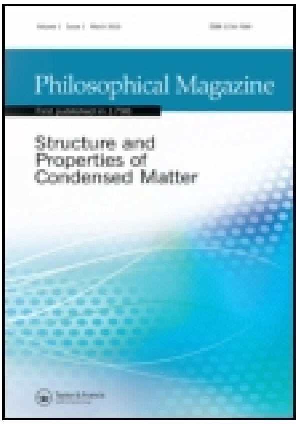

Philosophical Magazine
Publication details, including instructions for authors and subscription information:
http://www.tandfonline.com/loi/tphm20

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Version of record first published: 09 Dec 2011.

To cite this article: Adriano Barra, Francesco Guerra & Emanuele Mingione (2012): Interpolating the Sherrington-Kirkpatrick replica trick, Philosophical Magazine, 92:1-3, 78-97

To link to this article: http://dx.doi.org/10.1080/14786435.2011.637979

Philosophical Magazine
Vol. 92, Nos. 1–3, 1–21 January 2012, 78–97

Interpolating the Sherrington–Kirkpatrick replica trick

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(*Received 11 April 2011; final version received 7 October 2011*)

Interpolation techniques have become, in the past decades, a powerful approach to describe several properties of spin glasses within a simple mathematical framework. Intrinsically, for their construction, these schemes were naturally implemented in the cavity field technique, or its variants such as stochastic stability and random overlap structures. However the first and most famous approach to mean field statistical mechanics with quenched disorder is the replica trick. Among the models where these methods have been used (namely, dealing with frustration and complexity), probably the best known is the Sherrington–Kirkpatrick spin glass. In this paper we apply the interpolation scheme to the original replica trick framework and test it directly on the cited paradigmatic model. Although the problem, at a mathematical level, has been deeply investigated by Talagrand, it is still rich in information from a theoretical physics perspective; in fact, by treating the number of replicas $n \in (0, 1]$ as an interpolating parameter (far from its original interpretation) the proof of the attendant commutativity of the zero replica and the infinite volume limits can be easily obtained. Further, within this perspective, we can naturally think of n as a quenching temperature close to that introduced in off-equilibrium approaches to gain some new insight into our understanding of the off-equilibrium features encountered in equilibrium statistical mechanics of spin glasses.

Keywords: cavity method; spin glasses; replica trick

1. Introduction

Born as a sideline in the condensed matter division of modern theoretical physics, spin glasses soon became the "harmonic oscillators"[1] of the new paradigm of complexity: hundreds – if not thousands – of papers developed from (and on) this seminal model. Frustration, replica symmetry breaking, rough valleys of free energy, slow relaxational dynamics, aging and rejuvenation (and much more) created the mathematical and physical strands of a new approach to Nature, where the protagonists are no longer the subjects themselves but mainly the ways they interact. As a result, complex statistical mechanics is invading areas far beyond condensed matter physics, ranging from biology (e.g. neurology [1–3] and immunology [4,5]) to

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ISSN 1478–6435 print/ISSN 1478–6443 online

http://dx.doi.org/10.1080/14786435.2011.637979
http://www.tandfonline.com

human sciences (e.g. sociology [6,7] and economics [8,9]), and much more (see [10] for instance).

Although a crucial role has surely been played by the underlying graph theory (due to breakthroughs obtained even there, e.g. with the understanding of the small-world [11] and scale-free networks [12]), we would like to confer on the Sherrington–Kirkpatrick (SK) model (or its concrete variants on graphs, such as the Viana–Bray model [13,14] to cite just one) a crucial role in this new science of complexity.

Among the methods developed for solving its thermodynamics [15,16], interpolation techniques, even though not yet so strong in solving the problem in fully autonomy, soon played a key role in – at least – describing several properties of this system, working as a synergic alternative to the replica trick [17–19], which is actually the first and most famous approach to mean field statistical mechanics with quenched disorder. In fact, the interpolation scheme has been "naturally" implemented in the cavity field technique [20–22], and its variants such as stochastic stability [2,23,24] and random overlap structures [25,26].

In this paper we want to study this model by extending the interpolating scheme, from the original cavity perspective to the replica trick, in a way close to the work of Talagrand [27]. To allow this procedure we completely forget the original role played by the "number" of replicas in the replica trick (tuned by a parameter $n \in (0,1]$) and think of it directly as a real interpolating parameter. Interestingly this can intuitively be thought of as a quenching parameter coherently with its counterpart in glassy dynamics (i.e. FDT violations [28,29]). First, once the mathematical strategy has been introduced in complete generality, we use it to obtain a clear picture of the infinite volume and the zero replica limits at the replica symmetric level (by means of which the whole original SK theory is reproduced). Then, within the Parisi full replica symmetry breaking scenario, coupled with the broken replica bounds [30], other robustness properties dealing with the exchange of these two limits are achieved as well.

The paper is structured as follows. In Section 2 we briefly introduce the model (and the ideas behind the strategy of the replica trick) while in Section 3 we outline the strategy we want to apply to the model. All the other sections are then left to the implementation of the interpolation into this framework and for presenting the consequent results.

2. The Sherrington–Kirkpatrick mean field spin glass

2.1. *The model and its related definitions*

The generic configuration of the Sherrington–Kirkpatrick model [17,18] is determined by the N Ising variables $\sigma_i = \pm 1,\ i = 1, 2, \ldots, N$. The Hamiltonian of the model, in some external magnetic field h, is

$$H_N(\sigma, h; J) = -\frac{1}{\sqrt{N}} \sum_{1 \le i < j \le N} J_{ij}\sigma_i\sigma_j - h \sum_{1 \le i \le N} \sigma_i. \tag{1}$$

The first term in (1) is a long-range random two-body interaction, while the second term represents the interaction of the spins with the magnetic field h. The external quenched disorder is given by the $N(N-1)/2$ independent and identically distributed (i.i.d.) random variables (r.v.) J_{ij}, defined for each pair of sites. For the sake of simplicity, denoting the average over this disorder by $\mathbb{E}$, we assume each J_{ij} to be a centered unit Gaussian with averages

$$\mathbb{E}(J_{ij}) = 0, \quad \mathbb{E}(J_{ij}^2) = 1.$$

For a given inverse temperature[2] β, we introduce the disorder dependent partition function $Z_N(\beta, h; J)$, the quenched average of the free energy per site $f_N(\beta, h)$, the associated averaged normalized log-partition function $\alpha_N(\beta, h)$, and the disorder dependent Boltzmann–Gibbs state ω, according to the definitions

$$Z_N(\beta, h; J) = \sum_\sigma \exp(-\beta H_N(\sigma, h; J)), \tag{2}$$

$$-\beta f_N(\beta, h) = N^{-1}\mathbb{E} \ln Z_N(\beta, h) = \alpha_N(\beta, h), \tag{3}$$

$$\omega(A) = Z_N(\beta, h; J)^{-1} \sum_\sigma A(\sigma) \exp(-\beta H_N(\sigma, h; J)), \tag{4}$$

where A is a generic smooth function of σ.

Let us now introduce the important concept of replicas. We consider a generic number n of independent copies of the system, characterized by the spin configurations $\sigma^{(1)}, \dots, \sigma^{(n)}$, distributed according to the product state

$$\Omega = \omega^{(1)} \times \omega^{(2)} \times \cdots \times \omega^{(n)},$$

where each $\omega^{(\alpha)}$ acts on the corresponding $\sigma_i^{(\alpha)}$ variables, and all are subject to the *same* sample J of the external disorder.

The overlap between two replicas a, b is defined according to

$$q_{ab}(\sigma^{(a)}, \sigma^{(b)}) = \frac{1}{N} \sum_{1 \le i \le N} \sigma_i^{(a)} \sigma_i^{(b)}, \tag{5}$$

and satisfies the obvious bounds $-1 \le q_{ab} \le 1$.

For a generic smooth function A of the spin configurations on the n replicas, we define the average $\langle A \rangle$ as

$$\langle A \rangle = \mathbb{E}\Omega A\big(\sigma^{(1)}, \sigma^{(2)}, \dots, \sigma^{(n)}\big), \tag{6}$$

where the Boltzmann–Gibbs average Ω acts on the replicated σ variables and $\mathbb{E}$ denotes, as usual, the average with respect to the quenched disorder J.

2.2. *The replica trick in a nutshell*

The replica trick consists of evaluating the logarithm of the partition function through its power expansion, namely

$$\log Z = \lim_{n \to 0} \frac{Z^n - 1}{n} \Rightarrow \langle \log Z \rangle = \lim_{n \to 0} \frac{\langle Z^n \rangle - 1}{n} = \lim_{n \to 0} \frac{1}{n} \log \langle Z^n \rangle, \tag{7}$$

such that the (intensive) free energy can be written as

$$f_N(\beta, h) = \lim_{n\to 0} f_N(n, \beta, h), \tag{8}$$

where $f_N(n, \beta, h)$ is defined through

$$-\beta f_N(n, \beta, h) = \alpha_N(n, \beta, h) = \frac{1}{Nn}\log\langle Z^n\rangle. \tag{9}$$

By assuming the validity of the following commutativity of the n, N limits

$$\lim_{N\to\infty}\lim_{n\to 0}\alpha_N(n, \beta, h) = \lim_{n\to 0}\lim_{N\to\infty}\alpha_N(n, \beta, h) \tag{10}$$

both Sherrington and Kirkpatrick (at the replica symmetric level [17,18]) and Parisi (within the full RSB scenario [31–33]) gave a clear picture of the thermodynamics, which can be streamlined as follows. At the replica symmetric level (i.e. by assuming replica equivalence, namely $q_{ab} = q$ for $a \neq b$, 1 otherwise) we get

$$\alpha_{\mathrm{SK}}(\beta) = \min_q\{\alpha(\beta, h, q)\}, \tag{11}$$

where the trial function $\alpha(\beta, h, q)$ is defined as

$$\alpha(\beta, h, q) = \log 2 + \int \mathrm{d}\mu(z)\log\cosh\big(\beta(\sqrt{q}z + h)\big) + \frac{\beta^2}{4}(1-q)^2. \tag{12}$$

The self-consistency relation for q reads off as

$$q_{\mathrm{SK}} = \int \mathrm{d}\mu(z)\tanh^2\big(\beta(\sqrt{q_{\mathrm{SK}}}z + h)\big). \tag{13}$$

At the broken replica level we can write

$$\lim_{N\to\infty}\frac{1}{N}\mathbb{E}\log Z_N(\beta, J, h) = \alpha(\beta, h) = -\beta f(\beta, h) = \alpha_P(\beta, h), \tag{14}$$

where $\alpha_P(\beta, h)$, the fully broken replica solution, is defined as follows. Let us consider the functional

$$\alpha_P(\beta, h, x) = \log 2 + f(0, y; x, \beta)\,|_{y=h} - \frac{\beta^2}{2}\int_0^1 qx(q)\mathrm{d}q, \tag{15}$$

where $f(q, y;\ x, \beta) \equiv f(q, y)$ is a solution of the equation

$$\partial_q f + \frac{1}{2}\partial_y^2 f + \frac{1}{2}x(q)(\partial_y f)^2 = 0, \tag{16}$$

with boundary $f(1, y) = \log\cosh(\beta y)$. Then

$$\alpha_P(\beta, h) = \inf_{x\in\mathcal{X}}\alpha_P(\beta, h, x), \tag{17}$$

where $\mathcal{X}$ is the convex space of the piecewise constant functions as introduced for instance in [30].

3. The interpolating framework for the replica trick

In this section we present our strategy of investigation; namely we prove some theorems and propositions whose implications will be exploited in subsequent sections. For the sake of clarity we will omit some straightforward demonstrations.

We want to think of the mapping between the one-replica and zero-replica as an interpolation scheme, by the introduction of an auxiliary interpolating function, which we call the n-quenched free energy, which (non-trivially) bridges the system between $n = 1$ and $n = 0$, as

$$\varphi_N(n, \beta, h) = \frac{1}{Nn} \log \mathbb{E}(Z_N^n(\beta, J, h)), \tag{18}$$

where, for the sake of clarity $Z_N^n(\beta, J, h) \equiv (Z_N(\beta, J, h))^n$. This is the same object as that studied in [27] that here we want to analyze from a slightly different perspective. It is then worth stressing the next

Theorem 3.1: *The following relation, between the n-quenched free energy and the free energy, holds*

$$\lim_{n\to 0} \varphi_N(n, \beta, h) = \alpha_N(\beta, h); \tag{19}$$

furthermore

$$\varphi_N(n, \beta, h) \geq \alpha_N(\beta, h) \tag{20}$$

for any n.

Proof: We can expand the n-quenched free energy in a Taylor series in $n \in [0, 1]$ to get

$$\begin{aligned} &\log \mathbb{E}(Z_N^n(\beta, J, h)) = 0 + \mathbb{E}(\log Z_N(\beta, J, h))n + o(n^2) \Rightarrow \\ &\lim_{n\to 0^+} \varphi_N(n, \beta, h) = \lim_{n\to 0} \frac{1}{Nn}(\mathbb{E}(\log Z_N(\beta, J, h))n + o(n^2)) = \alpha_N(\beta, h). \end{aligned} \tag{21}$$

The Jensen inequality ensures the second statement of the theorem. □

Proposition 3.2: *Through Theorem* 3.1 *we immediately obtain*

$$\lim_{N\to\infty} \lim_{n\to 0} \varphi_N(n, \beta, h) = \alpha(\beta, h). \tag{22}$$

We now want to deepen the properties of $\varphi_N(n, \beta, h)$ following the strategy outlined in [34]:

Proposition 3.3: *Let* $i \in Q = \{1, \ldots, N\}$. *Introduce positive weights* $\forall i \to w_i \in \mathbb{R}^+$. *Let* $\forall i \to U_i$ *be a family of Gaussian random variables such that* $\mathbb{E}(U_i) = 0$ *and* $\mathbb{E}(U_i U_j) = S_{ij}$, *where* S_{ij} *is a positive definite symmetric matrix.*

For the functional $\varphi(n, t) = n^{-1} \log \mathbb{E}(Z_t^n)$, where $Z_t = \sum_i w_i \exp(\sqrt{t} U_i)$, the following relation holds

$$\frac{d}{dt}\varphi(n, t) = \frac{1}{2}\langle S_{ii}\rangle_n + \frac{(n-1)}{2}\langle S_{ij}\rangle_n, \tag{23}$$

where we have introduced the following

Definition 3.4: $\langle A\rangle_n = \mathbb{E}(Z_t^n \mathbb{E}(Z_t^n)^{-1}\Omega(A))$ is a deformed state on the 2-product Boltzmann state, namely

$$\Omega(A) = \sum_{i,j}^{N} (Z_t^{-1} w_i \exp\sqrt{t}U_i)(Z_t^{-1}\omega_j \exp\sqrt{t}U_j)A,$$

where A is an observable on $Q \times Q$,

$$\omega(A) = \sum_{i}^{N} (Z_t^{-1} w_i \exp\sqrt{t}U_i)A,$$

with $A \in \mathcal{A}(Q)$.

The following generalization, considering two families of random variables, can be easily obtained.

Proposition 3.5: *Let $i \in Q = \{1, \dots, N\}$ be a probability space and $\forall i \to w_i \in \mathbb{R}^+$ be a probability weight and $\forall i \to U_i$ a family of random Gaussian variables such that $\mathbb{E}(U_i) = 0$ and $\mathbb{E}(U_iU_j) = S_{ij}$, where S_{ij} is a positive definite symmetric matrix.*

Let $\forall i \to \tilde{U}_i$ be another family of random Gaussian variables such that $\mathbb{E}(\tilde{U}_i) = 0$ *and* $\mathbb{E}(\tilde{U}_i\tilde{U}_j) = \tilde{S}_{ij}$, where S_{ij} is a positive definite symmetric matrix. Let us further consider the functional $\varphi(n,t) = n^{-1}\log\mathbb{E}(Z_t^n)$ (where $Z_t = \sum_i w_i \exp(\sqrt{t}U_i + \sqrt{1-t}\tilde{U}_i)$). Then the following relation holds

$$\frac{\mathrm{d}}{\mathrm{d}t}\varphi(n,t) = \frac{1}{2}\langle S_{ii} - \tilde{S}_{ii}\rangle_n + \frac{(n-1)}{2}\langle S_{ij} - \tilde{S}_{ij}\rangle_n. \tag{24}$$

We can then formulate the following

Theorem 3.6: *If $\forall (i,j) \in Q \times Q$, $S_{ii} = \tilde{S}_{ii}$ and $S_{ij} \geq \tilde{S}_{ij}$, the following relation holds*

$$\varphi(n,1) \leq \varphi(n,0), \quad \forall n \in (0,1].$$

Proof: Integrating the functional between 0 and 1 we get $\varphi(n,1) - \varphi(n,0) = \frac{1}{2}(n-1)\int_0^1 \mathrm{d}t\langle S_{ij} - \tilde{S}_{ij}\rangle_n$, whose right-hand side is ≤ 0 for $n \in (0,1]$.

Obviously the following relation tacitely holds: $\lim_{n\to 0}\langle\cdot\rangle_n = \langle\cdot\rangle$. □

Focusing on the Sherrington–Kirkpatrick model, as introduced earlier, and by using the results of the previous section, we still think at the n-variation as an interpolation and we can state the following

Theorem 3.7: *Let us consider the functional $\psi_N(n,\beta,h) = n^{-1}\log\mathbb{E}(Z_N^n(\beta,J,h)) = N\varphi_N(n,\beta,h)$. Then $\psi_N(n,\beta,h)$ is super-additive in N, $\forall n \in (0,1]$. Furthermore*

$$\lim_{N\to\infty}\varphi_N(n,\beta,h) = \sup_N \varphi_N(n,\beta,h) = \varphi(n,\beta,h), \text{ for any } n.$$

We omit the proof as it is analogous to the one given in [35].

Corollary 3.8: *Remembering that for super-additive (and bounded) functions we can write*

$$\lim_{N\to\infty} \alpha_N(\beta,h) = \sup_N \alpha_N(\beta,h) = \alpha(\beta,h), \tag{25}$$

we get a lower bound for $\varphi(n,\beta,h)$ as $\varphi(n,\beta,h) \geq \alpha(\beta,h)$ and $\sup_N \varphi_N(n,\beta,h) \geq \sup_N \alpha_N(\beta, h)$.

4. Replica symmetric interpolation

For the upper bound we have to tackle the replica symmetric approximation by using a linearization strategy as follows.[3] We introduce and define an interpolating partition function with $t \in [0,1]$ as

$$Z_t = \sum_{\{\sigma\}} \exp(\beta \widetilde{H}(t,\sigma)) \exp\left(\beta h \sum_i^N \sigma_i\right), \tag{26}$$

where, labeling with $K(\sigma)$ standard $\mathcal{N}(0,1)$ indexed by the configurations σ and characterized by the covariance $\mathbb{E}(K(\sigma)K(\sigma')) = q^2_{\sigma\sigma'}$ we defined

$$\widetilde{H}(t,\sigma) = \sqrt{t}\sqrt{\frac{N}{2}}K(\sigma) + \sqrt{1-t}\sqrt{q}\sum_i J_i\sigma_i, \tag{27}$$

where q will play the role of the replica-symmetric overlap, and J_i are random Gaussians i.i.d. $\mathcal{N}[0,1]$ independent also of $K(\sigma)$ and such that

$$\mathbb{E}\left(\left(\beta\sqrt{q}\sum_i J_i\sigma_i\right)\left(\beta\sqrt{q}\sum_j J_j\sigma_j\right)\right) = \beta^2 N q q_{\sigma\sigma'}. \tag{28}$$

Lemma 4.1: *Let us consider the functional $\varphi(t) = (Nn)^{-1}\log\mathbb{E}(Z_t^n)$. We have that*

$$\varphi(1) = \frac{1}{Nn}\log\mathbb{E}(Z_1^n) = \varphi_N(n,\beta,h) \tag{29}$$

$$\varphi(0) = \log 2 + \frac{1}{n}\log\int d\mu(z)\cosh^n\big(\beta(\sqrt{q}z+h)\big). \tag{30}$$

We are ready to state the next

Theorem 4.2: $\forall n \in (0,1]$ *we have*

$$\varphi_N(n,\beta,h) \leq \log 2 + \frac{1}{n}\log\int d\mu(z)\cosh^n\big(\beta(\sqrt{q}z+h)\big) + \frac{\beta^2}{4}(1-2q-(n-1)q^2) \tag{31}$$

uniformly in N.

Proof: By applying Proposition 3.5 we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\varphi(t) = \frac{\beta^2}{4} - \frac{\beta^2}{2}q + \frac{(n-1)\beta^2}{4}\langle q_{\sigma\sigma'}^2 - 2qq_{\sigma\sigma'}\rangle_n.$$

Then, completing with q^2 the square on the right-hand side, and integrating back in $0, 1$ we get the thesis. □

In complete analogy with the original SK theory we can define

$$\alpha(n,\beta,h,q) = \log 2 + \frac{1}{n}\log\int \mathrm{d}\mu(z)\cosh^n\big(\beta(\sqrt{q}z+h)\big) + \frac{\beta^2}{4}(1-2q-(n-1)q^2),$$
$$\alpha_{RS}(n,\beta,h) = \min_q(\alpha(n,\beta,h,q)). \tag{32}$$

Then we immediately get the next

Theorem 4.3: $\forall n\in(0,1]$, $\varphi_N(n, \beta, h) \le \alpha_{\mathrm{SK}}(n, \beta, h)$ *uniformly in N.*

It is worth noting that the stationarity of q becomes

$$\frac{\partial}{\partial q}\alpha(n,\beta,h,q) = 0 \Rightarrow q_n = \frac{\int \mathrm{d}\mu(z)\cosh^n\theta\tanh^2\theta}{\int \mathrm{d}\mu(z)\cosh^n\theta} = \langle\tanh^2\theta\rangle_n \tag{33}$$

where we have emphasized the n-dependence of q via q_n, we have used $\theta = \beta(\sqrt{q_n}z + h)$ for the sake of clarity, $\mathrm{d}\mu$ as a standard Gaussian measure and the averages as

$$\langle F\rangle_n = E\left(\frac{Z^n}{\mathbb{E}(Z^n)}F\right) = \frac{\int \mathrm{d}\mu(z)\cosh^n\theta F}{\int \mathrm{d}\mu(z)\cosh^n\theta}.$$

This ensures the validity of the next

Theorem 4.4: *For all values of* $n\in(0,1]$ *we have*

$$\alpha_{\mathrm{SK}}(n,\beta,h) \ge \alpha_{\mathrm{SK}}(\beta,h), \quad \lim_{n\to 0}\alpha_{\mathrm{SK}}(n,\beta,h) = \alpha_{\mathrm{SK}}(\beta,h),$$
$$q_n \ge q_{\mathrm{SK}}, \quad \lim_{n\to 0} q_n = q_{\mathrm{SK}}.$$

Furthermore it is possible to show easily that, under specific conditions, Equation (33) defines a contraction, implicitly accounting for the high-temperature regime.[4] For this task we rewrite the latter as

$$q = \beta^2 q\frac{\int \mathrm{d}\theta\exp(-\frac{\theta^2}{2\beta^2 q})\cosh^n\theta\tanh^2\theta}{\int \mathrm{d}\theta\exp(-\frac{\theta^2}{2\beta^2 q})\cosh^n(\theta)(\theta - n\beta^2 q\tanh\theta)\theta}, \tag{34}$$

such that $\forall q\in\mathcal{R}\to \|q\|\equiv|q|$.

Let us introduce the operator $\mathbf{K}$: $q\to\mathbf{K}(q)$ defined via the original replica symmetric self-consistency relation and use for its norm $\|\mathbf{K}\|\equiv\sup_q(\|\mathbf{K}(q)\|/\|q\|)$.

So we can state that

Theorem 4.5: *$\exists(n,\beta):\mathbf{K}$ is a contraction in $\mathcal{R}$ and these are related by $\beta_c(n)=\sqrt{1+n}^{-1}$. Coherently with the previous results, criticality is recovered at $\beta_c=1$ when $n\to 0$.*

Proof: By definition

$$\|\mathbf{K}\| = \sup_q \left\{ \frac{\beta^2|q|}{|q|} \frac{|\int d\theta \exp(-\frac{\theta^2}{2\beta^2 q}) \cosh^n\theta \tanh^2\theta|}{|\int d\theta \exp(-\frac{\theta^2}{2\beta^2 q}) \cosh^n(\theta)(\theta - n\beta^2 q \tanh\theta)\theta|} \right\}.$$

By using the reversed triangular relation we get $|\tanh\theta|\le|\theta|\Rightarrow|\theta-n\beta^2 q\tanh\theta|\ge|(|\theta|-n\beta^2 q|\tanh\theta|)|\ge|\theta||1-n\beta^2 q|$ such that

$$\|\mathbf{K}\| \le \sup_q \left\{ \frac{\beta^2}{|1-n\beta^2 q|} \right\}; \quad q\in[0,1]\Rightarrow\|\mathbf{K}\|\le\frac{\beta^2}{|1-n\beta^2|}. \tag{35}$$

So if $\beta^2\le|1-n\beta^2|$, $\mathbf{K}$ is a contraction and $q=0$ is the only solution of the self-consistency relation. $\square$

5. Broken replica interpolation

To find an easy way to deal with the RSB scenario within an interpolating framework, we now rearrange the scaffolding introduced in [30,34] as follows. Beyond the structures outlines in Propositions 3.3 and 3.5, we introduce $K\in\mathbf{N}$ as an RSB-level counter such that, concretely, $\forall(a,i)$ with $a=1,\ldots,K$ and $i=1,\ldots,N$ we use a family B_i^a of i.i.d. $\mathcal{N}[0,1]$, independent even of the U_i and such that

$$\mathbb{E}(B_i^a B_j^b) = \delta_{ab}\widetilde{S}^a ij. \tag{36}$$

We introduce the averages with respect to the variables $B_i^K, B_i^{K-1},\ldots,B_i^1, U_i$ with the notation

$$\mathbb{E}_a(\cdot)=\int d\mu(B_i^a)(\cdot)\ \forall a=1,\ldots,K,\ \mathbb{E}_0(\cdot)=\int d\mu(U_i)(\cdot),\ \mathbb{E}(\cdot)=\mathbb{E}_0\mathbb{E}_1,\ldots,\mathbb{E}_K(\cdot),$$

and, $\forall n\in(0,1]$, a family of order parameters $(m_1,\ldots,m_K)_n$ with $n<m_a<1$ $\forall a=1,\ldots,K$, and – recursively – the following r.v.

$$Z_K(t)=\sum_i w_i \exp\left(\sqrt{t}U_i+\sqrt{1-t}\sum_{a=1}^K B_i^a\right),\quad Z_{a-1}^{m_a}=\mathbb{E}_a(Z_a^{m_a}),\quad f_a=\frac{Z_a^{m_a}}{\mathbb{E}_a(Z_a^{m_a})}$$

in perfect analogy with the path outlined in [30]. We are then ready to state the following

Proposition 5.1: *Let us consider the functional* $\varphi(n,t)=n^{-1}\log\mathbb{E}_0(Z_0^n)$. *The following relation holds*

$$\frac{\mathrm{d}}{\mathrm{d}t}\varphi(n,t)=\frac{1}{2}\langle S_{ii}-\widehat{S}_{ii}^{K}\rangle_K^n+\frac{1}{2}\sum_{a=0}^{K}(m_{a+1}-m_a)_n\langle S_{ij}-\widehat{S}_{ij}^{a}\rangle_a^n \tag{37}$$

where $\widehat{S}_{ij}^0=0,\ \widehat{S}_{ij}^a=\sum_{b=1}^{a}\widetilde{S}_{ij}^b$.

5.1. *Upper bound and Parisi solution*

We can apply Proposition 5.1 to the interpolant $Z_K\equiv Z_t\equiv Z_N(\beta,t,x)$, where

$$Z_N(\beta,t,x)=\sum_{\sigma_1,\dots,\sigma_N}\exp\left(\beta\sqrt{\frac{N}{2}}K(\sigma)+\beta\sqrt{1-t}\sum_{a=1}^{K}\sqrt{q_a-q_{a-1}}J_i^a\sigma_i\right)e^{\beta h\sum_i\sigma_i}$$

and the J_i^a are defined as the B_i^a (see Equation 36 and above) and x_n mirrors the broken replica steps, namely we introduce a convex space χ_n whose elements are the $x_n(q)$ piecewise functions x_n: $q\to[n,1]$ such that $x_n(q)=m_a(n)$ for $q_{a-1}<q\le q_a$ $\forall a=1,\dots,K$, with the prescription $q_0=0$, $q_K=1$. Note that in this sense we wrote $Z_N(\beta,t,x)$ even though there is no explicit dependence on x on the right-hand side

We then consider the functional

$$\varphi(n,t)=(Nn)^{-1}\log\mathbb{E}_0(Z_0^n) \tag{38}$$

and introduce the following:

Lemma 5.2:

$$\varphi(n,1)=\varphi_N(n,\beta,h),\quad \varphi(n,0)=\log 2+f(0,h;x_n,\beta),$$

where f satisfies the Parisi equation with x_n as introduced in Section 2.

Consequently the following theorem holds

Theorem 5.3: $\forall n\in(0,1]$ *the functional n-quenched free energy* $\varphi(n,t)$ *defined in Equation* (38) *respects the bound*

$$\varphi(n,1)=\varphi_N(n,\beta,h)\le\log 2+f(0,h;x_n,\beta)-\frac{\beta^2}{4}\left(1-\sum_{a=0}^{K}(m_{a+1}-m_a)_n q_a^2\right)$$

uniformly in N.

Proof: We can use Proposition 5.1, keeping in mind the relations

$$\mathbb{E}\left(\beta^2\frac{N}{2}K(\sigma)K(\sigma')\right)=\beta^2\frac{N}{2}q_{12}^2=S_{ij},$$

$$\mathbb{E}\left(\beta^2\sqrt{q_a-q_{a-1}}\sqrt{q_b-q_{b-1}}\sum_i J_i^a\sigma_i\sum_j J_j^a\sigma_j\right)=\beta^2N(q_a-q_{a-1})q_{12}=\widetilde{S}_{ij}^a. \tag{39}$$

to get

$$\frac{\mathrm{d}}{\mathrm{d}t}\varphi(n,t) = -\frac{\beta^2}{4} - \frac{\beta^2}{4}\sum_{a=0}^{K}(m_{a+1}-m_a)_n\langle q_{12}^2 - 2q_a q_{12}\rangle_a^n.$$

Completing with q^2 the square on the right-hand side we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\varphi(n,t) = -\frac{\beta^2}{4}\left(1-\sum_{a=0}^{K}(m_{a+1}-m_a)_n q_a^2\right) - \frac{\beta^2}{4}\sum_{a=0}^{K}(m_{a+1}-m_a)_n\langle (q_{12}-q_a)^2\rangle_a^n.$$

Lastly, it is enough to remember that

$$(m_{a+1}-m_a)_n \geq 0 \quad \forall a=0,\ldots,K \Rightarrow \varphi(n,1) \leq \varphi(n,0) - \frac{\beta^2}{4}\left(1-\sum_{a=0}^{K}(m_{a+1}-m_a)_n q_a^2\right),$$

to get the thesis. □

We can then define

$$\alpha_P(\beta,h,x_n) = \log 2 + n\frac{\beta^2}{4} + f(0,y;x_n,\beta)|_{y=h} - \frac{\beta^2}{2}\int_0^1 q x_n(q)\mathrm{d}q, \tag{40}$$

and write furthermore that

$$\frac{1}{2}\left(1-\sum_{a=0}^{K}(m_{a+1}-m_a)_n q_a^2\right) = \int_0^1 q x_n(q)\mathrm{d}q - \frac{n}{2}$$

to state the next

Theorem 5.4: *The following bounds hold*

$$\begin{aligned} &\lim_{N\to\infty}\varphi_N(n,\beta,h) = \varphi(n,\beta,h) \leq \alpha_P(\beta,h,x_n) \Rightarrow \varphi(n,\beta,h) \leq \inf_{x_n}\alpha_P(\beta,h,x_n), \\ &\lim_{n\to 0}\varphi(n,\beta,h) \leq \lim_{n\to 0}\inf_{x_n}\alpha_P(\beta,h,x_n) = \alpha_P(\beta,h), \end{aligned} \tag{41}$$

and clearly $\lim_{n\to 0}\alpha_P(\beta,h,x_n) = \alpha_P(\beta,h,x)$.

6. The commutativity of $n\to 0$ and $N\to\infty$

Let us now extend the interpolation to tackle two i.i.d. copies of the original Hamiltonian H_1, H_2 as

$$H_N(\sigma,t) = \sqrt{t}H_1(\sigma) + \sqrt{1-t}H_2(\sigma), \tag{42}$$

where we have omitted the N-dependence in H_1, H_2 for the sake of clarity.

We can define the corresponding partition function as

$$Z(\beta,t) = \sum_\sigma e^{-\beta H(\sigma,t)}, \tag{43}$$

and define the interpolating functional as

$$\psi(n,t)=\frac{1}{n}\log\mathbb{E}_1(\exp(n\mathbb{E}_2(\log Z(\beta,t)))) \tag{44}$$

where $\mathbb{E}_{1,2}$ averages respectively over the disorders of $H_{1,2}$.

It is straightforward to check that

$$\psi(n,1)=\frac{1}{n}\log\mathbb{E}_1(\exp(n\log Z(\beta,t=1)))\equiv\frac{1}{n}\log\mathbb{E}(\exp(n\log Z(\beta))), \tag{45}$$

$$\psi(n,0)=\mathbb{E}_2(\log Z(\beta,t=0))\equiv\mathbb{E}(\log Z(\beta)), \tag{46}$$

where $Z(\beta)$ is the partition function of the original Hamiltonian.

Proposition 6.1: *After introducing*

$$G(n,t)=\exp(n\mathbb{E}_2(\log Z(\beta,t))), \tag{47}$$

and the t-dependent Boltzmann weights as $p(\sigma,t)=e^{-\beta H(\sigma,t)}/Z(\beta,t)$, *the streaming of the functional* $\psi(n,t)$ *with respect to the interpolating parameter is*

$$\frac{\mathrm{d}\psi(n,t)}{\mathrm{d}t}=n\frac{\beta^2}{2}\frac{1}{\mathbb{E}_1(G(n,t))}\mathbb{E}_1\left(G(n,t)\sum_{\sigma,\tau}\mathcal{C}(\sigma,\tau)\mathbb{E}_2(p(\sigma,t))\mathbb{E}_2(p(\tau,t))\right). \tag{48}$$

Proof: By a direct evaluation we get

$$\frac{\mathrm{d}\psi(n,t)}{\mathrm{d}t}=\frac{\mathbb{E}_1\left(G(n,t)\mathbb{E}_2(\frac{\mathrm{d}Z(\beta,t)}{\mathrm{d}t}\frac{1}{Z(\beta,t)})\right)}{\mathbb{E}_1(G(n,t))},$$

where

$$\frac{\mathrm{d}Z(\beta,t)}{\mathrm{d}t}=-\frac{\beta}{2}\sum_{\sigma}\left(\frac{1}{\sqrt{t}}H_1(\sigma)-\frac{1}{\sqrt{1-t}}H_2(\sigma)\right)e^{-\beta H(\sigma,t)}.$$

Then we write

$$\frac{\mathrm{d}\psi(n,t)}{\mathrm{d}t}=-\frac{\beta}{2}\frac{1}{\mathbb{E}_1(G(n,t))}(A-B),$$

where

$$A=\mathbb{E}_1\left(G(n,t)\mathbb{E}_2\sum_{\sigma}\left(\frac{1}{\sqrt{t}}H_1(\sigma)\,p(\sigma,t)\right)\right), \tag{49}$$

$$B=\mathbb{E}_1\left(G(n,t)\mathbb{E}_2\sum_{\sigma}\left(\frac{1}{\sqrt{1-t}}H_2(\sigma)\,p(\sigma,t)\right)\right). \tag{50}$$

Introducing here the label τ with the usual meaning of another set of Ising spins $\tau_i = \pm 1$, $i \in (1, \ldots, N)$, by applying Wick's theorem to A (on the family of random $H_1(\sigma)$) and calling the covariance matrix of $H_1(\sigma)$ $\mathcal{C}(\sigma,\tau)$ we get

$$A = \frac{1}{\sqrt{t}} \sum_{\sigma} \mathbb{E}_1(H_1(\sigma) G(n,t) \mathbb{E}_2(p(\sigma,t))) \tag{51}$$

$$= \frac{1}{\sqrt{t}} \sum_{\sigma,\tau} \mathcal{C}(\sigma,\tau) \mathbb{E}_1 \left(\frac{\partial G(n,t)}{\partial H_1(\tau)} \mathbb{E}_2(p(\sigma,t)) \right) + G(n,t) \mathbb{E}_2 \left(\frac{\partial p(\sigma,t)}{\partial H_1(\tau)} \right). \tag{52}$$

We must then evaluate explicitly

$$\frac{\partial G(n,t)}{\partial H_1(\tau)} = -n\beta\sqrt{t} G(n,t) \mathbb{E}_2 \left(e^{-\beta H(\tau,t)} \frac{1}{Z(\beta,t)} \right) = -n\beta\sqrt{t} G(n,t) \mathbb{E}_2(p(\tau,t)),$$

and

$$\frac{\partial p(\sigma,t)}{\partial H_1(\tau)} = -\beta\sqrt{t}(\delta_{\sigma\tau} p(\sigma,t) + p(\sigma,t)\, p(\tau,t)).$$

Overall we can write

$$A = -\beta \mathbb{E}_1 \left(G(n,t) \sum_{\sigma,\tau} \mathcal{C}(\sigma,\tau) \Big[n \mathbb{E}_2(p(\sigma,t)) \mathbb{E}_2(p(\tau,t)) + \mathbb{E}_2(\delta_{\sigma\tau} p(\sigma,t)) + p(\tau,t) \Big] \right).$$

By applying Wick's theorem to B (on the family of random $H_2(\sigma)$) and calling again its covariance matrix $\mathcal{C}(\sigma,\tau)$ (as the two Hamiltonian are i.i.d.) we get

$$B = \mathbb{E}_1 \left(G(n,t) \mathbb{E}_2 \sum_{\sigma} \left(\frac{1}{\sqrt{1-t}} H_2(\sigma)\, p(\sigma,t) \right) \right) \tag{53}$$

$$= \frac{1}{\sqrt{1-t}} \mathbb{E}_1 \left(G(n,t) \sum_{\sigma,\tau} \mathcal{C}(\sigma,\tau) \mathbb{E}_2 \left(\frac{\partial p(\sigma,t)}{\partial H_2(\tau)} \right) \right). \tag{54}$$

Mirroring the previous calculations, we get

$$\frac{\partial p(\sigma,t)}{\partial H_2(\tau)} = -\beta\sqrt{1-t}(\delta_{\sigma\tau} p(\sigma,t) + p(\sigma,t)\, p(\tau,t)).$$

Pasting all these together we get the thesis. □

Remark 1: The proposition still holds even if we consider an external field coupled to the system and not only for $n \in [0, 1]$.

We are ready to state the next

Theorem 6.2: *Let us recall that, calling* $\mathcal{C}$ *the covariance matrix of the SK Hamiltonian, the model is thermodynamically stable* [23], *namely there exists a constant* $C < \infty$ *such that* $\lim_{N\to\infty}(1/N)\mathcal{C}(\sigma,\sigma) \le C$ *(and, as a consequence of the*

Schwartz inequality, $\lim_{N\to\infty}(1/N)\mathcal{C}(\sigma,\tau)\leq C$, *and that it admits a sensible thermodynamic limit* [35]. *Then*

$$\lim_{n\to 0^+}\lim_{N\to\infty}\frac{1}{N}\varphi_N(\beta,n)=\alpha(\beta).$$

Proof: It is immediate to check that $\varphi_N(\beta, n)$ is increasing in n for $n\in[0, 1]$ and this monotony is preserved in the thermodynamic limit, so that

$$\exists\lim_{n\to 0^+}\lim_{N\to\infty}\frac{1}{N}\varphi_N(\beta,n), \tag{55}$$

$$\lim_{N\to\infty}\frac{1}{N}\varphi_N(\beta,n)\geq\lim_{N\to\infty}\frac{1}{N}\alpha_N(\beta)=\alpha(\beta), \tag{56}$$

or simply

$$\lim_{n\to 0^+}\lim_{N\to\infty}\frac{1}{N}\varphi_N(\beta,n)\geq\alpha(\beta).$$

To proof the inverse inequality we use Proposition 6.1.

Let us consider

$$\psi_N(n,\beta,t)=\frac{1}{Nn}\log\mathbb{E}_1\exp(n\mathbb{E}_2(\log Z_N(\beta,t))).$$

Of course we have that

$$\psi_N(n,\beta,1)=\varphi_N(\beta,n), \tag{57}$$

$$\psi_N(n,\beta,0)=\alpha_N(\beta), \tag{58}$$

and we can write

$$\psi_N(n,\beta,1)-\psi_N(n,\beta,0)=\int_0^1 \mathrm{d}t\frac{\partial}{\partial t}\psi_N(n,\beta,t),$$

where

$$\begin{aligned}&\frac{\partial}{\partial t}\psi_N(n,\beta,t)\\&\quad=\frac{n}{N}\frac{\beta^2}{2}\frac{1}{\mathbb{E}_1(G_N(n,\beta,t))}\mathbb{E}_1\left(G_N(n,\beta,t)\sum_{\sigma,\tau}\mathcal{C}_N(\sigma,\tau)\mathbb{E}_2(p_N(\sigma,\beta,t))\mathbb{E}_2(p_N(\tau,\beta,t))\right).\end{aligned} \tag{59}$$

Bounding $\mathcal{C}_N(\sigma,\tau)$ with its sup and noticing that

$$\sum_{\sigma,\tau}\mathbb{E}_2(p_N(\sigma,\beta,t))\mathbb{E}_2(p_N(\tau,\beta,t))=1,$$

we have that

$$\frac{\partial}{\partial t}\psi_N(n,\beta,t) \leq \frac{n}{N}\frac{\beta^2}{2}\max_{\sigma,\tau}\mathcal{C}_N(\sigma,\tau).$$

We can use now the property of thermodynamic stability to obtain

$$\lim_{N\to\infty}\frac{1}{N}\varphi_N(\beta,n) - \lim_{N\to\infty}\frac{1}{N}\alpha_N(\beta) \leq n\frac{\beta^2}{2}C,$$

or simply

$$\lim_{n\to 0^+}\lim_{N\to\infty}\frac{1}{N}\varphi_N(\beta,n) - \alpha(\beta) \leq 0,$$

which is the inverse bound.

For the commutativity of $\lim_n$ and $\lim_N$ now it is enough to prove the inverse limit. This can be achieved immediately by applying l'Hopital's rule to $\varphi_N(\beta, n)$ in n to get

$$\lim_{n\to 0^+}\varphi_N(\beta,n) = \alpha_N(\beta),$$

such that

$$\lim_{N\to\infty}\frac{1}{N}\lim_{n\to 0^+}\varphi_N(\beta,n) = \alpha(\beta).$$

□

Remark 2: We stress that, although in this paper we limit ourselves to the investigation of the properties of the pure SK model, the methods exploited in this section apply to a broad range of models, as discussed for instance in [23].

Finally we enlarge the scheme introduced in this section to allow for telescopic broken bounds by defining the following functional

$$\psi(n,m,t) = \frac{1}{n}\log\mathbb{E}_1\Big(\exp\Big[\frac{n}{m}\log\mathbb{E}_2(\exp(m\log Z(t)))\Big]\Big), \tag{60}$$

where, as usual, $\mathbb{E}_{1,2}$ average over the disorder $H_{1,2}$, respectively.

Again it is straightforward to check that

$$\psi(n,m,1) = \frac{1}{n}\log\mathbb{E}_1(\exp(n\log Z(1))) \equiv \frac{1}{n}\log\mathbb{E}(\exp(n\log Z)) \tag{61}$$

$$\psi(n,m,0) = \frac{1}{m}\log\mathbb{E}_2(\exp(m\log Z(0))) \equiv \frac{1}{m}\log\mathbb{E}(\exp(m\log Z)) \tag{62}$$

and that the following generalization of Proposition 6.1 holds

$$\frac{\mathrm{d}\psi(n,m,t)}{\mathrm{d}t} = \frac{\beta^2}{2}\frac{(n-m)}{\mathbb{E}_1(G(n,m,t))}\mathbb{E}_1\left(G(n,m,t)\sum_{\sigma,\tau}\mathcal{C}(\sigma,\tau)\mathbb{E}_2(p(\sigma,t)b(m,t))\mathbb{E}_2(p(\tau,t)b(m,t))\right), \tag{63}$$

where

$$G(n, m, t) = \exp\left[\frac{n}{m}\log \mathbb{E}_2(\exp(m\log Z(t)))\right], \tag{64}$$

by which we can argue that the n-quenched free energy $\varphi_N(\beta, n)$ has Lipschitz constant equal to $L = C\beta^2/2$.

6.1. *The temperature of the disorder*

From the perspective described in this paper, by which n is thought of as a real interpolating parameter between the annealed and the quenched representations of the free energy, it is interesting to try to connect the latter with the effective temperatures investigated in the dynamics and, as a direct consequence, with the spreading of the timescales involved in the thermalization, regardless of the particular Hamiltonian under investigation.

For this task, in this section we want to try to emphasize the formal analogy between the ''real'' temperature β and an ''effective'' temperature n: by using capital letters to denote extensive quantities (e.g. the extensive free energy takes the label F), we start by noticing the formal relation between β and n as

$$F(\beta) \propto \frac{1}{\beta}\mathbb{E}\log\sum_\sigma e^{-\beta H(\sigma;J)}, \tag{65}$$

$$F(n) \propto \frac{1}{n}\log \mathbb{E}e^{n\log Z(J)}. \tag{66}$$

Interestingly for a connection with the dynamical properties of glasses [28,29,37,38], while the Boltzmann temperature β rules the overall energy fluctuations of the system, n seems to tackle the behavior inside the valleys of free energy themselves.

To deepen this point we revise here the powerful approach investigated by Sherrington, Coolen and coworkers in a series of papers [39–41]. First, let us introduce the average $\mathbb{E}_\sigma$ over the configurations as

$$Z(\beta, J) = \frac{1}{2^N}\sum_\sigma e^{-\beta H(\sigma,J)} = \mathbb{E}_\sigma e^{-\beta H(\sigma,J)},$$

by which the annealed and quenched free energies (f_A, f_Q, respectively) can be written as

$$f_A(\beta) = -\frac{1}{\beta N}\log \mathbb{E}_J(Z(\beta, J)) = -\frac{1}{\beta N}\log \mathbb{E}_J\mathbb{E}_\sigma e^{-\beta H(\sigma,J)}, \tag{67}$$

$$f_Q(\beta) = -\frac{1}{\beta N}\mathbb{E}_J\log Z(\beta, J) = -\frac{1}{\beta N}\mathbb{E}_J\log \mathbb{E}_\sigma e^{-\beta H(\sigma,J)}, \tag{68}$$

where $p(J)$ should not be confused with the a priori J-distribution that is included in $\mathbb{E}_J$, and such that in the annealed case ($n=1$) both the r.v. J and σ thermalize on the same timescale (related to β), while in the quenched case ($n=0$) the r.v. J is averaged after taking the logarithm, such that its dynamics is completely frozen with respect to the dynamics of the fast variables σ. Since, so far, we have used n as a real interpolating parameter, we want to see here whether it can be thought of as a quencher (tuning it from one to zero) for the J.

For this task let us consider (and implicitly define) the extended extensive free energy Boltzmann functional

$$F = \mathbb{E}_J\mathbb{E}_\sigma p(\sigma, J)\left(H(\sigma, J) + \frac{1}{\beta}\log p(\sigma, J)\right) \tag{69}$$

where $p(J, \sigma)$ is a properly introduced weight whose explicit expression we want to work out.

We restrict ourselves in searching for explicit expressions that allow the following decomposition

$$p(J, \sigma) = p(J)\, p(\sigma|J),$$

such that, by direct substitution, we can write

$$F = \mathbb{E}_J p(J)\left(F_{\text{eff}}(J) + \frac{1}{\beta}\log p(J)\right), \tag{70}$$

where $F_{\text{eff}}(J)$ is the standard extensive free energy:[5]

$$F_{\text{eff}}(J) = \mathbb{E}_\sigma p(\sigma|J)\left(H(\sigma, J) + \frac{1}{\beta}\log p(\sigma|J)\right). \tag{71}$$

Now, for fixed J, we can minimize $F_{\text{eff}}(J)$ with respect to $p(\sigma|J)$ with the constraint $\mathbb{E}_\sigma p(\sigma|J) = 1$ so as to obtain the classical expression

$$p(\sigma|J) \equiv p(\sigma|J, \beta) = \frac{1}{Z(\beta, J)} e^{-\beta H(\sigma, J)},$$

where $Z(\beta, J) = \mathbb{E}_\sigma e^{-\beta H(J,\sigma)}$ is the standard partition function and the extensive free energy assumes the familiar representation

$$F_{\text{eff}}(J) \equiv F_{\text{eff}}(J, \beta) = -\frac{1}{\beta}\log Z(\beta, J). \tag{72}$$

Now let us instead minimize $\mathcal{F}$ with respect to $p(J)$ with two constraints: the former being the normalization over $P(J)$, i.e. $\mathbb{E}_J p(J) = 1$, the latter being the choice of the entropy for the J variables, which we retain in the classical equilibrium form (implicitly assuming adiabaticity as in the seminal papers by Coolen [39,40])

$$-\frac{1}{\beta}\mathbb{E}_J p(J)\log p(J) = S(n, \beta).$$

Note that here we emphasize the n-dependence introduced in this further "entropy" due to the complexity of the J-distribution.[6] Note further that this entropy is tuned by β.

Let us use λ and μ for the Lagrange multipliers, such that the functional to be minimized can be read off as

$$F + \mu(\mathbb{E}_J p(J) - 1) + \lambda\left(\frac{1}{\beta}\mathbb{E}_J p(J)\log p(J) + S(n,\beta)\right). \tag{73}$$

By minimizing with respect to $p(J)$ we get

$$F_{\text{eff}}(J,\beta) + \left(\frac{\lambda+1}{\beta}\right) + \left(\frac{\lambda+1}{\beta}\right)\log p(J) + \mu = 0 \tag{74}$$

or simply

$$p(J) = e^{-\frac{\beta}{\lambda+1}F_{\text{eff}}(J)}e^{-\frac{\beta}{\lambda+1}\mu}.$$

Using the constraint over the normalization (the one ruled by μ) we immediately get

$$e^{\frac{\beta}{\lambda+1}\mu} = \mathbb{E}_J e^{-\frac{\beta}{\lambda+1}F_{\text{eff}}(\beta,J)}.$$

We are left with the determination of λ. For this task we can always choose the function $S(n,\beta)$ such that $\frac{1}{\lambda+1} = n$, so as to get

$$p(J) \equiv p(J,\beta,n) = \frac{1}{\widetilde{Z}(\beta,n)}e^{-\beta n F_{\text{eff}}(J,\beta)}, \tag{75}$$

where

$$\widetilde{Z}(\beta,n) = \mathbb{E}_J e^{-\beta n F_{\text{eff}}(J,\beta)}.$$

The explicit expression defining $S(n,\beta)$ becomes

$$S(n,\beta) = -\frac{1}{\beta}\mathbb{E}_J p(J,\beta,n)\log p(J,\beta,n), \tag{76}$$

such that, pasting the whole lot together, we get the explicit expression for the functional $F(\beta, n)$, namely the n-quenched free energy:

$$F(\beta,n) = -\frac{1}{\beta n}\log\widetilde{Z}(\beta,n) = -\frac{1}{\beta n}\log\mathbb{E}_J(Z(\beta,J)^n). \tag{77}$$

It is straightforward to check that, for instance, when considering the Curie–Weiss model, the n-dependence disappears, while with opportune limits, it assumes the classical meaning when dealing with the Sherrington–Kirkpatrick model (e.g. Equations 67 and 68).

Acknowledgments

The strategy outlined in this paper is part of the study supported by the Italian Ministry for Education and Research, FIRB grant number *RBFR*08*EKEV*, and partially by Sapienza Università di Roma. FG is partially funded by INFN (Istituto Nazionale di Fisica Nucleare) which is also acknowledged. The authors are pleased to thank Dmitry Panchenko and an anonymous referee for their kind suggestions. We dedicate this paper to David Sherrington on the occasion of his seventieth birthday.

Notes

1. We learned this beautiful metaphor from Ton Coolen, whom we thank.
2. Here and in the following, we set the Boltzmann constant k_B equal to one, so that $\beta = 1/(k_B T) = 1/T$.
3. This procedure is deeply related to the mean field nature of the interactions, which ultimately allows one to consider even the low-temperature regimes as expressed in terms of high-temperature solutions [36].
4. High temperature is the β-region where there is only one solution, i.e. $q = 0$, of the self-consistency relation. When this condition breaks down, a phase transition to a broken replica phase appears; we label β_c that particular value of the temperature.
5. We allow ourselves a little abuse of notation in forgetting the β dependence for now.
6. Of course for simple systems, such as for instance the Curie–Weiss model where $P(J) \sim \delta(J-1)$, this term does not contribute to the thermodynamics and there is no n-dependence.

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