1) $\frac{\partial u}{\partial t} = k \frac{\partial^2 u}{\partial x^2}$ $\quad u(x=0,t) = u(x=L,t) = \phi$

$x \in (0,L)$, $t \geq 0$

hp se sin PDE $\wedge$ BC $\to$ L $\wedge$ O

$u(x,t) = \varphi(x)\,G(t)$ ANSATZ di BERNOULLI

$\frac{\partial u}{\partial t} = k \frac{\partial^2 u}{\partial x^2} \Rightarrow \varphi \frac{dG}{dt} = kG \cdot \frac{d^2\varphi}{dx^2}$ $\quad \frac{1}{k\varphi G}$

$$\frac{1}{k\cancel{\varphi} G}\cancel{\varphi}\frac{dG}{dt} = \frac{\cancel{kG}}{\cancel{k}\varphi\cancel{G}}\frac{d^2\varphi}{dx^2}$$

$$\frac{1}{k}\frac{1}{G}\frac{dG}{dt} = \frac{1}{\varphi}\frac{d^2\varphi}{dx^2} = -\lambda$$

BC $u(x=0,t) = \phi \Rightarrow \varphi(x=0)\cdot G(t) = 0$

$\varphi(x=0) = \phi$

$$\boxed{\frac{1}{kG}\frac{dG}{dt} = -\lambda}$$

$$\boxed{\frac{1}{\varphi}\frac{d^2\varphi}{dx^2} = -\lambda, \quad \varphi(0) = \phi, \quad \varphi(L) = \phi}$$

$$\frac{d\sigma}{\sigma} = -\lambda k\, dt \rightarrow \boxed{\sigma(t) = \sigma_0\, e^{-\lambda k t}}$$

$\boxed{\lambda > 0}$ —— Su basi fisiche: xchè?
altri: Temp. cresce esponenzial.

$$\frac{d^2\varphi}{dx^2} = -\lambda\varphi \qquad \varphi(0) = \varphi(L) = 0$$

$$\varphi(x) = C_1 \cos(\sqrt{\lambda}\, x) + C_2 \sin(\sqrt{\lambda}\, x)$$

$$\varphi(0) = 0 \Rightarrow \boxed{C_1 = 0}$$

$$\varphi(L) = 0 \Rightarrow \varphi(x) = C_2 \sin(\sqrt{\lambda}\, x)$$

$$\varphi(x = L) = C_2 \sin(\sqrt{\lambda}\, L) = 0 \qquad \sqrt{\lambda}\, L = n\pi$$

$$\lambda_n = \left(\frac{n\pi}{L}\right)^2$$

$$\varphi(x) = C_2 \sin\left(\frac{n\pi}{L} x\right)$$

$$\left(\frac{n\pi}{L}\right)^2$$

$$\sigma(t) = \sigma_0\, e^{-k\lambda t} \qquad U = \varphi \cdot \sigma$$

$$U(x,t) = B \sin\left(\frac{n\pi}{L}x\right) e^{-k\left(\frac{n\pi}{L}\right)^2 t}$$

~~INIZIO~~

Capisco → $$\boxed{U(x,t) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi}{L}x\right) e^{-k\left(\frac{n\pi}{L}\right)^2 t}} \quad (1)$$

$$U(t=0, x) = f(x) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi}{L}x\right) \quad (2)$$

chiar Fourier: espando la CF in serie di F.

CUI PRODEST?

LEMMA

$$\int_0^L \sin\left(\frac{m\pi}{L}x\right) \sin\left(\frac{n\pi}{L}x\right) dx = \begin{cases} 0 & m \neq n \\ \frac{L}{2} & m = n \end{cases} = \frac{L}{2}\delta_{mn}$$

Proof simplest scenario $m = n$ uso: $\sin^2(y) = \frac{1}{2} - \frac{1}{2}\cos(2y)$

$$\int_0^L \sin^2\left(\frac{n\pi}{L}x\right) dx = \frac{L}{n\pi}\int_0^{n\pi} \sin^2 y \, dy =$$

$$\frac{n\pi}{L}x = y \rightarrow dx = dy\frac{L}{n\pi}$$

$$= \frac{L}{n\pi}\left[\int_0^{n\pi} \frac{1}{2} dy - \int_0^{n\pi} \frac{1}{2}\cos(2y) dy\right] =$$

$$= \frac{L}{2\cancel{n\pi}}\left(\cancel{n\pi} - 0\right) - \overset{b}{\cancel{0}} = \frac{L}{2}$$

Come uso questo lemma x Δ CI ?

$$f(x) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi}{L}x\right)$$

$$\int_0^L f(x) \sin\left(\frac{m\pi}{L}x\right)dx = \sum_{n=1}^{\infty} B_n \underbrace{\int_0^L \sin\left(\frac{n\pi}{L}x\right) \sin\left(\frac{m\pi}{L}x\right)dx}_{\frac{L}{2}\delta_{m,n}}$$

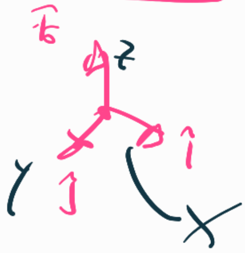
solo il termine m sopravvive

$$\Rightarrow \int_0^L f(x) \sin\left(\frac{m\pi}{L}x\right)dx = B_m \cdot \frac{L}{2}$$

$$\Rightarrow \boxed{B_m = \frac{2}{L}\int_0^L f(x) \sin\left(\frac{m\pi}{L}x\right)dx} \quad (3)$$

$\hat{i}\cdot\hat{j}=0$, $\hat{i}\cdot\hat{i}=1$

1 2 3 4 5 ... N — L



$$\bar{X}\cdot\bar{Y}=0 \rightarrow \sum_{i}^{N} X_i Y_i = 0 \Rightarrow \vec{X} \text{ è ortogonale a } \bar{Y}$$

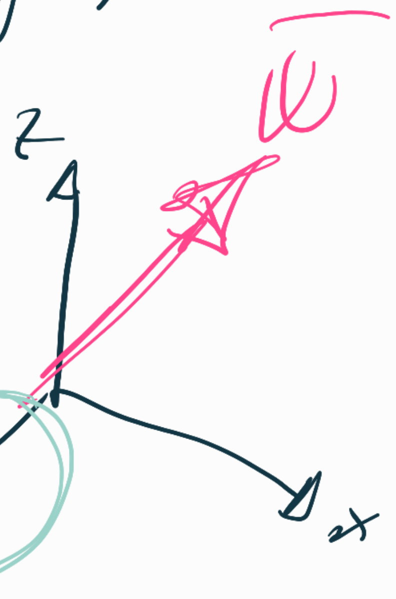
$$\hookrightarrow \int_0^L f(x)\cdot g(x) = 0$$

se e solo se $f(x) \perp g(x)$

$$f(x)\Big|_{x=x_i}$$

$$f(x_i)\, g(x_i)$$

$$\int_0^L A_n(x)\, A_m(x)\, dx = \delta_{n,m}$$



$$W_x = \frac{\bar{W}\cdot \hat{i}}{\hat{i}\cdot \hat{i}}$$

$\hat{i}, \hat{j}, \hat{k}$

$$\sin\left(\frac{(n=1)\pi}{L}x\right), \sin\left(\frac{(n=2)\pi}{L}x\right), \sin\left(\frac{(n=3)\pi}{L}x\right)$$

ESERCIZIO a $t=0 \to 100°C \to 0°C$

$$\frac{\partial U}{\partial t} = k\frac{\partial^2 U}{\partial x^2} \qquad t>0 \qquad x \in (0,L)$$

IC : $U(x, t=0) = 100°$

$$BC \begin{cases} U(x=0,t) = 0 \\ U(x=L,t) = 0 \end{cases}$$

$$U(x,t) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi}{L}x\right) e^{-k\left(\frac{n\pi}{L}\right)^2 t}$$

$$B_n = \frac{2}{L}\int_0^L f(x) \sin\left(\frac{n\pi}{L}x\right)dx \qquad f(x) = 100$$

$$\to B_n = \frac{200}{L}\int_0^L \sin\left(\frac{n\pi}{L}x\right)dx = \frac{200}{L}\left[\frac{-L}{n\pi}\cos\left(\frac{n\pi}{L}x\right)\right]_\phi^L$$

$$\to B_n = \frac{200}{n\pi}(1-\cos(n\pi)) \begin{cases} 0 & n \text{ pari} \\ \frac{400}{n\pi} & n \text{ dispari} \end{cases}$$

$$\boxed{U(x,t) = \sum_{n=1}^{\infty}\left[\frac{200}{n\pi}(1-\cos(n\pi))\right] e^{-k\left(\frac{n\pi}{L}\right)^2 t}}$$

Soluzione

$$\lim_{t\to\infty} U(x,t) = ?$$

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} \to \text{BC } U(0) = U(L) = \phi$$

$$\to \text{BC } \frac{dU}{dx}(x=0) = \frac{dU}{dx}(x=L) = \phi$$

SCALA LOGICA: "TO DO" STEPS

1) check che sia PDE che BCs siano LIO

2) inizialmente non vi curate di CI $U(x, t=0) = f(x)$

3) applicate S.V.: cercate sol. prodotto

4) Determinate gli autovalori ammiss. λ_n del problema in X

5) Risolvete l'altra ODE (quella in t) con i λ_n dati da X

6) Applicate principio di sovrapposizione x avere sol. x serie

7) espandete la CI f(x) nella stessa serie x determ. i coeff. B_n

---

ex 2.3.3c SSSSSP $x \in (0, L)$ $t \in \mathbb{R}^+$

$$\frac{\partial u}{\partial t} = k \frac{\partial^2 u}{\partial x^2} \qquad U(0,t) = U(L,t) = \phi$$

$$IC: U(x,0) = f(x) = 2\cos\left(\frac{3\pi}{L}x\right)$$

$$\boxed{U(x,t) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi}{L}x\right) e^{-k\left(\frac{n\pi}{L}\right)^2 t}}$$ ← sol. gen.

$$B_n = \frac{2}{L}\int_0^L f(x) \sin\left(\frac{n\pi}{L}x\right) dx$$

$$= \frac{2}{L}\int_0^L 2\cos\left(\frac{3\pi}{L}x\right) \sin\left(\frac{n\pi}{L}x\right) dx$$

EX 2.3.8

$x \in [0,L] \quad t \in \mathbb{R}^+$

$$\frac{\partial u}{\partial t} = K\frac{\partial^2 u}{\partial x^2} - \alpha u \qquad BCs \begin{cases} u(0,t) = \phi \\ u(L,t) = \phi \end{cases}$$

1) $\exists$ sol. staz. non banali se $\alpha > 0$?

2) risolv. la PDE con $U(x,0) = f(x)$, tenendo $\alpha > 0$.

3) studiare il limite a tempi lunghi.

PUNTO UNO

$$\frac{d^2 U}{dx^2} = \frac{\alpha U}{K} \Rightarrow U(x) = a\cosh\left(\sqrt{\frac{\alpha}{k}}x\right) + b\sinh\left(\sqrt{\frac{\alpha}{k}}x\right)$$

$U(0) = 0 \rightarrow \boxed{a = 0}$ $\quad U(L) = \phi \rightarrow$ $\boxed{b = 0}$

$\rightarrow U(x) = b\sinh\left(\sqrt{\frac{\alpha}{k}}x\right) \Rightarrow b \cdot \sinh\left(\sqrt{\frac{\alpha}{k}}L\right) = 0$

PUNTO DUE

dividendo per $\frac{1}{k\phi h}$

$$U(x,t) = \phi(x)\,h(t)$$

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} - \alpha u \rightarrow \boxed{\phi \cdot \frac{dh}{dt} = k h \frac{d^2\phi}{dx^2} - \alpha\phi h}$$

$$\frac{\phi}{k\phi h}\frac{dh}{dt} = \frac{\cancel{k}h}{\cancel{k}\phi h}\frac{d^2\phi}{dx^2} - \frac{\alpha\phi h}{k\phi h}$$

$$\frac{1}{k}\frac{1}{h}\frac{dh}{dt} = \frac{1}{\phi}\frac{d^2\phi}{dx^2} - \frac{\alpha}{k}$$

$$\underbrace{\frac{1}{k}\frac{1}{h}\frac{dh}{dt}+\frac{\alpha}{k}}_{\text{TEMPO}} = \underbrace{\frac{1}{\varphi}\frac{d^2\varphi}{dx^2}}_{\text{SPAZIO}} = -\lambda$$

Inizio dal problema nel tempo

$$\frac{1}{h}\frac{dh}{dt} + \cancel{\alpha} = -(\lambda k + \alpha)$$

$$\frac{dh}{h} = -(\lambda k + \alpha)\,dt \to \boxed{h(t) = h_0 e^{-\alpha t} e^{-\lambda k t}}$$

Sol. del tempo

ora approccio il problema nello spazio

$$\frac{d^2\varphi}{dx^2} = -\lambda\varphi \quad \text{con} \quad \begin{cases} \varphi(0) = \phi \\ \varphi(L) = \phi \end{cases}$$

$$\varphi(x) = C_1\cos(\sqrt{\lambda}x) + C_2\sin(\sqrt{\lambda}x)$$

$$\varphi(0) = \phi \Rightarrow \boxed{C_1 = 0} \to \varphi(x) = C_2\sin(\sqrt{\lambda}x)$$

$$\varphi(L) = \phi \Rightarrow C_2\sin(\sqrt{\lambda}L) = 0 \to \sqrt{\lambda}L = n\pi$$

→ quindi gli autovalori sono $\boxed{\lambda_n = \left(\frac{n\pi}{L}\right)^2}$

$$\phi(x) = C_2 \sin\left(\frac{n\pi}{L}x\right)$$

sol. prodotto

$$\boxed{U(x,t) = \phi(x)\, h(t) = A_0 \sin\left(\frac{n\pi}{L}x\right) e^{-\alpha t} e^{-\lambda_n t}}$$

3 PUNTO comportamento asintotico

$$\lim_{t\to+\infty} U(x,t) = \phi$$ che è vi ricordo $\Leftarrow$ la sol! stazionaria.

EX 235 ORTOGONALITÀ DEI SENI

prof: $$\int_0^L \sin\left(\frac{m\pi}{L}x\right)\sin\left(\frac{n\pi}{L}x\right)dx = \frac{L}{2}\delta_{m,n}$$

Trick trigonometrico $\sin(a)\sin(b) = \frac{1}{2}\left[\cos(a-b) - \cos(a+b)\right]$

1 caso $n=m$

$$\int_0^L \frac{1}{2}\left[\underbrace{\cos\left(\frac{\cancel{(n-m)}\pi}{L}x\right)}_{1} - \cos\left(\frac{\overbrace{(m+n)}^{2n}\pi}{L}x\right)\right]dx$$

$$\frac{1}{2}\cdot x\Big|_0^L - \frac{1}{2}\cancel{\sin(2n\pi)} = \frac{1}{2}L - \phi$$

2 caso $n \neq m$

Pongo $\frac{(n-m)\pi}{L}x = y$

$$\int_0^L \sin\left(\frac{n\pi}{L}x\right)\sin\left(\frac{m\pi}{L}x\right)dx =$$

$$= \frac{1}{2}\left[\int_0^L \cos\left(\frac{(n-m)\pi}{L}x\right)dx - \int_0^L \cos\left(\frac{(n+m)\pi}{L}x\right)dx\right]$$

$$= \frac{1}{2}\left\{\frac{L}{(n-m)\pi}\sin\left[\frac{(n-m)\pi}{L}x\right]_0^L - \frac{L}{(n+m)\pi}\sin\left[\frac{(n+m)\pi}{L}x\right]_0^L\right\}$$

$$= \phi - \phi + \phi - \phi = \phi$$

$$\boxed{\int_0^L \sin\left(\frac{n\pi}{L}x\right)\sin\left(\frac{m\pi}{L}x\right)dx = \frac{L}{2}\delta_{m,n}}$$

ORTOGONALITÀ ~~DEI~~ COSENI

$$\int_0^L \cos\left(\frac{n\pi}{L}x\right)\cos\left(\frac{m\pi}{L}x\right)dx \begin{cases} 0 & n \neq m \\ \frac{L}{2} & n = m \neq 0 \\ L & n = m = \phi \end{cases}$$

Trucco della TGMT:

$$\cos(a)\cos(b) = \frac{1}{2}\left[\cos(a+b) + \cos(a-b)\right]$$

1) Caso $n = m \neq 0$

$$\int_0^L \cos\left(\frac{n\pi}{L}x\right)\cos\left(\frac{m\pi}{L}x\right)dx = \frac{1}{2}\left[\int_0^L \cos\left(\frac{2m\pi}{L}x\right)dx + \int_0^L 1\,dx\right]$$

$$= \frac{1}{2}\left\{\underbrace{\left[\frac{L}{2\pi}\sin\left(\frac{2m\pi}{L}x\right)\right]_0^L}_{=0} + \underbrace{x\Big|_0^L}_{L-0}\right\} = \frac{1}{2}L$$

2) Caso $m = n = 0$ $\rightarrow$ $\int_0^L 1\,dx = L$

3) Caso $n \neq m$

$$\int_0^L \cos\left(\frac{n\pi}{L}x\right)\cos\left(\frac{m\pi}{L}x\right)dx =$$

$$= \frac{1}{2}\left[\frac{L}{(n+m)\pi}\sin\left(\frac{(n+m)\pi}{L}x\right)\Bigg|_0^L + \frac{L}{(n-m)\pi}\sin\left(\frac{(n-m)\pi}{L}x\right)\Bigg|_0^L\right]$$

0 0 0 0

TIRO LE SOMME → Sp. HILBERT

$\sin\left(\frac{3\pi}{L}x\right)$

$\sin\left(\frac{2\pi}{L}x\right)$ $\sin\left(\frac{\pi}{L}x\right)$

$$f(x) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi}{L}x\right)$$

$$U(x,t) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi}{L}x\right) \exp\left[-k\left(\frac{n\pi}{L}\right)^2 t\right]$$

$$\int f(x) \cdot \sin\left(\frac{m\pi}{L}x\right) dx =$$

$$\sum_{n=1}^{\infty} B_n \underbrace{\int \sin\left(\frac{n\pi}{L}x\right) \sin\left(\frac{m\pi}{L}x\right) dx}_{\frac{L}{2}\delta_{m,n}}$$

$$B_m = \frac{\int_0^L f(x) \sin\left(\frac{m\pi}{L}x\right)}{L/2}$$

STUDIO UNA COSA DIVERSA

DIAMO GUFOSI

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}, \quad u(x, t=\phi) = f(x),$$

BCs: $\underline{\frac{\partial u}{\partial x}(x=0)} = \underline{\frac{\partial u}{\partial x}(x=L)} = \phi$

Flusso di calore è $\phi \propto \frac{\partial u}{\partial x}$

$$u(x,t) = G(t)\,\varphi(x) \rightarrow \varphi\frac{dG}{dt} = kG\frac{d^2\varphi}{dx^2}$$

$$\rightarrow \boxed{\frac{1}{kG}\frac{dG}{dt} = \frac{1}{\varphi}\frac{d^2\varphi}{dx^2} = -\lambda}$$

STUDIO il TEMPO

$$\frac{dG}{dt} = -\lambda k G \rightarrow \boxed{G(t) = G_0 e^{-\lambda k t}}$$

STUDIO lo spazio

$$\frac{d^2\varphi}{dx^2} = -\lambda\varphi \quad \begin{cases} \lambda > 0 \\ \lambda = 0 \\ \lambda < 0 \end{cases}$$

studiare caso x caso

INIZIO DAL CASO $\lambda = 0$

$$\frac{d^2\varphi}{dx^2} = 0 \Rightarrow \varphi(x) = C_1 x + C_2 \rightarrow \boxed{\varphi(x) = C_2}$$

$$\frac{d\varphi}{dx}(x=0) = 0 \rightarrow \boxed{C_1 = 0} \quad \& \quad \frac{d\varphi}{dx}(x=L) = 0$$

ORA STUDIAMO IL CASO $\lambda > 0$

$$\frac{d^2\varphi}{dx^2} = -\lambda\varphi \qquad \varphi(x) = A\sin(\sqrt{\lambda}\,x) + B\cos(\sqrt{\lambda}\,x)$$

$$\frac{d\varphi}{dx} = \sqrt{\lambda}\left[A\cos(\sqrt{\lambda}\,x) - B\sin(\sqrt{\lambda}\,x)\right]$$

$$\frac{d\varphi}{dx}(x=0) = 0 \Rightarrow \boxed{A = 0} \Rightarrow \varphi = B\cos(\sqrt{\lambda}\,x)$$

$$\frac{d\varphi}{dx}(x=L) = 0 \Rightarrow \underbrace{\sqrt{\lambda}\,B}_{\neq 0}\sin(\sqrt{\lambda}\,L) = 0$$

$$\sqrt{\lambda}\,L = n\pi \Longrightarrow \boxed{\lambda_n = \left(\frac{n\pi}{L}\right)^2}$$

$$\boxed{\varphi(x) = B_n \cos\left(\frac{n\pi}{L} x\right)}$$

Caso $\lambda < 0$

$$\frac{d^2\varphi}{dx^2} = -\lambda\varphi \rightarrow A\sinh(\sqrt{\lambda}x) + B\cosh(\sqrt{\lambda}x)$$

$$\frac{d\varphi}{dx} = \sqrt{\lambda}\left[A\cosh(\sqrt{\lambda}x) + B\sinh(\sqrt{\lambda}x)\right]$$

$$\varphi'(0) = 0 \Rightarrow \boxed{A = 0} \wedge \varphi'(L) = 0$$

$$\rightarrow \sqrt{\lambda}\, B \sinh(\sqrt{\lambda}L) = 0$$

$$\rightarrow \boxed{B = 0}$$

$$U(x,t) = \underbrace{A_0}_{\lambda = 0} + \underbrace{\sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi}{L}x\right) e^{-k\left(\frac{n\pi}{L}\right)^2 t}}_{\lambda > 0}$$