) \\ &= \mathbb{E}_{J'}\log(1-\lambda-c)^{-\frac{1}{2}}\mathbb{E}_g\exp\left(\frac{\beta\sqrt{\bar{q}}}{(1-\lambda-c)^{\frac{1}{2}}}J'g\right) \\ &= \log(\sigma) + \frac{1}{2}\mathbb{E}_{J'}\beta^2\bar{q}\sigma^2J'^2 = \log(\sigma) + \frac{1}{2}\beta^2\bar{q}\sigma^2, \end{aligned} \tag{39}$$

[1] despite it will be transparent at the end of the section, it may result helpful to bear in mind that $\bar{q}$ will act as the replica symmetric approximation of the overlap.

with $\sigma = (1-\lambda-c)^{-\frac{1}{2}}$ and where g as usual is a $\mathcal{N}(0,1)$ random variable. Therefore $\varphi_N(t)$ fulfills the following boundary conditions:

$$\begin{aligned} \varphi_N(1) &= A_N(\beta,\lambda) \\ \varphi_N(0) &= \log(\sigma) + \frac{1}{2}\beta^2\bar{q}\sigma^2. \end{aligned} \tag{40}$$

Now we have to evaluate the $t-$derivative of $\varphi_N(t)$ in order to obtain the sum rule

$$\varphi_N(1) = \varphi_N(0) + \int_0^1 dt \frac{d}{dt}\varphi_N(t). \tag{41}$$

Using the notation $\langle . \rangle_t = \mathbb{E}\Omega_t(.)$, where $\Omega_t(.)$ is the replicated Boltzmann state encoded in the partition function (37), we can write

$$\begin{aligned} \frac{d}{dt}\varphi_N(t) &= \frac{1}{N}\frac{\beta}{2\sqrt{2tN}}\sum_{i,j=1}^N \mathbb{E}J_{ij}\omega_t(z_i z_j) - \frac{\beta^2}{4N^2}\left\langle (\sum_{i=1}^N z_i^2)^2 \right\rangle_t \\ &- \frac{1}{N}\frac{\beta\sqrt{\bar{q}}}{2\sqrt{1-t}}\sum_{i=1}^N \mathbb{E}J_i'\omega_t(z_i) - \frac{c}{2N}\sum_{i=1}^N \left\langle z_i^2 \right\rangle_t. \end{aligned} \tag{42}$$

A standard integration by parts over the external noise, as in (14), shows that

$$\begin{aligned} \frac{1}{N}\frac{\beta}{2\sqrt{2tN}}\sum_{i,j=1}^N \mathbb{E}J_{ij}\omega_t(z_i z_j) &= \frac{\beta^2}{4N^2}\left\langle (\sum_{i=1}^N z_i^2)^2 \right\rangle_t - \frac{\beta^2}{4}\left\langle q_{12}^2 \right\rangle_t \\ \frac{1}{N}\frac{\beta\sqrt{\bar{q}}}{2\sqrt{1-t}}\sum_{i=1}^N \mathbb{E}J_i'\omega_t(z_i) &= -\frac{\beta^2}{2}\bar{q}\left\langle q_{12} \right\rangle_t + \frac{\beta^2\bar{q}}{2N}\sum_{i=1}^N \left\langle z_i^2 \right\rangle_t. \end{aligned} \tag{43}$$

Inserting (43) into (42), we get

$$\frac{d}{dt}\varphi_N(t) = -\frac{\beta^2}{4}\left\langle q_{12}^2 \right\rangle_t + \frac{\beta^2}{2}\bar{q}\left\langle q_{12} \right\rangle_t + \frac{1}{2N}\sum_{i=1}^N (-\beta^2\bar{q} - c)\left\langle z_i^2 \right\rangle_t, \tag{44}$$

hence, adding and subtracting a term $\frac{\beta^2}{4}\bar{q}^2$ and with the choice $c = -\beta^2\bar{q}$, we finally obtain

$$\frac{d}{dt}\varphi_N(t) = \frac{\beta^2}{4}\bar{q}^2 - \frac{\beta^2}{4}\left\langle (q_{12} - \bar{q})^2 \right\rangle_t. \tag{45}$$

We have just proved the following

Theorem 4. *For every* $\bar{q} \in \mathcal{D}_{\beta,\lambda}(\bar{q}) \equiv \left\{\bar{q} \in \mathbb{R}^+ : 1-\lambda+\beta^2\bar{q} > 0\right\}$ *is defined*

$$\tilde{A}(\beta,\lambda,\bar{q}) = \log(\sigma) + \frac{1}{2}\beta^2\bar{q}\sigma^2 + \frac{\beta^2}{4}\bar{q}^2 \tag{46}$$

with $\sigma = (1-\lambda+\beta^2\bar{q})^{-\frac{1}{2}}$. *Then,* $\forall N$ *and* $\forall \bar{q} \in \mathcal{D}_{\beta,\lambda}(\bar{q})$, *the quenched free energy of the mean field gaussian spin glass model defined in* (2) *fulfills the sum rule*

$$A_N(\beta,\lambda) = -\beta f_N(\beta,\lambda) = \tilde{A}(\beta,\lambda,\bar{q}) - \frac{\beta^2}{4}\int_0^1 dt \left\langle (q_{12}-\bar{q})^2 \right\rangle_t . \tag{47}$$

Moreover, $\forall \bar{q} \in \mathcal{D}_{\beta,\lambda}(\bar{q})$, $\tilde{A}(\beta,\lambda,\bar{q})$ *is an upper bound for* $A_N(\beta,\lambda)$ *uniformly in* N, *i.e.*

$$A_N(\beta,\lambda) = -\beta f_N(\beta,\lambda) \leq \tilde{A}(\beta,\lambda,\bar{q}). \tag{48}$$

Since the bound (48) is uniform in N, then it is true also in the thermodynamic limit. The error term in (47) reduces to the overlap's fluctuations around $\bar{q}$. We can minimize this error, or equivalently optimize the estimate in (48), by taking the value of $\bar{q}$ that minimize $\tilde{A}(\beta,\lambda,\bar{q})$. For this purpose we state the following

Proposition 3. *We have*

$$\frac{\partial}{\partial\bar{q}}\tilde{A}(\beta,\lambda,\bar{q}) = 0 \quad \Leftrightarrow \quad \bar{q} = 0 \quad or \quad \bar{q} = \frac{\beta-(1-\lambda)}{\beta^2}.$$

The solution $\bar{q} = 0$ *is a minimum for* $\beta_\lambda < 1$, *i.e.* $\beta \leq 1-\lambda$). *Conversely, for* $\beta > 1-\lambda$ *the minimum is* $\bar{q} = \frac{\beta-(1-\lambda)}{\beta^2}$.

Proof. Since $\partial_{\bar{q}}\sigma = -\frac{\beta^2}{2}\sigma^3$ we have that

$$\begin{aligned}\frac{\partial}{\partial\bar{q}}\tilde{A}(\beta,\lambda,\bar{q}) &= -\frac{\beta^2}{2}\sigma^2 + \frac{\beta^2}{2}\sigma^2 + \frac{\beta^4}{2}\bar{q}\sigma^4 + \frac{\beta^2}{2}\bar{q} \\ &= \frac{\beta^2}{2}\bar{q}\left(1-\beta^2\sigma^4\right)\end{aligned}$$

with two roots $\bar{q} = 0$ and $\beta\sigma^2 = 1$, i.e. $\bar{q} = \frac{\beta-(1-\lambda)}{\beta^2}$. If we study $\tilde{A}(\beta,\lambda,\bar{q})$ as a function of $\bar{q}^2$ we see that

$$\frac{\partial}{\partial\bar{q}^2}\tilde{A}(\beta,\lambda,\bar{q}) = \frac{1}{2\bar{q}}\frac{\partial}{\partial\bar{q}}\tilde{A}(\beta,\lambda,\bar{q}) = \frac{\beta^2}{4}\left(1-\frac{\beta^2}{(1-\lambda+\beta^2\bar{q})^2}\right).$$

Since $\frac{\partial}{\partial \bar{q}^2}\tilde{A}(\beta,\lambda,\bar{q})$ is increasing, $\tilde{A}$ is a convex function of $\bar{q}^2$ and at $\bar{q}=0$ we have that

$$\frac{\partial}{\partial \bar{q}^2}\tilde{A}(\beta,\lambda,\bar{q}^2)|_{\bar{q}=0} = \frac{\beta^2}{4}\left(1-\frac{\beta^2}{(1-\lambda)^2}\right) = \frac{\beta^2}{4}\left(1-\beta_\lambda^2\right).$$

Due to the convexity of $\tilde{A}(\bar{q}^2)$, the minimum is achieved at $\bar{q}=0$ for $\beta_\lambda < 1$ and at $\bar{q}>0$ for $\beta_\lambda > 1$. □

By combining the information of Theorem 4 and Proposition 3 we have the proof of the following main result.

Theorem 5. *The replica symmetric approximation for the free energy is well defined by the following variational principle:*

$$A^{RS}(\beta,\lambda) = \inf_{\bar{q}\in\mathcal{D}_{\beta,\lambda}(\bar{q})} \tilde{A}(\beta,\lambda,\bar{q}), \tag{49}$$

where

$$\tilde{A}(\beta,\lambda,\bar{q}) = \log(\sigma) + \frac{1}{2}\beta^2\bar{q}\sigma^2 + \frac{\beta^2}{4}\bar{q}^2, \tag{50}$$

with $\sigma(\beta,\lambda,\bar{q})$ defined in (46). The minimum is achieved at $\bar{q}=0$ for $\beta \le 1-\lambda$ and at $\bar{q}=\frac{\beta-(1-\lambda)}{\beta^2}$ otherwise. Moreover the replica symmetric approximation is an upper bound for $A(\beta,\lambda)$, infact, uniformly in N,

$$A_N(\beta,\lambda) = -\beta f_N(\beta,\lambda) \le A^{RS}(\beta,\lambda). \tag{51}$$

For $\beta_\lambda < 1$ the replica symmetric free energy reduces to the annealed one, that, accordingly with Theorem 3, coincides with the thermodynamic limit of the true free energy in such a region. Note that $\bar{q}=\frac{\beta-(1-\lambda)}{\beta^2}$ is also the optimal value for $\lambda>1$, infact $1-\lambda+\beta^2\bar{q}=\beta>0$ such that $\bar{q}\in\mathcal{D}_{\beta,\lambda}(\bar{q})$ and the RS approximation is well defined. In this case we see that $\bar{q}\to\infty$ when $\beta\to 0$.

5 Fluctuation Theory for the Order Parameter

This section is dedicated to the study of the fluctuations of the (rescaled and centered) order parameter.
The general idea behind is that critical phenomena arises in presence of a divergence of the fluctuation of the order parameter of the model. As critical phenomena are interesting by themselves, this analysis deserve major depth. In particular, we want to bound the annealed region, namely where

$\bar{q} = \lim_{N\to\infty} \left\langle q^2_{12,N} \right\rangle = 0$, checking that the rescaled fluctuations of $\langle q_{12,N} \rangle$ diverge on the same critical line where the fluctuation of the annealed free energy are singular.
To this task we introduce and define the rescaled and centered overlap

$$\xi_{12,N} = \sqrt{N}(q_{12,N} - \bar{q}), \tag{52}$$

which, in the thermodynamic limit, converges to a Gaussian random variable, whose variance spreads up to infinity as far as the system approaches the critical line in the (β, λ) plane.
Once again the strategy we outline is the one developed for the Sherrington-Kirkpatrick model [11][15][16]. It is still based on the evaluation of overlap correlations at $t = 0$ with respect to the *Boltzmannfaktor* defined in (37): We can in fact evaluate the thermodynamical observables at $t = 0$ due to the lacking of correlation and then propagate the solution up to $t = 1$.
To this task we introduce the following proposition

Proposition 4. *For every smooth function F_s of the overlaps $\{q_{ab}\}_{1\le a<b\le s}$ among s replicas,*

$$\frac{d}{dt}\langle F_s\rangle_t = \frac{\beta^2}{2}\left(\sum_{1\le a<b\le s}\left\langle F_s\xi^2_{ab}\right\rangle_t - s\sum_{a=1}^{s}\left\langle F_s\xi^2_{as+1}\right\rangle_t + \frac{s(s+1)}{2}\left\langle F_s\xi^2_{s+1s+2}\right\rangle_t\right).$$

The proof is here omitted, since it can be easily obtained by long but direct calculation. We are interested in considering $F_s = \xi^2_{12}$, such that

$$\frac{d}{dt}\left\langle\xi^2_{12}\right\rangle_t = \frac{\beta^2}{2}\left(\left\langle\xi^4_{12}\right\rangle_t - 4\left\langle\xi^2_{12}\xi^2_{13}\right\rangle_t + 3\left\langle\xi^2_{12}\xi^2_{34}\right\rangle_t\right). \tag{53}$$

To understand how $\left\langle\xi^2_{12}\right\rangle_t$ behaves we need to tackle even the other two correlation functions $\langle\xi_{12}\xi_{13}\rangle_t$, and $\langle\xi_{12}\xi_{34}\rangle_t$. For the sake of simplicity, let us consider t as a time and put

$$A(t) = \left\langle\xi^2_{12}\right\rangle_t, \quad B(t) = \langle\xi_{12}\xi_{13}\rangle_t, \quad C(t) = \langle\xi_{12}\xi_{34}\rangle_t. \tag{54}$$

Under the Gaussian *Ansatz* for the high temperature behavior of ξ_{ab} we can apply Wick theorem and, by using Proposition (4), we can construct the following dynamical system for $A(t)$, $B(t)$ and $C(t)$:

$$\begin{aligned}
\dot{A} &= \beta^2(A^2 - 4B^2 + 3C^2),\\
\dot{B} &= \beta^2(2AB - 6BC + 6C^2 - 2B^2),\\
\dot{C} &= \beta^2(\frac{1}{2}AC + 4B^2 - 16BC + 10C^2),
\end{aligned}$$

which can be straightforwardly solved with the initial data

$$
\begin{aligned}
A(0) &= \mathbb{E}\omega^2(z_i^2) - \bar{q}^2, \\
B(0) &= \mathbb{E}\omega(z_i^2)\omega^2(z_i) - \bar{q}^2, \\
C(0) &= \mathbb{E}\omega(z_i)^4 - \bar{q}^2,
\end{aligned}
$$

where

$$
\begin{aligned}
\omega(z_i) &= \frac{\mathbb{E}_z z e^{\beta\sqrt{\bar{q}}Jz+\frac{(c+\lambda)}{2}z^2}}{\mathbb{E}_z e^{\beta\sqrt{\bar{q}}Jz+\frac{(c+\lambda)}{2}z^2}} = \frac{1}{\beta\sqrt{\bar{q}}}\partial_J \log \mathbb{E}_z \exp(\beta\sqrt{\bar{q}}Jz + \frac{(c+\lambda)}{2}z^2) \\
&= \frac{1}{\beta\sqrt{\bar{q}}}\partial_J(\log(\sigma) + \frac{1}{2}\beta^2\sigma^2\bar{q}J^2) = \beta\sqrt{\bar{q}}\sigma^2 J, \qquad (55)
\end{aligned}
$$

$$
\begin{aligned}
\omega(z_i^2) &= \frac{\mathbb{E}_z z^2 e^{\beta\sqrt{\bar{q}}Jz+\frac{(c+\lambda)}{2}z^2}}{\mathbb{E}_z e^{\beta\sqrt{\bar{q}}Jz+\frac{(c+\lambda)}{2}z^2}} = \frac{1}{\beta^2\bar{q}}\frac{\partial_J^2\mathbb{E}_z e^{\beta\sqrt{\bar{q}}Jz+\frac{(c+\lambda)}{2}z^2}}{\mathbb{E}_z e^{\beta\sqrt{\bar{q}}Jz+\frac{(c+\lambda)}{2}z^2}} \\
&= \frac{1}{\beta^2\bar{q}}\left(\partial_J^2(\log \mathbb{E}_z e^{\beta\sqrt{\bar{q}}Jz+\frac{(c+\lambda)}{2}z^2}) + (\partial_J \log \mathbb{E}_z e^{\beta\sqrt{\bar{q}}Jz+\frac{(c+\lambda)}{2}z^2})^2\right) \\
&= \sigma^2 + \beta^2\bar{q}\sigma^4 J^2. \qquad (56)
\end{aligned}
$$

As we are interested in finding criticality, the simplest procedure is approaching the critical line from the annealed regime, where $\bar{q} = 0$. This further simplifies the initial conditions as

$$
A(0) = \sigma^4, \quad B(0) = C(0) = 0. \qquad (57)
$$

So the solution of the dynamical system is trivial as $B(t)$ and $C(t)$ are identically zero, while $A(t)$ satisfies

$$
\dot{A} = \beta^2 A^2, \quad A(0) = \sigma^4, \qquad (58)
$$

whose solution is

$$
A(t) = \frac{1}{\sigma^{-4} - \beta^2 t}. \qquad (59)
$$

Remembering that $\sigma = (1-\lambda+\beta^2\bar{(}q))^{-\frac{1}{2}}$, propagating up to $t = 1$ we finally obtain

$$
\langle \xi_{12}^2 \rangle = A(1) = \frac{1}{(1-\lambda)^2 - \beta^2}, \qquad (60)
$$

that diverges when $\beta_\lambda = 1$, i.e. $\beta = 1 - \lambda$, in complete agreement with Theorem 3.

6 Broken Replica Symmetry Bound

In this section we go beyond the replica symmetric approximation and we show a different bound for the free energy density, that should in principle improve the previous one. First of all we introduce the convex space χ of functional order parameters x, as nondecreasing functions of the auxiliary variable q in the $[0,1]$ interval, i.e.

$$\chi \ni x : [0, Q] \ni q \to x(q) \in [0, 1], \tag{61}$$

and we have to think $x(q)$ as a possible distribution function for the overlap. We will consider the case of piecewise constant functional order parameters, characterized by an integer K and two sequence of numbers, $q_0, q_1, \ldots, q_K$ and $m_1, \ldots, m_K$, satisfying

$$0 = q_0 \le q_1 \ldots \le q_K = Q \quad 0 \le m_1 \ldots \le m_K \le 1, \tag{62}$$

such that $x(q) = m_i$ for $q \in [q_{i-1}, q_i]$. It is useful to define also $m_0 = 0$ and $m_{K+1} = 1$. The replica symmetric case correspond to $K = 2$, $q_1 = \bar{q}$, $m_1 = 0$ and $m_2 = 1$, where overlap selfaverages around $\bar{q}$; the case $K = 3$, with two possible value (q_1 and q_2) for the overlap, is the first level of replica symmetry breaking, and so on. Now, following the interpolation scheme in [13], we consider a generic piecewise constant $x(q)$ and we introduce the interpolating partition function

$$\begin{aligned}\tilde{Z}_N(t; x(q)) = \mathbb{E}_z \qquad & \exp\left(\beta\sqrt{\frac{t}{2N}}\sum_{i,j=1}^{N} J_{ij} z_i z_j - t\frac{\beta^2}{4N}(\sum_{i=1}^{N} z_i^2)^2\right. \\ & \left. +\beta\sqrt{1-t}\sum_{a=1}^{K}\sqrt{q_a - q_{a-1}}\sum_{i=1}^{N} J_i^a z_i + (1-t)\frac{C}{2}\sum_{i=1}^{N} z_i^2\right) \\ & \exp\left(\beta h\sum_{i=1}^{N} z_i + \frac{\lambda}{2}\sum_{i=1}^{N} z_i^2\right),\end{aligned} \tag{63}$$

where $t \in [0,1]$. Here we have introducted additional independent gaussian random variable $J_i^a \in \mathcal{N}(0,1)$, $a = 1, \ldots, K$, $i = 1, \ldots, N$. As in the previous section we will set C in order to optimize the approximation. Let us call $\mathbb{E}_a$ the average with respect all the random variables J_i^a, $i = 1, \ldots, N$ and $\mathbb{E}_0$ the average with respect all the J_{ij}. We denote with $\mathbb{E}$ the average with respect to all J. Now we define recursively the random variables

$$Z_K = \tilde{Z}_N; \; Z_{K-1} = (\mathbb{E}_K Z_K^{m_k})^{\frac{1}{m_k}}; \; \ldots Z_0 = (\mathbb{E}_1 Z_1^{m_1})^{\frac{1}{m_1}}, \tag{64}$$

where each Z_a depends only on the external noise J_{ij} and on the J_i^b for $b \leq a$. Finally we define the auxiliary interpolating function

$$\varphi_N(t; x(q)) = \frac{1}{N} \mathbb{E}_0 \log Z_0(t; x(q)), \tag{65}$$

that is completely averaged out with respect of all the external noises. Notice that, at $t = 1$, we recover the original $A_N(\beta, \lambda)$, while, at $t = 0$, we have a solvable one body interaction problem. Thus, we have the possibility to find an other sum rule for the free energy

$$A_N(\beta, \lambda) = \varphi_N(t = 0) + \int_0^1 dt \frac{d}{dt} \varphi_N(t), \tag{66}$$

after calculating the t-derivative of $\varphi_N(t, x(q))$. For this purpose we need some additional definitios. Let us introduce the random variables

$$f_a = \frac{Z_a^{m_a}}{\mathbb{E}_a Z_a^{m_a}} \qquad a = 1, \dots, K \tag{67}$$

and notice that they depend only on the J_i^b for $b \leq a$ and they are normalized, $\mathbb{E} f_a = 1$. Moreover we consider the t-dependent state ω associated to the *Boltzmannfaktor* defined in (63) and its replicated Ω. A very important rule is played by the following states $\tilde{\omega}_a$, with $a = 1, \dots, K$, and its replicated $\tilde{\Omega}_a$, defined as

$$\tilde{\omega}_K(.) = \omega(.); \quad \tilde{\omega}_a = \mathbb{E}_{a+1} \dots \mathbb{E}_K (f_{a+1} \dots f_K \omega(.)). \tag{68}$$

Finally we define the generalized $\langle . \rangle_a$ average as

$$\langle . \rangle_a = \mathbb{E}(f_1 \dots f_a \tilde{\Omega}_a(.)). \tag{69}$$

The basic motivation for the introduction of an interpolating function like $\varphi(t; x(q))$ and the reason cause we tell about a broken replica symmetry bound, is the following

Theorem 6. *The t-derivative of $\varphi_N(t)$, defined in* (65)*, is given by*

$$\begin{aligned} \frac{d}{dt} \varphi_N(t) &= \frac{\beta^2}{4} \sum_{a=1}^{K} (m_{a+1} - m_a) q_a^2 \\ &- \frac{\beta^2}{4} \sum_{a=1}^{K} (m_{a+1} - m_a) \left\langle (q_{12} - q_a)^2 \right\rangle_a , \end{aligned} \tag{70}$$

if we set the value $C = -\beta^2 \sum_{a=1}^{K} (q_a - q_{a-1}) = -\beta^2 Q$.

Theorem 7. *In the thermodynamic limit, for every functional order parameter $x(q)$ of the type* (62)*, the following sum rule holds*

$$
\begin{aligned}
A(\beta,\lambda) &= \varphi(0;x) + \frac{\beta^2}{4}\sum_{a=1}^{K}(m_{a+1}-m_a)q_a^2 \\
&- \frac{\beta^2}{4}\sum_{a=1}^{K}(m_{a+1}-m_a)\int_0^1 \left\langle (q_{12}-q_a)^2 \right\rangle_a dt \qquad (71)
\end{aligned}
$$

and, consequently, we have the following bound for the free energy density:

$$
-\beta f(\beta,\lambda) = A(\beta,\lambda) \le \varphi(0;x) + \frac{\beta^2}{4}\sum_{a=1}^{K}(m_{a+1}-m_a)q_a^2. \qquad (72)
$$

Clearly, Theorem 7 follows from Theorem 6 by taking into account (66) and noting that the error term, containing overlap's fluctuation around every q_a, is negative defined.
Now let us go to Theorem 6. The proof is straightforward and we will indicate only the main points. We begin with

Lemma 4.

$$
\frac{d}{dt}\varphi(t;x) = \frac{1}{N}\mathbb{E}(f_1 \dots f_K Z_K^{-1}\partial_t Z_K)
$$

where

$$
\begin{aligned}
Z_K^{-1}\partial_t Z_K &= \tilde{Z}_N^{-1}\partial_t \tilde{Z}_N \\
&= \frac{\beta}{2\sqrt{2tN}}\sum_{i,j=1}^{N} J_{ij}\omega(z_i z_j) - \frac{\beta^2}{4N}\omega((\sum_{i=1}^{N} z_i^2)^2) \\
&- \frac{\beta}{2\sqrt{1-t}}\sum_{a=1}^{K}\sqrt{q_a - q_{a-1}}\sum_{i=1}^{N} J_i^a \omega(z_i) - \frac{C}{2}\sum_{i=1}^{N}\omega(z_i^2)
\end{aligned}
$$

Proof. From the definition (67) we get , for $a = 0, 1, \dots K-1$

$$
Z_a^{-1}\partial_t Z_a = \mathbb{E}_{a+1}(f_{a+1} Z_{a+1}^{-1}\partial_t Z_{a+1})
$$

and the proof follows iterating this formula. □

Now, using a standard integration by parts on the external noise, we get

$$
\begin{aligned}
\mathbb{E}(J_{ij} f_1 \dots f_K \omega(z_i z_j)) &= \sum_{a=1}^{K}\mathbb{E}(f_1 \dots \partial_{J_{ij}} f_a \dots f_K \omega(z_i z_j)) \\
&+ \mathbb{E}(f_1 \dots f_K \partial_{J_{ij}} \omega(z_i z_j))
\end{aligned}
$$

$$
\begin{aligned}
\mathbb{E}(J_i^a f_1 \dots f_K \omega(z_i)) &= \sum_{b=1}^{K} \mathbb{E}(f_1 \dots \partial_{J_i^a} f_b \dots f_K \omega(z_i)) \\
&+ \mathbb{E}(f_1 \dots f_K \partial_{J_i^a} \omega(z_i))
\end{aligned} \tag{73}
$$

that can be completely evaluated using the following

Lemma 5. *For the J-derivative we have*

$$
\begin{aligned}
\partial_{J_{ij}} \omega(z_i z_j) &= \beta \sqrt{\frac{t}{2N}} (\omega(z_i^2 z_j^2) - \omega^2(z_i z_j)), && (74) \\
\partial_{J_i^a} \omega(z_i) &= \beta \sqrt{1-t} \sqrt{q_a - q_{a-1}} (\omega(z_i^2) - \omega^2(z_i)), && (75) \\
\partial_{J_{ij}} f_a &= \beta \sqrt{\frac{t}{2N}} m_a f_a (\tilde{\omega}_a(z_i z_j) - \tilde{\omega}_{a-1}(z_i z_j)), && (76) \\
\partial_{J_i^a} f_b &= 0, \quad \textit{if } b < a, && (77) \\
\partial_{J_i^a} f_b &= \beta \sqrt{1-t} \sqrt{q_a - q_{a-1}} m_a f_a \tilde{\omega}_a(z_i), \quad \textit{if } b = a, && (78) \\
\partial_{J_i^a} f_b &= \beta \sqrt{1-t} \sqrt{q_a - q_{a-1}} m_b f_b (\tilde{\omega}_b(z_i) - \tilde{\omega}_{b-1}(z_i)), \textit{if } b > a && (79)
\end{aligned}
$$

Proof. Eq. (74), (75) follow from standard calculations, while Eq. (76) comes from the definition (67) and the easily established

$$
\begin{aligned}
\partial_{J_{ij}} Z_a^{m_a} &= m_a Z_a^{m_a} Z_a^{-1} \partial_{J_{ij}} Z_a \\
Z_a^{-1} \partial_{J_{ij}} Z_a &= \mathbb{E}_{a+1}(f_{a+1} Z_{a+1}^{-1} \partial_{J_{ij}} Z_{a+1}) \ \text{ for } a = 0, \dots, K-1 \\
Z_K^{-1} \partial_{J_{ij}} Z_K &= \beta \sqrt{\frac{t}{2N}} \omega(z_i z_j) \\
Z_a^{-1} \partial_{J_{ij}} Z_a &= \beta \sqrt{\frac{t}{2N}} \mathbb{E}_{a+1}(f_{a+1} \dots f_K \omega(z_i z_j)) = \beta \sqrt{\frac{t}{2N}} \tilde{\omega}_a(z_i z_j)
\end{aligned}
$$

In the same way we get Eq. (78), (79), where we have to remember that f_b does not depend on J_i^a if $b < a$. □

If we use Lemma 4 and Lemma 5, after some straightforward calculations, we obtain

$$
\begin{aligned}
\frac{d}{dt} \varphi_N(t) &= \frac{\beta^2}{4} \sum_{a=1}^{K} (m_{a+1} - m_a) q_a^2 \\
&- \frac{\beta^2}{4} \sum_{a=1}^{K} (m_{a+1} - m_a) \left\langle (q_{12} - q_a)^2 \right\rangle_a \\
&+ \frac{1}{2N} (-\beta^2 Q - C) \sum_{i=1}^{N} \left\langle z_i^2 \right\rangle_K
\end{aligned} \tag{80}
$$

and we complete the proof of Theorem 6 setting $C = -\beta^2 Q$.
Now we should find a general expression for $\varphi_N(0;x)$, as in the following

Theorem 8. *For any choice of the piecewise functional order parameter x, the initial condition $\varphi_N(0;x)$ is given by*

$$\varphi_N(0;x) = \log \sigma(Q) + f(0,0;x), \tag{81}$$

where $f(q,y;x)$ is the solution of the Parisi equation, i.e. the nonlinear antiparabolic partial differential equation

$$\partial_q f(q,y) + \frac{1}{2}\big(f''(q,y) + x(q) f'^2(q,y)\big) = 0, \tag{82}$$

with final condition at $q = Q$

$$f(Q,y) = \frac{\beta^2}{2}\sigma^2(Q) y^2. \tag{83}$$

Proof. Since the *Boltzmannfaktor* factorizes at $t = 0$, we have that

$$\begin{aligned}
\tilde{Z}_N(0;x) &= \mathbb{E}_z \exp\left(\frac{(\lambda - \beta^2 Q)}{2}\sum_{i=1}^N z_i^2\right)\exp\left(\beta\sum_{a=1}^K \sqrt{q_a - q_{a-1}}\sum_{i=1}^N J_i^a z_i\right) \\
&= \prod_{i=1}^N \sigma(Q)\exp\left(\frac{\beta^2}{2}\sigma(Q)^2 (\sum_{a=1}^K \sqrt{q_a - q_{a-1}} J_i^a)^2\right) \\
&\equiv \prod_{i=1}^N \sigma(Q)\exp\left(f(Q, \sum_{a=1}^K \sqrt{q_a - q_{a-1}} J_i^a)\right).
\end{aligned} \tag{84}$$

From the definition (65) of the interpolating function $\varphi_N(t;x)$, we note that, due to the $1/N$ factor, we can evaluate the (84) on a single site only. The $\sigma(Q)$ goes to form the $\log\sigma(Q)$ term and what remains is just the solution of the Parisi equation, evaluated at $y = 0$, and propagated from $q = Q$ to $q = 0$ through a series of gaussian integration as in [13]. □

Now we can exactly solve equation (82) with final condition (83) to find $f(0,0;x)$ and so $\varphi_N(0;x)$. Infact we give the following

Lemma 6. *For any functional order parameter $x \in \mathcal{X}$, the solution of equation (82) with final condition (83), evaluated at $y = 0$ and $q = 0$ is given by*

$$f(0,0;x) = \frac{1}{2}\beta^2\sigma^2(Q)\int_0^Q \frac{dq}{1 - \beta^2\sigma^2(Q)\int_q^Q x(q')dq'} \tag{85}$$

Proof. We look for a solution of (82) of the form $f(q,y) = a(q) + \frac{1}{2}b(q)y^2$. Since f must fulfill final condition (83), it has to be $a(Q) = 0$ and $b(Q) = \beta^2\sigma^2(Q)$. If we want $f(q,y)$ to be a solution of (82)

$$\begin{aligned} & \partial_q f(q,y) + \frac{1}{2}\big(f''(q,y) + x(q)f'^2(q,y)\big) \\ = \quad & a'(q) + \frac{1}{2}b(q) + \frac{1}{2}y^2\left(b'(q) + x(q)b^2(q)\right) = 0, \end{aligned} \tag{86}$$

i.e. $f(q,y)$ is a solution of (82) if $a(q)$ and $b(q)$ are solutions of the ordinary differential equation's system

$$a'(q) + \frac{1}{2}b(q) \quad = \quad 0 \tag{87}$$

$$b'(q) + x(q)b^2(q) \quad = \quad 0, \tag{88}$$

with final conditions $a(Q) = 0$ and $b(Q) = \beta^2\sigma^2(Q)$. Integrating equation (88) we obtain

$$\frac{1}{b(q)} = \frac{1}{\beta^2\sigma^2(Q)} - \int_q^Q x(q')dq'. \tag{89}$$

Putting (89) into equation 87 and integrating, we have the proof. □

Finally, from the continuity of $f(q,y;x)$ respect to the choice of the functional order parameter x (see [13], [12]) and noticing that

$$\frac{\beta^2}{4}\sum_{a=1}^{K}(m_{a+1} - m_a)q_a^2 = \frac{\beta^2}{4}Q^2 - \frac{\beta^2}{2}\int_0^Q qx(q)dq, \tag{90}$$

and we can use 7 for stating our main result

Theorem 9. *The pressure of the model is defined by the following variational principle:*

$$A(\beta,\lambda) = \inf_{x\in\mathcal{X}} \hat{A}(\beta,\lambda;x), \tag{91}$$

with

$$\begin{aligned} \hat{A}(\beta,\lambda;x) \quad = \quad & \log\sigma(Q) + \frac{1}{2}\beta^2\sigma^2(Q)\int_0^Q \frac{dq}{1 - \beta^2\sigma^2(Q)\int_q^Q x(q')dq'} \\ + \quad & \frac{\beta^2}{4}Q^2 - \frac{\beta^2}{2}\int_0^Q qx(q)dq, \end{aligned} \tag{92}$$

Moreover the infimum is attaiened at the RS functional order parameter $x = 0$, $0 \le q < q^{RS}$, $x = 1$ *elsewhere.*

Proof. The first part of the theorem is a direct conseguence of all the results in this section; so we will focus our attention only on its last part, that is

$$A^{RS}(\beta,\lambda) \leq \inf_{x\in\mathcal{X}} \hat{A}(\beta,\lambda;x).$$

Let x_ε a family of functional order parameter parametrized by ε and consider $\hat{A}(\beta,\lambda;x_\varepsilon)$. We will find the infimum of $\hat{A}(\beta,\lambda;x)$ imposing that $\frac{d}{d\varepsilon}\hat{A}(\beta,\lambda;x_\varepsilon)|_{\varepsilon=0} = 0$ for any family x_ε passing through $x_0 = x$ when $\varepsilon = 0$. Using (92) and defining $\eta(q) = \frac{d}{d\varepsilon}x_\varepsilon(q)|_{\varepsilon=0}$, the infimum is achieved in x satisfying

$$\begin{aligned}
\frac{d}{d\varepsilon}\hat{A}(\beta,\lambda;x_\varepsilon)|_{\varepsilon=0} &= -\frac{\beta^2}{2}\int_0^Q q\eta(q)dq + \frac{1}{2}\beta^4\sigma^4(Q)\int_0^Q dq\int_q^Q dq'\frac{\eta(q')}{\bar{b}^2(q)} \\
&= -\frac{\beta^2}{2}\int_0^Q q\;\eta(q)dq + \frac{1}{2}\beta^4\sigma^4(Q)\int_0^Q dq\;\eta(q)\int_0^q \frac{dq'}{\bar{b}^2(q')} \\
&= -\frac{\beta^2}{2}\int_0^Q dq\;\eta(q)\left(q - \beta^2\sigma^4(Q)\int_0^q \frac{dq'}{\bar{b}^2(q')}\right) = 0,
\end{aligned}$$

where we have defined $\bar{b}(q) = 1 - \beta^2\sigma^2\int_q^Q x(q')dq'$. From the arbitrariness of $\eta(q)$, it has to be

$$q - \beta^2\sigma^4(Q)\int_0^q \frac{dq'}{\bar{b}^2(q')} = \int_0^q dq'\left(1 - \frac{\beta^2\sigma^4(Q)}{\bar{b}^2(q')}\right) = 0, \tag{93}$$

for every q and consequently $1 - \frac{\beta^2\sigma^4(Q)}{\bar{b}^2(q')}$=0, i.e., recalling the definition of $\bar{b}(q)$,

$$1 - \beta^2\sigma^2(Q)\int_q^Q x(q')dq' = \beta\sigma^2(Q), \tag{94}$$

that has the solution $x = 0$ and $Q = q^{RS}$, such that $\beta\sigma^2(q^{RS}) = 1$. □

So we have completed the RSB scheme, showing that the RSB bound for the free energy gives the same result of the RS appressimation. Anyway we have not yet proven that the replica symmetric solution is the exact infinite volume free energy of the model. That will be the subject of our last section.

7 The Reverse Bound

In this section we will show a lower bound for the gaussian pressure by using a well known method in statistical mechanics. The idea, coming from the proof of the equivalence between microcanonical and canonical ensemble [19], is to cut the space into spherical shells of thickness η in such a way the integral over the whole space in the partition function is just the sum over all the shell; taking just one single shell then, we obtain a lower bound. Namely, we fix $R \in [0, \infty)$ and define

$$S^{\eta}_{R_N} = \{z \in \mathbb{R}^N : R\sqrt{N} - \eta \le \|z\| \le R\sqrt{N}\}, \tag{95}$$

such that,

$$\begin{aligned} Z^g_N(\beta,\lambda,J) &= \int_{\mathbb{R}^N} dz \frac{e^{-\|z\|^2/2}}{(2\pi)^{N/2}} e^{\left(-\beta H_N(z,J) - \frac{\beta^2}{4N}\|z\|^4 + \frac{\lambda}{2}\|z\|^2\right)} \\ &\ge \int_{S^{\eta}_{R_N}} dz \frac{e^{-\|z\|^2/2}}{(2\pi)^{N/2}} e^{\left(-\beta H_N(z,J) - \frac{\beta^2}{4N}\|z\|^4 + \frac{\lambda}{2}\|z\|^2\right)} \\ &= \frac{\eta S_{R_N}}{(2\pi)^{\frac{N}{2}}} \ \frac{1}{\eta} \int_{S^{\eta}_{R_N}} \frac{dz}{S_{R_N}} e^{\left(-\beta H_N(z,J) - \frac{\beta^2}{4N}\|z\|^4 + \frac{(\lambda-1)}{2}\|z\|^2\right)} \\ &\ge \frac{\eta S_{R_N}}{(2\pi)^{\frac{N}{2}}} \ e^{-\frac{\beta^2 R^4 N}{4} + \frac{(\lambda-1)R^2 N}{2} + O\left(\eta\sqrt{N}\right)} \ \frac{1}{\eta} \int_{S^{\eta}_{R_N}} \frac{dz}{S_{R_N}} e^{-\beta H_N(z,J)} \end{aligned}$$

Taking $\frac{1}{N}\mathbb{E}\log$ we get

$$A^g_N(\beta,\lambda) \ge \frac{1}{N}\log(\frac{S_{R_N}}{(2\pi)^{\frac{N}{2}}}) - \frac{\beta^2 R^4}{4} + \frac{(\lambda-1)R^2}{2} + A^{sh}_{N,\eta}(\beta, R_N) + O\left(\frac{1}{\sqrt{N}}\right). \tag{96}$$

Since we can exchange the limits $\eta \to 0$ and $N \to \infty$ (as it is easy to prove), taking the infinite volume limit of both sides we obtain

$$A^g(\beta,\lambda) \ge A^{sf}(\beta,R) - \frac{\beta^2 R^4}{4} + \frac{(\lambda-1)R^2}{2} + \log(R) + \frac{1}{2}, \tag{97}$$

where an easy computation show that $\frac{1}{N}\log(\frac{S_{R_N}}{(2\pi)^{\frac{N}{2}}})$ tends to $\log(R) + 1/2$. Now we can take the supremum over $R \in [0, \infty]$ obtaining

$$A^g(\beta,\lambda) \ge \sup_{R\in(0,\infty)} \left[A^{sf}(\beta,R) - \frac{\beta^2 R^4}{4} + \frac{(\lambda-1)R^2}{2} + \log R + \frac{1}{2}\right]. \tag{98}$$

In what follows we will see that the right side of (98) is just the replica symmetric approximation A^g_{RS} and thus we have just the lower bound we need.

As it is well known [17][8][18], the free energy of the spherical model can be expressed as the variational principle

$$A^{sf}(\beta, R=1) = \min_{q\in[0,1]} \frac{1}{2}\left(\frac{q}{1-q} + \log(1-q) + \frac{\beta^2}{2}(1-q^2)\right),$$

where the minimum is achieved at $q=0$, for $\beta<1$, and at $q = 1-\frac{1}{\beta}$ otherwise. Furthermore, it's easy to check (through a change of variable) that, in the spherical model, a shift on the radius is equivalent to a temperature rescaling, *i.e.*

$$A^{sf}(\beta, R) = A^{sf}(\beta R^2, 1) = \begin{cases} \frac{\beta^2 R^4}{4}, & \beta R^2 < 1 \\ \beta R^2 - \log R - \frac{1}{2}\log\beta - \frac{3}{4}. & \beta R^2 > 1. \end{cases} \tag{99}$$

First, we note that the function we want to to maximize

$$f(\beta, R) = \begin{cases} \frac{(\lambda-1)R^2}{2} + \frac{1}{2}\log(R^2) + \frac{1}{2}, & \beta R^2 < 1 \\ \beta R^2 - \frac{\beta^2 R^4}{4} + \frac{(\lambda-1)R^2}{2} - \frac{1}{2}\log\beta - \frac{1}{4}, & \beta R^2 > 1 \end{cases} \tag{100}$$

is continuous in $R \in (0,\infty)$, goes to $-\infty$ at the interval's extremes and is concave in R^2 such that it must have a finite unique maximum. It is more useful to extremalize in R^2, such that

$$\frac{\partial}{\partial R^2} f(\beta, R) = \begin{cases} (\lambda-1) + \frac{1}{2R^2}, & \beta R^2 < 1 \\ \beta - \frac{\beta^2 R^2}{2} + (\lambda-1), & \beta R^2 > 1 \end{cases} \tag{101}$$

and

$$\frac{\partial}{\partial R^2} f(\beta, R) = 0 \Leftrightarrow \begin{cases} R^2 = \frac{1}{1-\lambda}, & \beta R^2 < 1 \\ R^2 = \frac{2\beta+(\lambda-1)}{\beta^2}, & \beta R^2 > 1 \end{cases} \tag{102}$$

Thanks to the concavity of f, for each value of (β, λ), only one critical point $\bar{R}^2$ can exists: if $\beta < (1-\lambda)$, $\bar{R}^2 = 1/(1-\lambda)$, otherwise $\bar{R}^2 = 2\beta + (\lambda-1)/\beta^2$ and we obtain

$$A^g(\beta, \lambda) \geq \begin{cases} -\frac{1}{2}\log(1-\lambda) & \beta < 1-\lambda \\ -\frac{1}{2}\log(\beta) + \frac{\beta\bar{q}}{2} + \frac{\beta^2\bar{q}^2}{4}, & \beta > 1-\lambda \end{cases} \tag{103}$$

with $\bar{q}(\beta, \lambda) = \frac{\beta-(1-\lambda)}{\beta^2}$. Right side of equation (103) is exactly the replica symmetric approximation of the model. We can put all information we have found in the following final

Theorem 10. *In the thermodynamic limit, the pressure of the gaussian spin glass model satysfies the following inequality*

- $A^g(\beta,\lambda) \le A^g_{RS}(\beta,\lambda)$
- $A^g(\beta,\lambda) \ge \sup_{R\in(0,\infty)} \left[A^{sf}(\beta,R) - \frac{\beta^2 R^4}{4} + \frac{(\lambda-1)R^2}{2} + \log R + \frac{1}{2}\right].$

Moreover,

$$\sup_{R\in(0,\infty)} \left[A^{sf}(\beta,R) - \frac{\beta^2 R^4}{4} + \frac{(\lambda-1)R^2}{2} + \log R + \frac{1}{2}\right] = A^g_{RS}(\beta,\lambda)$$

thus

$$A^g(\beta,\lambda) = A^g_{RS}(\beta,\lambda) = \sup_{R\in(0,\infty)} \left[A^{sf}(\beta,R) - \frac{\beta^2 R^4}{4} + \frac{(\lambda-1)R^2}{2} + \log R + \frac{1}{2}\right].$$

8 Conclusions and Outlooks

In this paper we introduced and solved a model of Gaussian spin glass. We have shown how to regularize the soft spins in order to tackle a right control of the thermodynamic observables and extend the tecniques developed for the Sherrington-Kirkpatrick model.
In this way we have achieved existence of thermodynamic limit of the free energy; we have computed the annealed free energy and, both looking at its own fluctuations, both looking at the fluctuations of the order parameter, we have found its region of validity in the (β,λ) plane, so finding the critical line of the model.
Furthermore, we have studied the Replica Symmetric approximation, and, by a deeper analysis through the RSB scheme, we have found that it actually give the same bound to the free energy of the model of the RS one. By further analysis we have shown that RS solution is infact exact.
This was in a sense aspected, essentially for two reasons: the supposed relation with the Hopfield Model and the intimate connection with the Spherical Spin Glass [8][20][18][21]. On the first one, that is our main matter of investigation, and also the reason why we need to introduce this model [3], we have already spent some words in the Introduction. About the second, there is a clear *a priori* hint given by concentration of gaussian measure argoument that the two models are equivalent, strengthened by the *a posteriori* fact that they share the same structure of the distribution of the overlap (both

replica symmetric). This feature is not surprising, and was pointed out also in the final step of our proof, in the previous section. The relationship between the two can be deepen more, studing model with even more general noise. We plan to report soon about this topic.

Acknowledgements
The author are pleased to thank Andrea Crisanti and Michel Talagrand for useful conversations.
This work belongs to the strategy of investigation funded by the MIUR trough the FIRB grant RBFR08EKEV which is acknowledged together with a contribute by Sapienza Universit'a di Roma.
FG is further grateful to INFN (Istituto Nazionale di Fisica Nucleare).
AB is further grateful to GNFM (Gruppo Nazionale per la Fisica Matematica).
GG would like to thank Francis Comets, Giambattista Giacomin and all the members of LPMA Universitè Denis Diderot Paris 7, for kind hospitality and fruitful discussions.

References

[1] A. Barra, G. Genovese, F. Guerra, *Equilibrium statistical mechanics of bipartite spin systems*, J. Phys. A: Math. Theor. **44**, 245002 (2011).

[2] A. Barra, G. Genovese, F. Guerra, *The replica symmetric behavior of the analogical neural network*, J. Stat. Phys. 140, 784 (2010).

[3] A. Barra, G. Genovese, F. Guerra, D. Tantari, *How glassy is neural network?*, preprint (2012).

[4] A. Barra, F. Guerra, *Ergodicity in the analogical Hopfield model for neural networks*, J. Math. Phys. 49, 125127 (2008).

[5] G. Ben Arous, A. Dembo, A. Guionnet, *Aging of spherical spin glasses*, Prob. Theor. Related Fields 120, 1, (2001).

[6] T. H. Berlin, M. Kac, *The Spherical Model of a Ferromagnet*, Phys. Rev. 86, 821 (1952).

[7] A. Bovier, A. Klymovskiy,*The Aizenman-Sims-Starr and Guerra's schemes for the SK model with multidimensional spins*, Electron. J. Probab. 14,161 (2009).

[8] A. Crisanti, H. -J. Sommers, *The spherical p-spin interaction p-spin glass model: the statistics*, Z. Phys. B Condensed Matter 83, 341, (1992).

[9] G. Genovese, A. Barra, *A mechanical approach to mean field spin models*, J. Math. Phys. **50**, 365234 (2009).

[10] Y. V. Fyodorov, H. -J- Sommers *Classical Particle in a Box with Random Potential: exploiting rotational symmetry of replicated hamiltonian*, Nuclear Physics B 764,128 (2007).

[11] F. Guerra, *Sum rules for the free energy in the mean field spin glass model*, in *Mathematical Physics in Mathematics and Physics: Quantum and Operator Algebraic Aspects*, Fields Institute Communications **30**, 161 (2001).

[12] F. Guerra, *An Introduction to Mean Field Spin Glass Theory: Methods and Results*, A. Bovier et al. eds, Les Houches, Session LXXXIII, (2005).

[13] F.Guerra, *Broken Replica Symmetry Bounds in the Mean Field Spin Glass Model*, Comm. Math. Phys. **233**, 1 (2003).

[14] F. Guerra, F. L. Toninelli, *The Thermodynamic Limit in Mean Field Spin Glass Models, Comm. Math. Phys.* **230**, 71 (2002).

[15] F. Guerra, F. L. Toninelli, *Quadratic replica coupling in the Sherrington Kirkpatrick mean field spin glass model, J. Math. Phys.* **43**, 3704 (2002).

[16] F. Guerra, F. L. Toninelli, *Central limit theorem for fluctuations in the high temperature region of the Sherrington Kirkpatrick spin glass model, J. Math. Phys.* **43**, 6224 (2002).

[17] J. M. Kosterlitz, D. J. Thouless, Raymund C. Jones, *Spherical model of a spin glass*, Phys. Rev. Lett. 36, 1217 (1976).

[18] D. Panchenko, M. Talagrand *On the overlap in the multiple spherical models*, Ann. Probab. 35, 2321 (2007).

[19] D. Ruelle. *Statistical Mechanics. Rigorous results*, W.A. Benjamin Inc., New York, 1969.

[20] M. Talagrand, *Free energy of the spherical mean field model*, Prob. Theor. Related Fields 134, 339 (2006).

[21] M. Talagrand, *Spin glasses: a challenge for mathematicians. Cavity and mean field models*, Springer Verlag (2003).