NOTES ON FERROMAGNETIC DILUTED p-SPIN MODEL

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In this paper we develop the interpolating cavity field technique for the mean field ferromagnetic p-spin. The model we introduce is a natural extension of the diluted Curie–Weiss model to $p > 2$ spin interactions. Several properties of the free energy are analyzed and, in particular, we show that it recovers the expressions already known for $p = 2$ models and for $p > 2$ fully connected models. Further, as the model lacks criticality, we present extensive numerical simulations to evidence the presence of a first-order phase transition and deepen the behaviour at the transition line. Overall, a good agreement is obtained among analytical results, numerics and previous works.

Keywords: Ising model, random graphs, statistical mechanics.

1. Introduction

Born as a theoretical background for thermodynamics, statistical mechanics provides nowadays a flexible approach to several scientific problems whose depth and wideness increases continuously. In fact, in the last decades statistical mechanics has invaded fields as diverse as spin glasses [22], neural networks [1], protein folding [20], immunological memory [26, 8], social networks [2], theoretical economy [13] and urban planning [10]. As a consequence, an always increasing need for models and proper techniques must be fulfilled. Coherently, recently, several models have been systematically tackled via the smooth cavity field by the authors, namely the Curie–Weiss model [6], the fully connected p-spin model [7], the Sherrington–Kirkpatrick model [5], its diluted counterpart Viana–Bray model [9] and the diluted ferromagnetic model [3]. All these models can just be seen as different components of a more general class including models based on binary agents with mean field interactions (Fig. 1). Now, in order to complete the analysis of the free energies for the whole class, the X-OR-SAT (of the Random Optimization Theory [23]) and

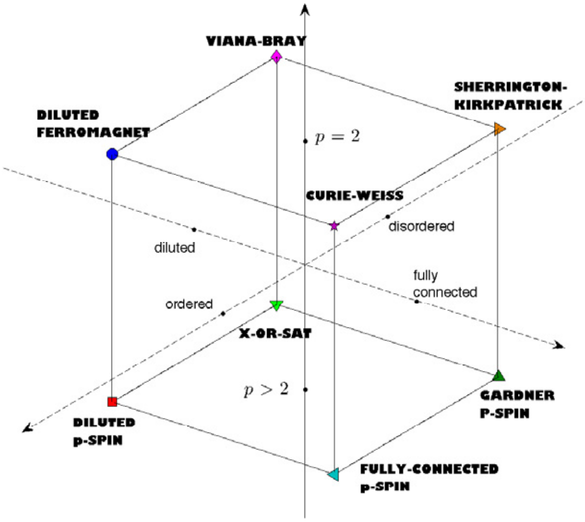

Fig. 1. Schematic representation of the connections among different models based on mean field interactions between variables endowed with discrete symmetry.

the diluted ferromagnetic p-spin model, are still missing; this paper is devoted to the study of the latter.

In a nutshell, the system is a ferromagnet in which the interactions happen in p-plets, instead of more classical couples (see also [16]), and the interacting agents live on a diluted random network, i.e. the Erdös–Rényi graph [31]. In general, the graph can be specified by fixing the number of nodes N and its "connectivity" α, which represents the average number of nearest neighbours per site.

As standard ferromagnets, the model is shown to exhibit two phases, a paramagnetic one and a (replica symmetric) ferromagnetic one, on the the other hand, as a difference with respect to the standard ferromagnet, the phase transition does not display criticality for $p > 2$. The model is investigated by means of cavity field technique and extensive numerical simulations.

While a rigorous proof of the existence of the thermodynamic limit for spin structures defined on random graphs (as the one we are considering) is still to be achieved [14, 15], research on their properties continue, and, in this sense, we assume such an existence and work out our framework where we find an expression for the free energy as a function of p, of the network connectivity α and of the (inverse) temperature β, showing that it is consistent with known results. In particular, by properly tuning p and α we recover the Curie–Weiss model [6], the diluted Ising model [3] and the fully-connected p-spin model [7]; moreover, regardless of the (finite) dilution, for $p = 2$ criticality is restored. Full agreement with Monte-Carlo simulations is obtained both on the absence of the critical behaviour and on the free energy structure.

The paper is organized as follows: In Section 2 the model is introduced and some of its properties worked out together with the introduction of a proper statistical mechanics machinery, while in Section 3 its equilibrium is solved via the smooth cavity field technique. Section 4 deals with the properties of the free energy and its consistency with well-known models, while in Section 5 our numerical analysis is presented. Section 6 is left for a summary and outlook. Finally, Appendix contains the detailed proofs of the theorems introduced.

2. The diluted even-p-spin ferromagnet

In this section we explore the properties of a diluted even-p-spin ferromagnet: we restrict ourselves only to even values of p for mathematical convenience as the investigation with the cavities is much simpler. However, due to monotonicity of all the observables in p, such restriction does not imply any loss of generality, as confirmed also by numerical simulations performed on both even and odd values of p.

Before proceeding, it is worth recalling some concepts concerning the diluted random network where the magnetic system is set. Such a network is an Erdös–Rényi (ER) graph [31] defined as follows: given a number N of nodes, we introduce connections between them in such a way that each pair of vertices i, j has a connecting link with independent probability equal to α/N, with $0 \leq \alpha \leq N$. As a result, the probability distribution for the number of links per node (or coordination number) is binomial with average α. Hence, the parameter α provides a measure of the "degree of connectivity" of the graph itself: the smaller α the more diluted the system; for $\alpha = 0$ and $\alpha = N$ the extreme cases of fully disconnected and fully connected graphs, respectively, are recovered. Notice that in the thermodynamic limit $N \to \infty$ the binomial distribution converges towards the Poisson distribution [11].

The ER graph can be algebraically described by the so-called adjacency matrix $\mathbf{A}$ which is an $N \times N$ symmetric matrix whose entry A_{ij} is 1 if $i \neq j$ and the two nodes are connected together, otherwise it is zero.

We now associate to each node i a binary variable $\sigma_i = \pm 1$, $i \in [1, N]$, and we introduce p families $\{i_\nu^1\}, \{i_\nu^2\}, \dots, \{i_\nu^p\}$ of i.i.d. random variables uniformly distributed on the previous interval. Then, the Hamiltonian is given by the following expression,

$$H_N(\sigma, \gamma) = -\sum_{\nu=1}^{k_{\gamma N}} \sigma_{i_\nu^1} \sigma_{i_\nu^2} \dots \sigma_{i_\nu^p}, \tag{1}$$

where $k_{\gamma N}$ represents the number of connected p-plets present in the graph. Reflecting the underlying network, $k_{\gamma N}$ is a Poisson distributed random variable with mean value γN. The relation among the coordination number α and γ is $\gamma \propto \alpha^{p-1}$: this will be easily understood a few lines later by a normalization argument coupled with the high connectivity limit of this mean field model.

The quenched expectation of the model is given by the composition of the Poissonian average with the uniform one performed over the families $\{i_\nu\}$,

$$\mathbb{E}[\cdot] = E_P E_i[\cdot] = \sum_{k=0}^{\infty} \frac{e^{-\gamma N}(\gamma N)^k}{k! N^p} \sum_{i_\nu^1, \dots, i_\nu^p}^{1,N} [\cdot], \tag{2}$$

where the term $N^p \approx N!/(N-p)!$ accounts for the number of possible ordered p-plets.

As they will be useful in our derivation, it is worth stressing the following properties of the Poisson distribution: Let us consider a function $g : \mathbb{N} \to \mathbb{R}$, and a Poisson variable k with mean γN, whose expectation is denoted by $\mathbb{E}$.

It is easy to verify that

$$\mathbb{E}[kg(k)] = \gamma N \mathbb{E}[g(k-1)], \tag{3}$$

$$\partial_{\gamma N} \mathbb{E}[g(k)] = \mathbb{E}[g(k+1) - g(k)], \tag{4}$$

$$\partial^2_{(\gamma N)^2} \mathbb{E}[g(k)] = \mathbb{E}[g(k+2) - 2g(k+1) + g(k)]. \tag{5}$$

The Hamiltonian written as in Eq. (1) has the advantage that it is the sum of (a random number of) i.i.d. terms. To see the connection to a more familiar Hamiltonian written in terms of adjacency matrix elements, we first notice that being α/N the probability that two nodes are connected, among the N^p possible p-plets, the number of connected p-plets is Poisson-distributed with average $\alpha^{p-1}N + O(\sqrt{N})$ for large N. We now define the adjacency tensor $A_{i_1,\dots,i_p} \equiv A_{i_1,i_2} A_{i_1,i_3} \dots A_{i_1,i_p}$ which equals 1 whenever the p-plet $i_1, \dots, i_p$ occurs to be connected; $A_{i_1,\dots,i_p}$ is Poisson distributed and has mean $\gamma N/N^p \sim (\alpha/N)^{p-1}$. Hence, we can write the following Hamiltonian which is thermodynamically equivalent to $H_N(\sigma, \gamma)$ appearing in Eq. (1),

$$-H_N(\sigma;\gamma) \sim -\hat{H}_N(\sigma;\mathbf{A}) = \sum_{i_1,\dots,i_p}^{N} A_{i_1,\dots,i_p} \sigma_{i_1} \dots \sigma_{i_p}. \tag{6}$$

Then, it is enough to consider the streaming of the following interpolating free energy (whose structure proves the statement a priori by its thermodynamic meaning), depending on the real parameter $t \in [0, 1]$,

$$\phi(t) = \frac{\mathbb{E}}{N} \ln \sum_{\sigma} e^{\beta(\sum_{\nu=1}^{k} \sigma_{i_\nu^1} \dots \sigma_{i_\nu^p} + \sum_{i_1,\dots,i_p}^{N} A_{i_1,\dots,i_p} \sigma_{i_1} \dots \sigma_{i_p})},$$

where k is a Poisson random variable with mean $\gamma N t$ and $A_{i_1,\dots,i_p}$ are random Poisson variables with mean $(1-t)\gamma/N^{p-1}$. In this way the two separated models are recovered in the two extremals of the interpolation (for $t = 0, 1$). By computing the t-derivative, we get

$$\frac{1}{\gamma} \frac{d\phi(t)}{dt} = \mathbb{E} \ln(1 + \Omega(\sigma_{i_0^1} \dots \sigma_{i_0^p}) \tanh(\beta)) \tag{7}$$

$$- \frac{1}{N^p} \sum_{i_1,\dots,i_p}^{N} \ln(1 + \Omega(\sigma_{i_1} \dots \sigma_{i_p}) \tanh(\beta)) = 0,$$

where the label 0 in i_0^k stands for a new spin, born in the derivative, according to the Poisson property (4); as the i_0's are independent of the random site indices in the t-dependent Ω measure, the equivalence is proved.

Following a statistical mechanics approach, we know that the macroscopic behaviour, versus the connectivity α and the inverse temperature $\beta = 1/T$, is described by the following free energy density (often called *quenched pressure*),

$$A(\alpha, \beta) = \lim_{N\to\infty} A_N(\alpha, \beta) \qquad (8)$$
$$= \lim_{N\to\infty} \frac{1}{N}\mathbf{E}\ln Z_N(\gamma, \beta),$$

where

$$Z_N(\gamma, \beta) = \sum_{\{\sigma\}} e^{-\beta H_N(\sigma,\gamma)} \qquad (9)$$

is the partition function. Taken $g(\sigma)$ as a generic function, the Boltzmann state is therefore given by

$$\omega(g(\sigma)) = \frac{1}{Z_N(\gamma, \beta)} \sum_{\{\sigma_N\}} g(\sigma) e^{-\beta H_N(\sigma,\gamma)}, \qquad (10)$$

with its replicated form

$$\Omega(g(\sigma)) = \prod_s \omega^{(s)}(g(\sigma^{(s)})) \qquad (11)$$

and the total average $\langle g(\sigma)\rangle$ is defined as

$$\langle g(\sigma)\rangle = \mathbf{E}[\Omega(g(\sigma))]. \qquad (12)$$

Let us introduce further, as order parameters of the theory, the multi-overlaps

$$q_{1\ldots n} = \frac{1}{N}\sum_{i=1}^{N} \sigma_i^{(1)} \ldots \sigma_i^{(n)}, \qquad (13)$$

with a particular attention at the magnetization $m = q_1 = (1/N)\sum_{i=1}^N \sigma_i$ and to the two replica overlap $q_{12} = (1/N)\sum_{i=1}^N \sigma_i^1\sigma_i^2$.

The normalization constant of the quenched pressure can be checked by performing the expectation value of the cost function:

$$\mathbb{E}[H] = -\gamma N m^p,$$
$$\mathbb{E}[H^2] - \mathbb{E}^2[H] = \gamma^2 N^2\left[(q_{12}^p - m^p) + O\left(\frac{1}{N}\right)\right], \qquad (14)$$

by which it is easy to see that the model is well defined, in particular it is linearly extensive in the volume N. Then, in the high connectivity limit each agent interacts with all the others ($\alpha \sim N$) and, in the thermodynamic limit, $\alpha \to \infty$. Now, such a high-connectivity limit, i.e. a linear divergence of α, is properly recovered for any finite p, $p < N$. In particular, if $p = 2$ the amount of couples in the summation scales as $N(N-1)/2$ and $\gamma = 2\alpha$; if $p = 3$ the amount of triples scales as $N(N-1)(N-2)/3!$ with $\gamma = 3!\alpha^2$.

Before starting our free energy analysis, we want to point out also the connection between this diluted version and the fully connected counterpart. Let us remember that the Hamiltonian of the fully connected p-spin model (FC) can be written as [7]

$$H_N^{FC}(\sigma) = \frac{p!}{2N^{p-1}} \sum_{1 \le i_1 < \cdots < i_p \le N} \sigma_{i_1}\sigma_{i_2}\ldots\sigma_{i_p}, \tag{15}$$

and let us consider the trial function $\hat{A}(t)$ defined as follows,

$$\hat{A}(t) = \frac{1}{N}\mathbb{E}\ln\sum_{\sigma}\exp\Big[\beta\sum_{\nu}^{k_{\gamma N t}}\sigma_{i_\nu^1}\sigma_{i_\nu^2}\ldots\sigma_{i_\nu^p} + (1-t)\frac{\beta' N}{2}m^p\Big], \tag{16}$$

which interpolates between the fully connected p-spin model and the diluted one, such that for $t = 0$ only the fully connected survives, while the opposite happens for $t = 1$. Let us work out the derivative with respect to t to obtain

$$\begin{aligned}\partial_t \hat{A}(t) = &(p-1)\alpha^{p-1}\ln\cosh(\beta) \\ &- (p-1)\alpha^{p-1}\sum_n \frac{-1^n}{n}\theta^n\langle q_n^p\rangle - \frac{\beta'}{2}\langle m^p\rangle,\end{aligned} \tag{17}$$

by which we see that the correct scaling, in order to recover the proper infinite connectivity model, is obtained when $\alpha \to \infty$, $\beta \to 0$ and $\beta' = 2(p-1)\alpha^{p-1}\tanh(\beta)$ is held constant.

REMARK. It is worth noting that for $p = 2$ we recover the correct scaling of the diluted Curie-Weiss model [3], furthermore the diluted p-spin model reduces to the fully connected one, in the infinite connectivity limit, uniformly in the size of the system.

3. The smooth cavity approach

In this section we want to look for an iterative expression of the free energy density by using a version of the cavity strategy [5, 6] that we briefly recall: the idea behind the cavity techniques [18, 22], which, for our purposes, resembles the stochastic stability approach [12, 25], is that information concerning the free energy density can be extrapolated when looking at the incremental extensive free energy given by the addition of a spin.

In diluted models, this additional spin changes also (infinitesimally in the high N limit) the connectivity and, in evaluating how the free energy density varies conformingly with this, we are going to prove that it can be written by a cavity function and such a connectivity shift. So the behaviour of the system is encoded into these two parts. The latter is simpler as it is made up only by stochastically stable terms (a proper definition of these terms will follow in the current section). Conversely, the former term needs to be expressed via these terms and this must be achieved by iterative expansions.

At first we show how the free energy density can be decomposed via these two parts (the cavity function and the connectivity shift). Then, we analyze each term separately. We will see that they can be expressed by the momenta of the magnetization and of the multi-overlaps, weighted in a perturbed Boltzmann state, which recovers the standard one in the thermodynamic limit.

THEOREM 1. *In the thermodynamic limit, the quenched pressure of the even p-spin diluted ferromagnetic model is given by the expression*

$$A(\alpha,\beta)=\ln 2-\frac{\alpha}{p-1}\frac{d}{d\alpha}A(\alpha,\beta)+\Psi(\alpha,\beta,t=1), \tag{18}$$

where the cavity function $\Psi(\alpha,\beta,t)$ *is introduced as*

$$\mathbb{E}\Bigg[\ln\frac{\sum_{\{\sigma\}}e^{\beta\sum_{\nu=1}^{k_{\tilde\gamma N}}\sigma_{i_\nu^1}\sigma_{i_\nu^2}\dots\sigma_{i_\nu^p}}\,e^{\beta\sum_{\nu=1}^{k_{2\tilde\gamma t}}\sigma_{i_\nu^1}\sigma_{i_\nu^2}\dots\sigma_{i_\nu^{p-1}}}}{\sum_{\{\sigma\}}e^{\beta\sum_{\nu=1}^{k_{\tilde\gamma N}}\sigma_{i_\nu^1}\sigma_{i_\nu^2}\dots\sigma_{i_\nu^p}}}\Bigg]$$

$$=\mathbb{E}\Bigg[\ln\frac{Z_{N,t}(\tilde\gamma,\beta)}{Z_N(\tilde\gamma,\beta)}\Bigg]=\Psi_N(\tilde\gamma,\beta,t), \tag{19}$$

with $\tilde\gamma=\gamma(1+N^{-1})$ *and*

$$\Psi(\gamma,\beta,t)=\lim_{N\to\infty}\Psi_N(\tilde\gamma,\beta,t). \tag{20}$$

For the sake of clearness and to avoid interrupting the paper with long technical calculations, the proof of the theorem is reported in Appendix.

Thanks to the previous theorem, it is possible to figure out an expression for the pressure by studying the properties of the cavity function $\Psi(\alpha,\beta,1)$ and the connectivity shift $\partial_\alpha A(\alpha,\beta)$. Using the properties of the Poisson distribution (3) and (4), we can write

$$\frac{d}{d\alpha}A(\alpha,\beta)=\frac{(p-1)}{N}\alpha^{p-2}\frac{d}{d\gamma}\mathbb{E}\Big[\ln Z_N(\gamma,\beta)\Big]$$
$$=(p-1)\alpha^{p-2}\mathbb{E}\Big[\ln\sum_{\{\sigma\}}e^{\beta\sum_{\nu=1}^{k+1}\sigma_{i_\nu^1}\dots\sigma_{i_\nu^p}}--\ln\sum_{\{\sigma\}}e^{\beta\sum_{\nu=1}^{k}\sigma_{i_\nu^1}\dots\sigma_{i_\nu^p}}\Big].$$

Now considering the relation (and definition)

$$e^{\beta\sigma_{i_0^1}\dots\sigma_{i_0^p}}=\cosh\beta+\sigma_{i_0^1}\dots\sigma_{i_0^p}\sinh\beta, \tag{21}$$
$$\theta=\tanh\beta, \tag{22}$$

we can write

$$\frac{d}{d\alpha}A(\alpha,\beta) = (p-1)\alpha^{p-2}\Big[\ln\cosh\beta + \mathbb{E}[\ln(1+\omega(\sigma_{i_\nu^1}\dots\sigma_{i_\nu^p})\theta)]\Big]. \tag{23}$$

At the end, expanding the logarithm, we obtain

$$\frac{d}{d\alpha}A(\alpha,\beta) = (p-1)\alpha^{p-2}\ln\cosh\beta - (p-1)\alpha^{p-2}\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\theta^n\langle q_{1,\dots,n}^p\rangle. \tag{24}$$

With the same procedure it is possible to show that

$$\frac{d}{dt}\Psi(\tilde{\alpha},\beta,t) = 2\tilde{\alpha}^{p-1}\ln\cosh\beta - 2\tilde{\alpha}^{p-1}\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\theta^n\langle q_{1,\dots,n}^{p-1}\rangle_{\tilde{\alpha},t}, \tag{25}$$

where

$$\tilde{\alpha} = \alpha\Big[\frac{N}{N+1}\Big]^{\frac{1}{p-1}}.$$

Now, by Eq. (25), we see that even the cavity function, once the r.h.s. of Eq. (25) is integrated back against t, can be expressed via all the order parameters of the model,

$$\Psi(\tilde{\alpha},\beta,t) = 2\tilde{\alpha}^{p-1}\bigg(t\cdot\ln\cosh(\beta) - \sum_{n=1}^{\infty}\frac{(-\theta)^n}{n}\int_0^t\langle q_{1,\dots,n}^{p-1}\rangle_{\tilde{\alpha},t}\bigg).$$

So, as expected, we can understand the properties of the free energy by analyzing the properties of the order parameters: magnetization and overlaps, weighted in their extended Boltzmann state $\tilde{\omega}_t$.

Further, as we expect that the order parameters are able to describe thermodynamics even in the true Boltzmann states ω, Ω [21], accordingly to the following definitions, we are going to show that *filled* order parameters (the ones involving even numbers of replicas) are stochastically stable or, in other words, are independent of the t-perturbation in the thermodynamic limit, while the others become filled, again in this limit (such that even for them $\omega_t \to \omega$ in the high N limit and thermodynamics is recovered). The whole is explained in the following definitions and theorems of this section.

DEFINITION. We define the *t-dependent Boltzmann state* $\tilde{\omega}_t$ as

$$\tilde{\omega}_t(g(\sigma)) = \frac{1}{Z_{N,t}(\gamma,\beta)}\sum_{\{\sigma\}}g(\sigma)e^{\beta\sum_{\nu=1}^{k_{\tilde{\gamma}N}}\sigma_{i_\nu^1}\dots\sigma_{i_\nu^p}+\beta\sum_{\nu=1}^{k_{2\tilde{\gamma}t}}\sigma_{i_\nu^1}\dots\sigma_{i_\nu^{p-1}}}, \tag{26}$$

where $Z_{N,t}(\gamma,\beta)$ extends the classical partition function in the same spirit of the numerator of Eq. (26) itself.

We see that the original Boltzmann state of an N-spin system is recovered as t approached 0, while, in the limit $t\to 1$ and gauging the spins, it is possible to build a Boltzmann state of an $N+1$ spins, with a little shift both in α, β, which vanishes in the $N\to\infty$ limit.

Now, coherently with the implication of thermodynamic limit (by which $A_{N+1}(\alpha,\beta) - A_N(\alpha,\beta) = 0$ for $N \to \infty$), we are going to define the *filled* overlap monomials and show their independence (stochastic stability) with respect to the perturbation encoded by the interpolating parameter t. These parameters are already "good" order parameters describing the theory, while the others (the fillable ones) must be expressed via the formers, and this will be achieved by expanding them with a suitably introduced streaming equation.

DEFINITION. We can split the class of monomials of the order parameters in two families:

- We define *filled* or equivalently *stochastically stable* those overlap monomials with all the replicas appearing an even number of times (i.e. q_{12}^2, m^2, $q_{12}q_{34}q_{1234}$).
- We define *fillable* those overlap monomials with at least one replica appearing an odd number of times (i.e. q_{12}, m, $q_{12}q_{34}$).

We are going to show three theorems that will play a guiding role for our iteration: as this approach has been deeply developed in similar contexts (as fully connected Ising and p-spin models [6, 7], fully connected spin glasses [5] or diluted ferromagnetic models [3, 11], which are the "boundaries" of the model of this paper) we will not show all the details of the proof, but we sketch them in Appendix as they are really intuitive. The interested reader will find a clear derivation in Appendix and can deepen this point by looking at the original works.

THEOREM 2. *In the thermodynamic limit and setting $t = 1$ we have*

$$\tilde{\omega}_{N,t}(\sigma_{i_1}\sigma_{i_2}\ldots\sigma_{i_n}) = \tilde{\omega}_{N+1}(\sigma_{i_1}\sigma_{i_2}\ldots\sigma_{i_n}\sigma_{N+1}^n). \tag{27}$$

THEOREM 3. *Let Q_{ab} be a fillable monomial of the overlaps* (*this means that $q_{ab}Q_{ab}$ is filled*), *where $a, b \in \mathbb{N}$. We have*

$$\lim_{N\to\infty}\lim_{t\to 1}\langle Q_{ab}\rangle_t = \langle q_{ab}Q_{ab}\rangle, \tag{28}$$

(*examples*: *for $N \to \infty$ we get $\langle m_1\rangle_t \to \langle m_1^2\rangle$, $\langle q_{12}\rangle_t \to \langle q_{12}^2\rangle$, $\langle q_{12}q_{34}\rangle_t \to \langle q_{12}q_{34}q_{1234}\rangle$*).

THEOREM 4. *In the $N \to \infty$ limit, the averages $\langle\cdot\rangle$ of the filled polynomials are t-independent in β average.*

4. Properties of the free energy

In this section we are going to address various points: at first we work out the constraints that the model must fulfill, which are in agreement both with a self-averaging behaviour of the magnetization and with the replica-symmetric behaviour of the multi-overlaps [27]; then we write an iterative expression for the free energy density and its links with known models as diluted ferromagnets ($p \to 2$ limit) and fully connected p-spin models ($\alpha \to \infty$ limit).

With the definition

$$\tilde{\beta} = 2(p-1)\tilde{\alpha}^{p-1}\theta \tag{29}$$
$$= 2(p-1)\alpha^{p-1}\frac{N}{N+1}\theta \quad \stackrel{N\to\infty}{\longrightarrow} 2(p-1)\alpha^{p-1}\theta = \beta',$$

we show (and prove in Appendix) the streaming of replica functions, by which fillable multi-overlaps can be expressed via filled ones.

PROPOSITION 1. *Let F_s be a function of s replicas. Then the following streaming equation holds*:

$$\frac{\partial\langle F_s\rangle_{t,\tilde{\alpha}}}{\partial t} = \tilde{\beta}\Big[\sum_{a=1}^{s}\langle F_s m_a^{p-1}\rangle_{t,\tilde{\alpha}} - s\langle F_s m_{s+1}^{p-1}\rangle_{t,\tilde{\alpha}}\Big] \tag{30}$$
$$+ \tilde{\beta}\theta\Big[\sum_{a<b}^{1,s}\langle F_s q_{a,b}^{p-1}\rangle_{t,\tilde{\alpha}} - s\sum_{a=1}^{s}\langle F_s q_{a,s+1}^{p-1}\rangle_{t,\tilde{\alpha}}$$
$$+ \frac{s(s+1)}{2!}\langle F_s q_{s+1,s+2}^{p-1}\rangle_{t,\tilde{\alpha}}\Big] + O(\theta^2).$$

REMARK. We stress that, at the first two levels of approximation presented here, the streaming has the structure of a θ-weighted linear sum of the Curie–Weiss streaming (θ^0 term) [6] and the Sherrington–Kirkpatrick streaming (θ^1 term) [5], providing mathematical structures of disordered systems with a certain degree of independence with respect to the kind of quenched noise (frustration or dilution).

It is now immediate to obtain the linear order parameter constraints (often known as Aizenman–Contucci polynomials [4, 6, 11]) of the theory: in fact, the generator of such a constraint is the streaming equation when applied on each filled overlap monomial (or equivalently it is possible to apply the streaming on a fillable one and then gauge the obtained expression; for the sake of clearness both the methods will be exploited, the former for q_2 and the latter for m).

As examples, dealing with the terms m^{p-1} and q_2^{p-1}, it is straightforward to check that

$$0 = \lim_{N\to\infty}\frac{\partial\langle m_N^{p-1}\rangle_{t,\tilde{\alpha}}}{\partial t} = \tilde{\beta}\Big(\langle m_1^{2(p-1)}\rangle - \langle m_1^{p-1}\rangle^2\Big)$$
$$+ \tilde{\beta}\theta\Big(\langle m_1^{p-1}q_2^{p-1}\rangle - \langle m_1^{p-1}\rangle\langle q_2^{p-1}\rangle\Big) + O(\theta^3),$$

then, by gauging the above expression, in the thermodynamic limit, (as $\lim_{N\to\infty}\langle m_N^{p-1}\rangle_t \to \langle m^p\rangle$), we get

$$\Big((\langle m_1^{2p}\rangle - \langle m_1^p\rangle^2) + \theta(\langle q_2^{2p}\rangle - \langle q_2^p\rangle^2)\Big) = 0, \qquad \forall\theta\in\mathbb{R}^+.$$

The fact that the previous expression holds for every θ suggests self-averaging for the energy (by which all the linear constraints can be derived [11]) due to the first term, as well as replica symmetric behaviour of the two replica overlap due to the last one. Analogously, the contribution of the $\langle q_2^2 \rangle$ generator is

$$0 = \Big((\langle q_{12}^{p-1} m_1^{p-1} \rangle + \langle q_{12}^{p-1} m_2^{p-1} \rangle - 2\langle q_{12}^{p-1} m_3^{p-1} \rangle) \\ + \theta(\langle q_{12}^{p-1} q_{12}^{p-1} \rangle - 4\langle q_{12}^{p-1} q_{23}^{p-1} \rangle + 3\langle q_{12}^{p-1} q_{34}^{p-1} \rangle) \Big),$$

which shows replica symmetric behaviour of the magnetization by the first term and the classical Aizenman–Contucci relations [4, 11] by the latter.

Furthermore, turning now our attention to the free energy, it is easy to see that the streaming equation allows to generate all the desired overlap functions coupled to every well behaved F_s. In this way, if F_s is a fillable overlap, we can always expand recursively it into a filled one, the only price to pay given by the θ order that has to be reached or, which is equivalent, the number of derivatives that have to be performed.

Let us now remember the t-derivative of the cavity function (25), showing explicitly the first two terms of its expansion,

$$\frac{d}{dt}\Psi(\tilde{\alpha}, \beta, t) = 2\tilde{\alpha}^{p-1} \ln\cosh\beta + \tilde{\beta}\langle m_1^{p-1} \rangle_{\tilde{\alpha},t} \\ - \frac{\tilde{\beta}}{2}\theta\langle q_{12}^{p-1} \rangle_{\tilde{\alpha},t} - 2\tilde{\beta}^{p-1} \sum_{n=3}^{\infty} \frac{-1^n \theta^n}{n} \langle q_{1,\dots,n}^{p-1} \rangle_{\tilde{\alpha},t}. \tag{31}$$

As derivative of fillable terms involves filled ones, we can arrive to an analytical form of $\Psi(\alpha, \beta)$ if we calculate it as the t-integral of its t-derivative, together with the obvious relation $\Psi(t = 0) = 0$. Hence, if we apply the streaming equation machinery to the overlaps constituting Eq. (31), we are able to fill them and to remove their t-dependence in the thermodynamic limit. In this way we are allowed to bring them out from the final t-integral.

In fact, without gauging (so, not only in the ergodic regime, where symmetries are preserved), we can expand the streaming of $\langle m^{p-1} \rangle_t$,

$$\frac{d\langle m_1^{p-1} \rangle_t}{dt} = \tilde{\beta}\Big[\langle m_1^{2(p-1)} \rangle - \langle m_1^{p-1} m_2^{p-1} \rangle_t \Big] \\ - \tilde{\beta}\theta \Big[\langle m_1^{p-1} q_{12}^{p-1} \rangle_t - \langle m_1^{p-1} q_{23}^{p-1} \rangle_t \Big] + O(\theta^2).$$

We can note the presence of the filled monomial $\langle m_1^{2(p-1)} \rangle$, whose t-dependence has been omitted explicitly to underline its stochastic stability, while the overlaps $\langle m_1^{p-1} m_2^{p-1} \rangle_t$ and $\langle m_1^{p-1} q_{12}^{p-1} \rangle_t$ can be saturated in two steps of streaming. This will be sufficient, wishing to have a fourth-order expansion for the cavity function.

We now derive these two functions and apply the same scheme to all the overlaps that appear and that have to be necessary filled in order to obtain the desired result,

$$\frac{d\langle m_1^{p-1}m_2^{p-1}\rangle_t}{dt} = 2\tilde{\beta}\Big[\langle m_1^{2(p-1)}m_2^{p-1}\rangle_t - \langle m_1^{p-1}m_2^{p-1}m_3^{p-1}\rangle_t\Big] + \theta\tilde{\beta}\Big[\langle m_1^{p-1}m_2^{p-1}q_{12}^{p-1}\rangle - 4\langle m_1^{p-1}m_2^{p-1}q_{13}^{p-1}\rangle_t + 3\langle m_1^{p-1}m_2^{p-1}q_{34}^{p-1}\rangle_t\Big], \tag{32}$$

$$\frac{d\langle m_1^{2(p-1)}m_2^{p-1}\rangle_t}{dt} = 2\tilde{\beta}\Big[\langle m_1^{2(p-1)}m_2^{2(p-1)}\rangle_t\Big] + \tilde{\beta}\Big[\text{fillable terms}\Big] + O(\theta^2). \tag{33}$$

Integrating back in t and neglecting higher order terms we have

$$\langle m_1^{2(p-1)}m_2^{p-1}\rangle_t = \tilde{\beta}\Big[\langle m_1^{2(p-1)}m_2^{2(p-1)}\rangle\Big]t, \tag{34}$$

and we can write

$$\langle m_1^{p-1}m_2^{p-1}\rangle_t = \tilde{\beta}\theta\langle m_1^{p-1}m_2^{p-1}q_{12}^{p-1}\rangle t + \tilde{\beta}^2\langle m_1^{2(p-1)}m_2^{2(p-1)}\rangle t^2. \tag{35}$$

Let us take a look now at the other overlap $\langle m_1^{p-1}q_{12}^{p-1}\rangle_t$,

$$\frac{d\langle m_1^{p-1}q_{12}^{p-1}\rangle_t}{dt} = \tilde{\beta}\Big[\langle m_1^{2(p-1)}q_{12}^{p-1}\rangle_t - \langle m_1^{p-1}m_2^{p-1}q_{12}^{p-1}\rangle_t - 2\langle m_1^{p-1}m_2^{p-1}m_3^{p-1}q_{12}^{p-1}\rangle_t\Big] + O(\theta^2), \tag{36}$$

that gives

$$\langle m_1^{p-1}q_{12}^{p-1}\rangle_t = \tilde{\beta}\langle m_1^{p-1}m_2^{p-1}q_{12}^{p-1}\rangle t + O(\theta^2). \tag{37}$$

At this point we can write for $\langle m_1^{p-1}\rangle_{t,\tilde{\alpha}}$ (and consequently for $\langle q_{12}^{p-1}\rangle_{t,\tilde{\alpha}}$)

$$\begin{aligned}\langle m_1^{p-1}\rangle_{t,\tilde{\alpha}} &= \tilde{\beta}\langle m_1^{2(p-1)}\rangle t - \frac{\tilde{\beta}^3}{3}\langle m_1^{2(p-1)}m_2^{2(p-1)}\rangle t^3 \\ &\quad - \tilde{\beta}^2\theta\langle m_1^{p-1}m_2^{p-1}q_{12}^{p-1}\rangle t^2 + O(\theta^3), \\ \langle q_{12}^{p-1}\rangle_{t,\tilde{\alpha}} &= \tilde{\beta}\theta\langle q_{12}^{2(p-1)}\rangle t + \tilde{\beta}^2\langle m_1^{p-1}m_2^{p-1}q_{12}^{p-1}\rangle t^2 + O(\theta^3).\end{aligned}$$

With these relations, Eq. (31) becomes

$$\begin{aligned}\frac{d}{dt}\Psi_N(\alpha,\beta,t) &= 2\alpha^{p-1}\ln\cosh\beta + \tilde{\beta}^2\langle m_1^{2(p-1)}\rangle t \\ &\quad - \frac{\tilde{\beta}^2\theta^2}{2}\langle q_{12}^{2(p-1)}\rangle t - \frac{3\tilde{\beta}^3\theta}{2}\langle m_1^{p-1}m_2^{p-1}q_{12}^{p-1}\rangle t^2 \\ &\quad - \frac{\tilde{\beta}^4}{3}\langle m_1^{2(p-1)}m_2^{2(p-1)}\rangle t^3 + O(\theta^5),\end{aligned}$$

which ultimately allows us to write an iterated expressions for Ψ evaluated at $t = 1$,

$$\Psi_N(\alpha, \beta, 1) = 2\alpha^{p-1} \ln\cosh\beta + \frac{\tilde{\beta}^2}{2}\langle m_1^{2(p-1)}\rangle - \frac{\tilde{\beta}^2\theta^2}{4}\langle q_{12}^{2(p-1)}\rangle - \frac{\tilde{\beta}^3\theta}{2}\langle m_1^{p-1} m_2^{p-1} q_{12}^{p-1}\rangle - \frac{\tilde{\beta}^4}{12}\langle m_1^{2(p-1)} m_2^{2(p-1)}\rangle t^3 + O(\theta^5). \tag{38}$$

Overall the result we were looking for, namely a Landau-like polynomial form for the free energy, reads off as

$$A(\alpha, \beta) = \ln 2 + \alpha^{p-1} \ln\cosh\beta + \frac{\beta'}{2}\Big(\beta'\langle m^{2(p-1)}\rangle - \langle m^p\rangle\Big) + \frac{\beta'\theta}{4}\Big(\beta'\theta\langle q_{12}^{2(p-1)}\rangle - \langle q_{12}^p\rangle\Big) + O(\theta^5). \tag{39}$$

Now, several conclusions can be addressed from the expression (39). In fact, as we are going to see immediately through remarks, this formula can bridge free-energies of quite different models (diluted versus nondiluted, critical versus uncritical) and acts as a general free energy expression close to the phase transition.

REMARK. At first let us note that, by constraining the interaction to be pairwise, critical behaviour should arise [21]. Coherently, we see that for $p = 2$ we can write the free energy expansion as

$$A(\alpha, \beta)_{p=2} = \ln 2 + \alpha \ln\cosh(\beta) - \frac{\beta'}{2}(1 - \beta')\langle m^2\rangle - \frac{\beta'\theta}{4}\langle q_2^2\rangle,$$

which coincides with the one of the diluted ferromagnet [3] and displays criticality at $2\alpha\theta = 1$, where the coefficient of the second-order term vanishes, in agreement with previous results [3] and Landau theory [21].

REMARK. The free energy density of the fully connected p-spin model is [7] $A(\beta') = \ln 2 + \ln\cosh(\beta m^{p-1}) - (\beta/2)m^p$, which coincides with the expansion (39) in the limit of $\alpha \to \infty$ and $\beta \to 0$ with $\beta' = 2(p-1)\alpha^{p-1}\theta$.

REMARK. It is worth noting that the connectivity no longer plays a linear role in contributing to the free energy density, as it does happen for the diluted two body models [3, 19]. This is interesting in applications to economic networks, where, for high values of coordination number it may be interesting to develop strategies with more than one coupling [28].

5. Numerics

We now analyze the system described in the previous section, from the numerical point of view by performing extensive Monte-Carlo simulations. Within this approach it is more convenient to use the second Hamiltonian introduced (see Eq. (6)),

$$\hat{H}_N(\sigma, \mathbf{A}) = -\sum_{i_i}^{N} \sigma_{i_1} \sum_{i_2<i_3<\dots<i_p=1}^{N} A_{i_1,\dots,i_p} \sigma_{i_2}\sigma_{i_3}\dots\sigma_{i_p}. \tag{40}$$

The product between the elements of the adjacency tensor ensures that the $p-1$ spins considered in the second sum are joined by a link with i_1.

The evolution of the magnetic system is realized by means of a single spin-flip dynamics based on the Metropolis algorithm [24]. At each time step a spin is randomly extracted and updated whenever its coordination number is larger than $p-1$. For α large enough (at least above the percolation threshold, as obviously holds for the results found previously) and $p = 3, 4$ this condition is generally satisfied. The updating procedure for a spin σ_i works as follows: First, we calculate the energy variation ΔE_i due to a possible spin flip, which for $p = 3$ and $p = 4$ reads, respectively,

$$\Delta E_i = 2\sigma_i \sum_{j<k=1}^{N} A_{i,j} A_{i,k} \sigma_j \sigma_k, \tag{41}$$

$$\Delta E_i = 2\sigma_i \sum_{j<k<w=1}^{N} A_{i,j} A_{i,k} A_{i,w} \sigma_j \sigma_k \sigma_w. \tag{42}$$

Now, if $\Delta E_i < 0$, the spin-flip $\sigma_i \to -\sigma_i$ is realized with probability 1, otherwise it is realized with probability $e^{-\beta \Delta E}$.

The cases $p = 3, 4$ were studied in detail, while for $p = 2$ we refer to [3]. Our investigations are aimed to provide the evidence for the existence of a phase transition and its nature, as well as to highlight a proper scaling for the temperature as the parameter α is tuned.

Concerning the first point, we measured the so-called Binder cumulants defined as

$$G_N(T(\alpha)) \equiv 1 - \frac{\langle m^4 \rangle_N}{3\langle m^2 \rangle_N^2}, \tag{43}$$

where $\langle \cdot \rangle_N$ indicates the statistical average obtained for a system of size N and $T = \beta^{-1}$ [29]. The study of Binder cumulants is particularly useful to locate and catalogue the phase transition. In fact, in the case of continuous phase transitions, $G_N(T)$ takes a universal positive value at the critical point T_c, namely all the curves obtained for different system sizes N cross each other. On the other hand, for a first-order transition $G_N(T)$ exhibits a minimum at $T_{\min}$, whose magnitude diverges as N. Moreover, a crossing point at T_{cross} can be as well detected when curves pertaining to different sizes N are considered [30]. Now, $T_{\min}$ and T_{cross} scale as $T_{\min} - T_c \propto N^{-1}$ and $T_{\text{cross}} - T_c \propto N^{-2}$, respectively.

In Fig. 2 we show data for $G_N(T)$ obtained for $p = 3$ and considering systems of different sizes, namely $N = 400$, $N = 500$, and $N = 800$, but equal connectivity, namely $\alpha = 50$ (left panel) and $\alpha = 80$ (right panel). The existence of a minimum is clear and it occurs at $T \approx 625$ and $T \approx 1600$, respectively; notice that such

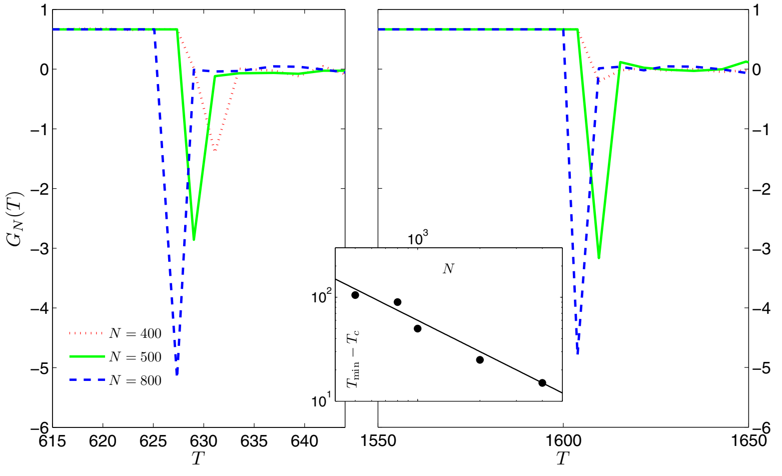

Fig. 2. Binder cumulants $G_N(T)$ for systems with $p = 3$ and different size N, as shown in the legend, and connectivity $\alpha = 50$ (left panel) and $\alpha = 80$ (right panel). The inset shows $T_{\min} - \alpha^2/4$ versus the system size N in a logarithmic scale plot, again $\alpha = 80$; data points from numerical simulations (•) are fitted by the scaling law N^{-1}.

temperatures scale like α^{p-1}, in agreement with the analytic results above. Analogous results are found also for $p = 4$ and they all highlight the existence of a first-order phase transition (hence lack of criticality) at a temperature which depends on the connectivity α.

More precisely, focusing the attention on the case $p = 3$, according to Fig. 2 we can derive $T_c = \alpha^2/4$, which was compared with $T_{\min}$ measured for several choices of α and N: Indeed, $T_{\min}$ asymptotically approaches $\alpha^2/4$ as N gets larger. We also find that the scaling behaviour of $T_{\min} - \alpha^2/4$ is consistent with $\sim N^{-1}$; the inset of Fig. 2 shows the special case $\alpha = 80$.

In order to deepen the role of connectivity in the evolution of the system we measure the macroscopic observable $\langle m \rangle$ and its (normalized) fluctuations $\langle m^2 \rangle - \langle m \rangle^2$, studying their dependence on T and on α. Data for different choices of size and dilution are shown in Fig. 3 for $p = 3$ and in Fig. 4 for $p = 4$.

The profile of the magnetization, with an abrupt jump, and the corresponding peak found for its fluctuations confirm the existence of a first-order phase transition at a well-defined temperature T_c whose value depends on the dilution α. More precisely, by properly normalizing the temperature in agreement with analytical results, namely $\tilde{\beta} \equiv \beta \alpha^{p-1}$, we found a very good collapse of all the curves considered. Hence, we have agreement among analytic and numerics concerning the scaling of the temperature as α^{p-1}. Moreover our data clearly indicate that the critical temperature can be written as $T_c = f(p)\alpha^{p-1}$, where $f(p)$ is a monotonic decreasing function of p. In particular, numerical hints suggest $f(3) = 1/4$ and $f(4) \approx 0.08$.

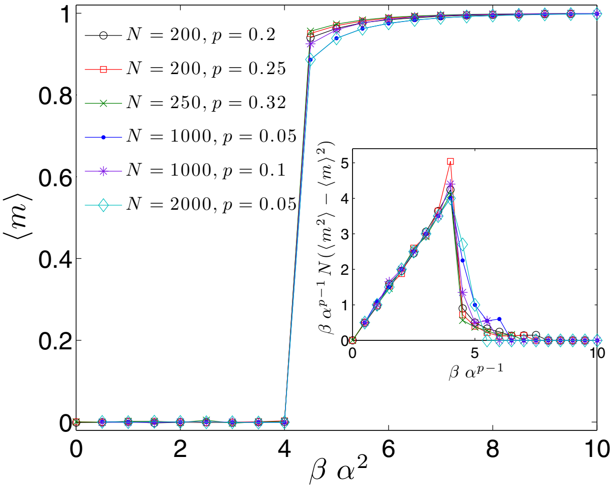

Fig. 3. Magnetization (main figure) and its normalized fluctuations (inset) for 3-spin systems of different sizes and different dilution as a function of $\beta\ \alpha^{p-1}$. The collapse of all the curves provides a strong evidence for the scaling of the temperature.

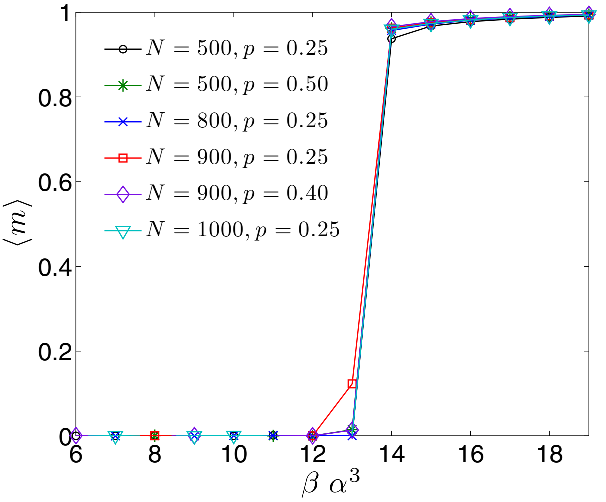

Fig. 4. Magnetization for 4-spin systems of different sizes and different dilution as a function of $\beta\ \alpha^{p-1}$. The collapse of all the curves provides a strong evidence for the scaling of the temperature.

6. Conclusions

In this paper we performed an analysis of the ferromagnetic diluted p-spin model via cavity field technique and numerical simulations. Several questions have been addressed, including an expression for the free energy, the self-averaging families for the order parameters and a study of the phase transition among a paramagnetic and a ferromagnetic regime. Despite a rigorous picture for the lacking of replica symmetry breaking in diluted ferromagnet is still unavailable, we supported strong evidence toward a full replica symmetric behaviour in the whole phase diagram. In particular, we showed the vanishing of criticality for $p > 2$ and we found a proper scaling for the transition temperature as a function of the system dilution, namely $T_c \sim \alpha^{p-1}$.

Further development should be two-fold: from one side the same analysis is still to be performed on the X-OR-SAT model which constitutes another element making up the class of models based on binary agents with mean field interaction. On the other side, the whole mathematical architecture still suffers a not exhaustive development; in fact the difference among even and odd p model, at least for large p, is thermodynamically almost irrelevant, while the lacking of the gauge symmetry in the latter rules out the method at this stage. Moreover, it is highlighted the need to develop a Hamilton–Jacobi technique [17] in order to handle this kind of problem to avoid the iteration procedure implied by the cavity method.

Finally, we underline that our analysis has been carried on assuming the existence of the thermodynamic limit for spin-structures defined on diluted networks, although a rigorous proof is still missing; we plan to investigate the existence of the thermodynamic limit in the future.

Appendix: Analytical proofs

In this section the proofs of all the theorems and Proposition 1 are reported.

Proof of Theorem 1:

Bridging a system made of $N + 1$ spins with one made of N spins implies the definition of rescaled γ, α parameters, accordingly to [3, 11]

$$\tilde{\gamma} = \gamma \frac{N}{N+1} \quad \stackrel{N\to\infty}{\longrightarrow} \quad \gamma, \tag{44}$$

$$\tilde{\alpha} = \alpha \left[\frac{N}{N+1}\right]^{\frac{1}{p-1}} \quad \stackrel{N\to\infty}{\longrightarrow} \quad \alpha. \tag{45}$$

We have, in distribution, the Hamiltonian of a system made of $N + 1$ particles writable as

$$\begin{aligned} H_{N+1}(\sigma, \gamma) &= - \sum_{\nu=1}^{k_{\gamma(N+1)}} \sigma_{i_\nu^1}\sigma_{i_\nu^2}\dots\sigma_{i_\nu^p} \\ &\sim - \sum_{\nu=1}^{k_{\tilde{\gamma}N}} \sigma_{i_\nu^1}\sigma_{i_\nu^2}\dots\sigma_{i_\nu^p} - \sum_{\nu=1}^{k_{2\tilde{\gamma}}} \sigma_{i_\nu^1}\sigma_{i_\nu^2}\dots\sigma_{i_\nu^{p-1}}\sigma_{N+1}, \end{aligned} \tag{46}$$

that we may rewrite as

$$H_{N+1}(\sigma, \gamma) = H_N(\sigma, \tilde{\gamma}) + \hat{H}_N(\sigma, 2\tilde{\gamma}). \tag{47}$$

Following the above decomposition, let us consider the partition function of the same $N+1$ spin model and let us introduce the gauge transformation $\sigma_i \to \sigma_i \sigma_{N+1}$ which is a symmetry of the Hamiltonian known as *spin-flip*,

$$
\begin{aligned}
Z_{N+1}(\gamma,\beta) &\sim \sum_{\{\sigma_{N+1}\}} e^{-\beta H_N(\sigma,\tilde{\gamma})-\beta \hat{H}_N(\sigma,\tilde{\gamma})\sigma_{N+1}} \qquad (48)\\
&= \sum_{\{\sigma_{N+1}\}} e^{\beta H_N(\sigma,\tilde{\gamma})+\beta\sum_{\nu=1}^{k_{2\tilde{\gamma}}} \sigma_{i_\nu^1}\cdots\sigma_{i_\nu^{p-1}}\sigma_{N+1}}\\
&= 2\sum_{\{\sigma_N\}} e^{\beta\sum_{\nu=1}^{k_{\tilde{\gamma}N}} \sigma_{i_\nu^1}\cdots\sigma_{i_\nu^p}+\beta\sum_{\nu=1}^{k_{2\tilde{\gamma}}} \sigma_{i_\nu^1}\cdots\sigma_{i_\nu^{p-1}}}\\
&= 2Z_N(\tilde{\gamma},\beta)\tilde{\omega}(e^{-\beta\hat{H}_N}),
\end{aligned}
$$

where the new Boltzmann state $\tilde{\omega}$, and its replicated $\tilde{\Omega}$, are introduced as

$$
\tilde{\omega}(g(\sigma)) = \frac{\sum_{\{\sigma_N\}} g(\sigma)e^{-\beta H_N(\tilde{\gamma},\sigma)}}{\sum_{\{\sigma_N\}} e^{-\beta H_N(\tilde{\gamma},\sigma)}}, \qquad (49)
$$

$$
\tilde{\Omega}(g(\sigma)) = \prod_i \tilde{\omega}^{(i)}(g(\sigma^{(i)})). \qquad (50)
$$

To continue the proof we now take the logarithm of both sides of the last expression in Eq. (48), apply the expectation $\mathbb{E}$ and subtract the quantity $\mathbb{E}[\ln Z_{N+1}(\tilde{\gamma},\beta)]$. We obtain

$$
\mathbb{E}[\ln Z_{N+1}(\gamma,\beta)] - \mathbb{E}[\ln Z_{N+1}(\tilde{\gamma},\beta)] = \ln 2 - \mathbb{E}\left[\ln\frac{Z_{N+1}(\tilde{\gamma},\beta)}{Z_N(\tilde{\gamma},\beta)}\right] + \Psi_N(\tilde{\gamma},\beta,1), \qquad (51)
$$

For large N, the left-hand side gives

$$
\begin{aligned}
&\mathbb{E}[\ln Z_{N+1}(\gamma,\beta)] - \mathbb{E}[\ln Z_{N+1}(\tilde{\gamma},\beta)] \qquad (52)\\
&\quad = (\gamma-\tilde{\gamma})\frac{d}{d\gamma}\mathbb{E}[\ln Z_{N+1}(\gamma,\beta)]|_{\gamma=\tilde{\gamma}}\\
&\quad = \gamma\frac{1}{N+1}\frac{d}{d\gamma}\mathbb{E}[\ln Z_{N+1}(\gamma,\beta)]|_{\gamma=\tilde{\gamma}} =\\
&\quad = \gamma\frac{d}{d\gamma}A_{N+1}(\gamma,\beta). \qquad (53)
\end{aligned}
$$

Considering the α dependence of γ, we have

$$
\partial_\gamma \propto \frac{1}{(p-1)\alpha^{p-2}}\partial_\alpha \quad \Rightarrow \quad \gamma\frac{d}{d\gamma}A \propto \frac{\alpha}{p-1}\frac{d}{d\alpha}A,
$$

where the symbol $\propto$ instead of $=$ reflects the arbitrariness by which we include the $p!$ term, multiplying α, inside the definition of γ, or directly in α.

Performing now the thermodynamic limit, we see that at the right-hand side we have

$$\lim_{N\to\infty} \mathbb{E}[\ln \frac{Z_{N+1}(\alpha,\beta)}{Z_N(\tilde{\alpha},\beta)}] \longrightarrow A(\alpha,\beta) \tag{54}$$

and the theorem is proved. □

Proofs of Theorems 2, 3, 4:

In this sketch we are going to show how to get Theorem 2 in some detail. It automatically has as a corollary Theorem 3 which ultimately gives, as a simple consequence when applied on filled monomials, Theorem 4.

Let us assume for a generic overlap correlation function Q, of s replicas, the following representation

$$Q = \prod_{a=1}^{s} \sum_{i_l^a} \prod_{l=1}^{n^a} \sigma^a_{i_l^a} I(\{i_l^a\}),$$

where a labels the replicas, the internal product takes into account the spins (labeled by l) which contribute to the a-part of the overlap $q_{a,a'}$ and runs to the number of time that the replica a appears in Q. The external product takes into account all the contributions of the internal one and the I factor fix the constraints among different replicas in Q; so, for example, $Q = q_{13}q_{23}$ can be decomposed in this form noting that $s = 3$, $n^1 = n^3 = 1, n^2 = 2$, $I = N^{-2}\delta_{i_1^1,i_1^3}\delta_{i_1^2,i_2^3}$, where the δ functions fixes the links between replicas $1,3 \to q_{1,3}$ and $2,3 \to q_{2,3}$. The averaged overlap correlation function is

$$\langle Q\rangle_t = \mathbf{E}\sum_{i_l^a} I(\{i_l^a\}) \prod_{a=1}^{s} \omega_t\Big(\prod_{l=1}^{n^a} \sigma^a_{i_l^a}\Big).$$

Now if Q is a fillable polynomial, and we evaluate it at $t = 1$, let us decompose it, using the factorization of the ω state on different replica, as

$$\langle Q\rangle_t = \mathbf{E}\sum_{i_l^a,i_l^b} I(\{i_l^a\},\{i_l^b\}) \prod_{a=1}^{u} \omega_a\Big(\prod_{l=1}^{n^a} \sigma^a_{i_l^a}\Big) \prod_{b=u}^{s} \omega_b\Big(\prod_{l=1}^{n^b} \sigma^b_{i_l^b}\Big),$$

where u stands for the number of the fillable replicas inside the expression of Q. So we split the measure Ω into two different subsets ω_a and ω_b: in this way the replica belonging to the b subset are always in even number, while the ones in the a subset are always odd. Applying the gauge $\sigma_i^a \to \sigma_i^a\sigma_{N+1}^a, \forall i \in (1, N)$ the even measure is unaffected by this transformation ($\sigma_{N+1}^{2n} \equiv 1$) while the odd measure takes a σ_{N+1} inside the Boltzmann measure.

$$\langle Q\rangle = \sum_{i_l^a,i_l^b} I(\{i_l^a\},\{i_l^b\}) \prod_{a=1}^{u} \omega\Big(\sigma^a_{N+1}\prod_{l=1}^{n^a} \sigma^a_{i_l^a}\Big) \prod_{b=u}^{s} \omega\Big(\sigma^b_{N+1}\prod_{l=1}^{n^b} \sigma^b_{i_l^b}\Big). \tag{55}$$

At the end we can replace in the last expression the index $N+1$ of σ_{N+1} by k

for any $k \neq \{i_l^a\}$ and multiply by one as $1 = N^{-1}\sum_{k=0}^{N}$. Up to orders $O(1/N)$, which go to zero in the thermodynamic limit, we have the proof.

It is now immediate to understand that Theorem 2 on a fillable overlap monomial has the effect of multiplying it by its missing part to be filled (Theorem 3), while it has no effect if the overlap monomial is already filled (Theorem 4). □

Proof of Proposition 1:

The proof works by direct calculation:

$$\frac{\partial \langle F_s\rangle_{t,\tilde{\alpha}}}{\partial t} = \frac{\partial \mathbb{E}}{\partial t}\left[\frac{\sum_{\{\sigma\}} F_s e^{\sum_{a=1}^{s}(\beta\sum_{\nu=1}^{k_{\tilde{\gamma}N}}\sigma^a_{i^1_\nu}\dots\sigma^a_{i^p_\nu}+\beta\sum_{\nu=1}^{k_{2\tilde{\gamma}t}}\sigma^a_{i^1_\nu}\dots\sigma^a_{i^{p-1}_\nu})}}{\sum_{\{\sigma\}} e^{\sum_{a=1}^{s}(\beta\sum_{\nu=1}^{k_{\tilde{\gamma}N}}\sigma^a_{i^1_\nu}\dots\sigma^a_{i^p_\nu}+\beta\sum_{\nu=1}^{k_{2\tilde{\gamma}t}}\sigma^a_{i^1_\nu}\dots\sigma^a_{i^{p-1}_\nu})}}\right] \tag{56}$$

$$= 2\tilde{\alpha}^{p-1}\mathbb{E}\left[\frac{\tilde{\Omega}_t(F_s e^{\sum_{a=1}^{s}(\beta\sigma^a_{i^1_0}\dots\sigma^a_{i^{p-1}_0})})}{\tilde{\Omega}_t(e^{\sum_{a=1}^{s}(\beta\sigma^a_{i^1_0}\dots\sigma^a_{i^{p-1}_0})})}\right] - 2\tilde{\alpha}^{p-1}\langle F_s\rangle_{t,\tilde{\alpha}}$$

$$= 2\tilde{\alpha}\mathbb{E}\left[\frac{\tilde{\Omega}_t(F_s\Pi_{a=1}^{s}(\cosh\beta + \sigma^a_{i^1_0}\dots\sigma^a_{i^{p-1}_0}\sinh\beta))}{\tilde{\Omega}_t(\Pi_{a=1}^{s}(\cosh\beta + \sigma^a_{i^1_0}\dots\sigma^a_{i^{p-1}_0}\sinh\beta))}\right] - 2\tilde{\alpha}^{p-1}\langle F_s\rangle_{t,\tilde{\alpha}}$$

$$= 2\tilde{\alpha}^{p-1}\left(\mathbb{E}\left[\frac{\tilde{\Omega}_t(F_s\Pi_{a=1}^{s}(1 + \sigma^a_{i^1_0}\dots\sigma^a_{i^{p-1}_0}\theta))}{(1+\tilde{\omega}_t(\sigma^a_{i^1_0}\dots\sigma^a_{i^{p-1}_0})\theta)^s}\right] - \langle F_s\rangle_{t,\tilde{\alpha}}\right),$$

Now noting that

$$\Pi_{a=1}^{s}(1+\sigma^a_{i^1_0}\dots\sigma^a_{i^{p-1}_0}\theta) = 1 + \sum_{a=1}^{s}\sigma^a_{i^1_0}\dots\sigma^a_{i^{p-1}_0}\theta + \sum_{a<b}^{1,s}\sigma^a_{i^1_0}\dots\sigma^a_{i^{p-1}_0}\sigma^b_{i^1_0}\dots\sigma^b_{i^{p-1}_0}\theta^2 + \cdots,$$

$$\frac{1}{(1+\tilde{\omega}_t\theta)^s} = 1 - s\tilde{\omega}_t\theta + \frac{s(s+1)}{2!}\tilde{\omega}_t^2\theta^2 + \cdots$$

we obtain

$$\frac{\partial \langle F_s\rangle_{t,\tilde{\alpha}}}{\partial t} = 2\tilde{\alpha}^{p-1}\Bigg(\mathbb{E}\Bigg[\tilde{\Omega}_t\Bigg(F_s(1+\sum_{a=1}^{s}\sigma^a_{i^1_0}\dots\sigma^a_{i^{p-1}_0}\theta + \sum_{a<b}^{1,s}\sigma^a_{i^1_0}\dots\sigma^a_{i^{p-1}_0}\sigma^b_{i^1_0}\dots\sigma^b_{i^{p-1}_0}\theta^2+\dots)\Bigg)\times \times\left(1 - s\tilde{\omega}_t\theta + \frac{s(s+1)}{2!}\tilde{\omega}_t^2\theta^2+\dots\right)\Bigg] - \langle F_s\rangle_{t,\tilde{\alpha}}\Bigg), \tag{57}$$

from which our thesis follows. □

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