$U(t)$ stazionario

$R_1 = 1\,m \qquad R_2 = 4\,m$

$T_1 = 0°C \qquad T_2 = 80°C$

$$\nabla^2 U = \boxed{\frac{1}{r^2}\partial_r\left(r^2\frac{\partial U}{\partial r}\right)} + \cancel{\frac{1}{r^2 \sin\varphi}\partial_\varphi\left(\sin\varphi\,\partial_\varphi U\right)} + \cancel{\frac{1}{r^2\sin^2\varphi}\frac{\partial^2 U}{\partial \theta^2}}$$

Sol.

$$\frac{\partial U}{\partial t} = K\nabla^2 U \Rightarrow \boxed{K\nabla^2 U = 0} \text{ se } \frac{\partial U}{\partial t} = 0$$

$$\boxed{\frac{d}{dr}\left[r^2\frac{dU}{dr}\right] = 0 \quad \text{con} \quad \begin{matrix} U(R_1 = 1) = 0°C \\ U(R_2 = 4) = 80°C \end{matrix}}$$

$$r^2\frac{dU}{dr} = C_1 \Rightarrow dU = C_1\frac{dr}{r^2}$$

$$\Rightarrow \boxed{U(r) = -\frac{C_1}{r} + C_2}$$

$\Leftarrow$ Troviamo C_1 e C_2

$$U(R_1 = 1) = \emptyset = -\frac{C_1}{1} + C_2 \Rightarrow \boxed{C_1 = C_2}$$

$U(R_2 = 4) =$

$$U(R=4) = 80 = -\frac{C_1}{4} + C_1 = \frac{3C_1}{4} = 80$$

$$320 = 3C_1 \to \boxed{C_1 = \frac{320}{3}} \quad \boxed{C_2 = \frac{320}{3}}$$

$$\boxed{U(r) = \frac{320}{3}\left[1 - \frac{1}{r}\right]}$$

Tecnica di Separazione delle Variabili

Deve essere tutto lineare!

$$\boxed{\frac{\partial U}{\partial t} = K\frac{\partial^2 U}{\partial x^2}}$$

$x \in (0, L)$

$t \in \mathbb{R}^+$

$$BC \begin{cases} U(0,t) = \phi \\ U(L,t) = \phi \end{cases}$$

$IC: U(x, t=0) = f(x)$

Lo cauchy

LINEARITÀ

Def] $\mathcal{L}$ operatore $\mathcal{L}$ è lineare $\Leftrightarrow$

$$\boxed{\mathcal{L}\left[c_1 u_1(x,t) + c_2 u_2(x,t)\right] = c_1 \mathcal{L} u_1(x,t) + c_2 \mathcal{L} u_2(x,t)}$$

$$\frac{d}{dx}\left[3f(x) + 4g(x)\right] = 3\frac{df}{dx} + 4\frac{dg}{dx}$$

$$\boxed{\mathcal{F} = \frac{\partial}{\partial t} - k\frac{\partial^2}{\partial x^2} \text{ è } \mathcal{L}}$$

$$\mathcal{F}\left[c_1 u_1 + c_2 u_2\right] = c_1 \mathcal{F} u_1 + c_2 \mathcal{F} u_2$$

$$\frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial y^2} = 0 \qquad \text{si}$$

ES considerato l'op.

$$\mathcal{L}(u) = \frac{\partial}{\partial x}\left[k_0(x)\frac{\partial}{\partial x} u\right]. \quad \mathcal{L} \text{ è lineare?}$$

$$L[c_1 u_1 + c_2 u_2] = c_1 L u_1 + c_2 L u_2$$

$$L u(x) = \frac{\partial k_0}{\partial x}\frac{\partial u}{\partial x} + k_0 \frac{\partial^2 u}{\partial x^2}$$

proof. de L é linear:

$$L[\alpha u_1(x) + \beta u_2(x)] =$$

$$= \frac{\partial k_0}{\partial x}\frac{\partial}{\partial x}[\alpha u_1 + \beta u_2] + k_0(x)\left[\alpha\frac{\partial^2 u_1}{\partial x^2} + \beta \frac{\partial^2 u_2}{\partial x^2}\right]$$

$$L[\alpha u_1 + \beta u_2] =$$

$$= \frac{\partial k_0}{\partial x}\frac{\partial}{\partial x}[\alpha u_1 + \beta u_2] + k_0(x)\left(\alpha \frac{\partial^2 u_1}{\partial x^2} + \beta \frac{\partial^2 u_2}{\partial x^2}\right)$$

$$= \alpha\left[\frac{\partial k_0}{\partial x}\frac{\partial u_1}{\partial x} + k_0(x)\frac{\partial^2 u_1}{\partial x^2}\right] + \beta\left[\frac{\partial k_0}{\partial x}\frac{\partial u_2}{\partial x} + k_0(x)\frac{\partial^2 u_2}{\partial x^2}\right]$$

$$= \boxed{\alpha L u_1 + \beta L u_2} \qquad \square$$

PRINCIPIO DI SOVRAPPOSIZIONE

Se $U_1 \to \mathcal{L}U_1 = 0 \wedge U_2 \to \mathcal{L}U_2 = 0$

$$\Rightarrow U_3 = \alpha U_1 + \beta U_2 \to \boxed{\mathcal{L}U_3 = 0}$$

$$\boxed{mg - \beta v = m\frac{dv}{dt}} \to$$ Tecnica Sep. Variabili x ODE

$$\frac{\partial U}{\partial t} = k\frac{\partial^2 U}{\partial x^2} \qquad x \in (0,L) \quad t \in \mathbb{R}^+$$ ① ②

$U(0,t) = U(L,t=0) = 0$ ②

$IC = f(x)$ ③ Dirichlet...

$$\boxed{U(x,t) = \phi(x)G(t)}$$ & ansatz Bernoulli

$$\frac{\partial U}{\partial t} = \boxed{\phi\frac{\partial G}{\partial t}} \wedge \frac{\partial^2 U}{\partial x^2} = \boxed{G\frac{\partial^2 \phi}{\partial x^2}}$$

Ma quindi $\frac{\partial U}{\partial t} = k\boxed{\frac{\partial^2 U}{\partial x^2}} \to$

$$\varphi \frac{\partial G}{\partial t} = K G \frac{\partial^2 \varphi}{\partial x^2}$$

$$\frac{1}{KG}\frac{\partial G}{\partial t} = \frac{1}{\varphi}\frac{\partial^2 \varphi}{\partial x^2} \rightarrow \boxed{\underbrace{\frac{1}{KG}\frac{dG}{dt}}_{\text{solo fun } t} = \underbrace{\frac{1}{\varphi}\frac{d^2\varphi}{dx^2}}_{\text{solo fun } x} = -\lambda}$$

Costante di separazione

1) $\boxed{\frac{1}{KG}\frac{dG}{dt} = -\lambda}$ 2) $\boxed{\frac{1}{\varphi}\frac{d^2\varphi}{dx^2} = -\lambda}$

$U = G \cdot \varphi \rightarrow$ che ripercussioni ha sulle BCs!

1) $U(0,t) \rightarrow \varphi(0)G(t) = \emptyset \rightarrow \varphi(0) = \emptyset$

2) $U(L,t) = \emptyset \rightarrow \varphi(L)G(t) = \emptyset \rightarrow \varphi(L) = \emptyset$

Risoluto

INDIPENDENTE DAL TEMPO

$$\underbrace{\frac{dG}{dt} = -\lambda K G}_{} \rightarrow G(t) = e^{rt}$$

$$\begin{cases} \dfrac{dG}{dt} = r e^{rt} \end{cases}$$

$$\boxed{r = -\lambda k}$$

$$\boxed{G(t) = c e^{-\lambda k t}}$$

$\lambda > 0$?

$\lambda < 0$?

su considerazioni fisiche si capisce che $\lambda > 0$

Risolviamo nello spazio

$$\frac{d^2\varphi}{dx^2} = -\lambda \varphi \quad \text{con} \quad \begin{cases} \varphi(0) = 0 \\ \varphi(L) = 0 \end{cases}$$

$$\varphi(x) = e^{rx} \rightarrow \frac{d\varphi}{dx} = r e^{rx}, \quad \frac{d^2\varphi}{dx^2} = r^2 e^{rx}$$

$$\boxed{r^2 = -\lambda}$$

1) $\lambda > 0 \to r = \pm i\sqrt{\lambda}$ ∃ 2 radici complesse coniugate

2) $\lambda = 0 \to r = (0,0)$ 2 radici (coincidenti)

3) $\lambda < 0 \to r = \pm\sqrt{-\lambda}$ 2 radici reali $<$ pos / neg

$$\frac{d^2\varphi}{dx^2} = -\lambda\varphi$$

$\boxed{\text{Analizziamo il caso } \lambda > 0}$

$\lambda > 0 \to r = \pm i\sqrt{\lambda}$

$$\varphi(x) = C_1\cos(\sqrt{\lambda}x) + C_2\sin(\sqrt{\lambda}x)$$

$$\varphi(x) = C_3 e^{i\sqrt{\lambda}x} + C_4 e^{-i\sqrt{\lambda}x}$$

$\varphi(0) = 0 \wedge \varphi(L) = 0$

$\to \boxed{C_1 = 0} \to \varphi(x) = C_2\sin(\sqrt{\lambda}x)$

$\varphi(L) = 0 \to C_2\sin(\sqrt{\lambda}L) = 0$

esclusa la sol. banale $(C_1 = 0, C_2 = 0)$
la è lo steady state

$$\boxed{\sqrt{\lambda}\, L = n\pi} \rightarrow \boxed{\lambda_n = \left(\frac{n\pi}{L}\right)^2}$$

$$\boxed{\phi(x) = C_2 \sin\left(\frac{n\pi}{L} x\right)}$$

$$U(x,t) = G(t)\,\phi(x)$$

Analizziamo $\lambda = \phi \rightarrow r = (\phi, \phi)$

$$\frac{d^2\phi}{dx^2} = \lambda\phi = \phi \rightarrow \phi(x) = C_1 + C_2 x$$

$$\phi(x=0) = 0 \rightarrow C_1 = 0$$

$$\phi(x=L) = 0 \rightarrow C_2 \cdot L = 0 \rightarrow C_2 = \phi$$

$$\boxed{C_1 = C_2 = \phi}$$

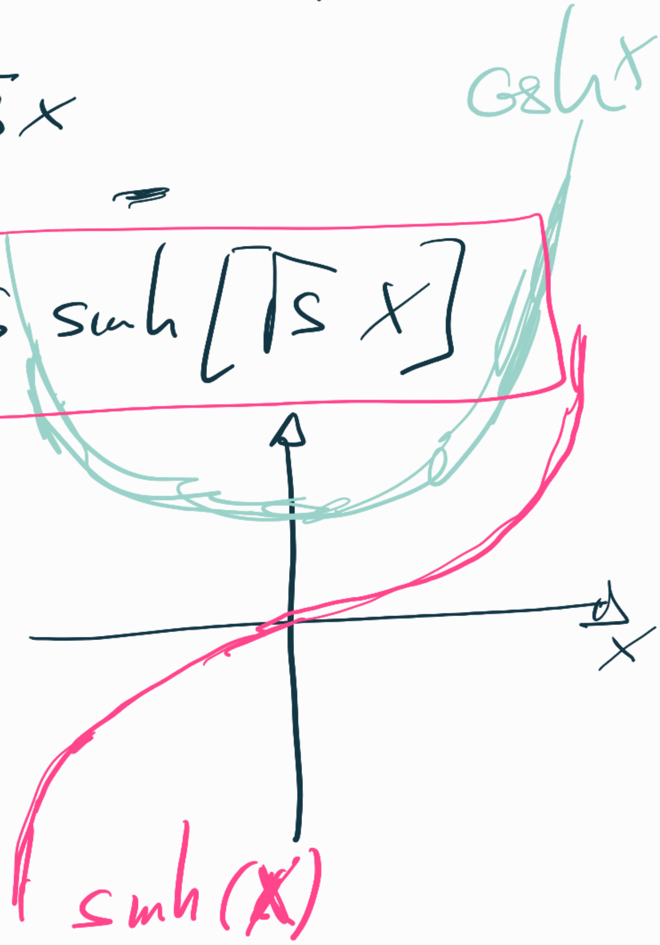
Analizziamo il caso $\lambda<0$

se $\lambda<0 \rightarrow \dfrac{d^2\varphi}{dx^2} = -\lambda\varphi = s\varphi$, $s=-\lambda$

$e^{x}+e^{-x} = 2\cosh(x)$

$$\varphi(x) = C_1 e^{+\sqrt{s}x} + C_2 e^{-\sqrt{s}x} =$$

$$\boxed{\varphi(x) = C_3 \cosh[\sqrt{s}\,x] + C_2 \sinh[\sqrt{s}\,x]}$$



$\rightarrow \varphi(0)=0 \rightarrow \boxed{C_3=0}$

$\rightarrow \varphi(L)=0 \rightarrow C_4 \sin(\sqrt{s}L)=0$

$\rightarrow \boxed{C_4=0}$

Riassunto

se $\lambda=0$ ~~o~~ $\lambda<0 \Longrightarrow \nexists$ soluzioni [illegible] banale

$\boxed{\text{l'unico caso di interesse è } \lambda>0}$

$$\begin{cases} U_t = kU_{xx} \\ U(0,t) = U(L,t) = \phi \end{cases} \qquad U(x,t) = \phi(x)\,G(t)$$

$$G(t) = C e^{-\lambda_n t} \wedge \phi(x) = C_2 \sin\left(\frac{n\pi}{L}x\right)$$

$$\boxed{U(x,t) = B_n \sin\left(\frac{n\pi}{L}x\right) e^{-k\left(\frac{n\pi}{L}\right)^2 t}}$$

con $B_n = C C_2$

che dicusto su: B_n ? $\to$ CI

OS $\boxed{\frac{\partial U}{\partial t}} = k\frac{\partial^2 U}{\partial x^2}$ con BC $\begin{cases} U(0,t) = \phi \\ U(L,t) = \phi \end{cases}$

e con IC. $U(x,0) = 4\sin\left(\frac{3\pi}{L}x\right)$

Sol. Gen. $U(x,t) = B\sin\left(\frac{n\pi}{L}x\right) e^{-k\left(\frac{n\pi}{L}\right)^2 t}$

$\boxed{t \to t}$

$$\boxed{B = 4 \wedge n = 3}$$

$$U(x,t) = 4\sin\left(\frac{3\pi}{L}x\right) e^{-k\left(\frac{3\pi}{L}\right)^2 t}$$ ok!

Esercizio 3.3

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} \qquad U(0,t) = U(L,t) = \phi$$

$$IC: \; U(x,0) = 4\sin\left(\frac{3\pi}{L}x\right) + 7\sin\left(\frac{8\pi}{L}x\right)$$

$$\boxed{U(x,t) = 4\sin\left(\frac{3\pi}{L}x\right)e^{-k\left(\frac{3\pi}{L}\right)^2 t} + 7\sin\left(\frac{8\pi}{L}x\right)e^{-k\left(\frac{8\pi}{L}\right)^2 t}}$$

$$\frac{\partial u}{\partial t} = -k\left(\frac{3\pi}{L}\right)^2\left[4\sin(3)e^{(3)}\right] - k\left(\frac{8\pi}{L}\right)^2\left[7\sin(8)e^{(8)}\right]$$

$$\frac{\partial u}{\partial x} = \left(\frac{3\pi}{L}\right)\left[4\cos(3)e^{(3)}\right] + \left(\frac{8\pi}{L}\right)\left[7\cos(8)e^{(8)}\right]$$

$$k\frac{\partial^2 u}{\partial x^2} = -k\left(\frac{3\pi}{L}\right)^2\left[4\sin(3)e^{(3)}\right] - k\left(\frac{8\pi}{L}\right)^2\left[7\sin(8)e^{(8)}\right]$$

$$-k\left(\frac{3\pi}{L}\right)^2\left[4\sin(3)e^{(3)} - 4\sin(3)e^{(3)}\right]$$

$$-k\left(\frac{8\pi}{L}\right)^2\left[7\sin(8)e^{(8)} - 7\sin(8)e^{(8)}\right] = 0$$

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}$$

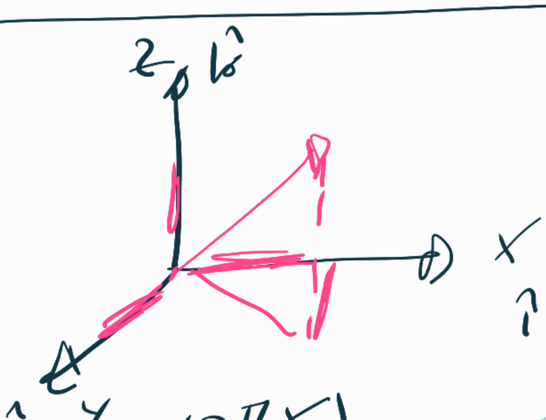
$$\frac{\partial u}{\partial t} - k\frac{\partial^2 u}{\partial x^2} = 0$$

Sol. GEN.

$$\boxed{U(x,t)=\sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi}{L}x\right) e^{-k\left(\frac{n\pi}{L}\right)^2 t}}$$

$$U(x,t=0)=f(x)=\sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi}{L}x\right)$$

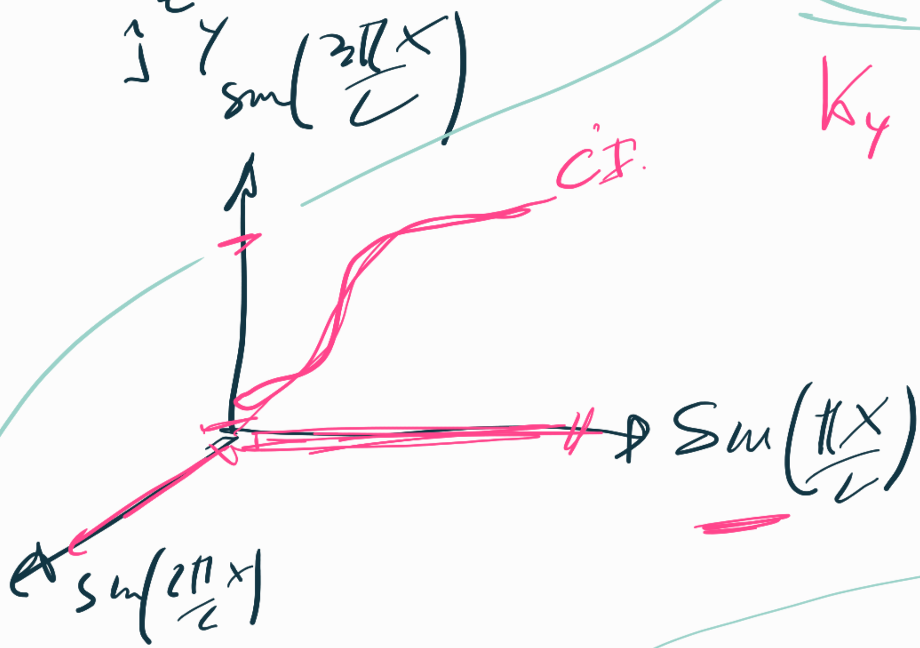
LE



$\vec{k}=(k_x, k_y, k_z)$

$k_x=\frac{\vec{k}\cdot\hat{\imath}}{\hat{\imath}\hat{\imath}}=\vec{k}\cdot\hat{\imath}$

$k_y=\frac{\vec{k}\cdot\hat{y}}{\hat{y}\hat{y}}$



$CI=f(x)$

Discreto

$\sum_{i=1}^{N} k_i \; \hat{\imath}=\begin{pmatrix}1\\0\\0\end{pmatrix}$ $\quad k_x \to \int dx\, f(x)\, \sin\left(\frac{n\pi}{L}x\right)$

$f(x)=\sum_i f_i \hat{\imath}$

Serie di Taylor

Serie di Fourier