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Notes on ferromagnetic *p*-spin and REM

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SUMMARY

In this paper we apply some of the recent mathematical techniques (mainly based on interpolation) developed in the spin glass theory to the ferromagnetic *p*-spin model. We introduce two Hamiltonians and derive their thermodynamics. This is a second step toward an alternative and rigorous formulation of the statistical mechanics of simple systems on lattice. A first step has been performed in *J. Stat. Phys.* (2007; arXiv:0712.1344) where the techniques have been tested on the two-body Ising model. For completeness the adaptation of the well-known random energy model to the context of the ferromagnetism is presented. At the end a discussion on the extension of these techniques to Gaussian-disordered *p*-spin models is also briefly outlined. Copyright © 2008 John Wiley & Sons, Ltd.

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1. INTRODUCTION

Owing to long time waited breakthrough in mathematical methods, in the last decade the statistical mechanics of disordered systems earned an highly increasing weight as a powerful framework to analyze complex systems. As a matter of fact several interesting methods, alternative to the well-known *replica trick* [1], in the so-called *spin glass* field of research [2, 3], were developed. In a recent previous study [4] the whole machinery built in [4–10] has been applied to the paradigmatic two-body Ising model for ferromagnetism [11]. However, its generalization to (even) *p*-spin interactions, which makes the model (both qualitatively and quantitatively) different from the (two interacting body) Ising model, was not taken into account as well as its limit of infinitely many interacting particles ($\lim p \to \infty$), which turns out to be an adaptation to non-frustrated

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systems of the so-called random energy model (REM) [12], introduced in the spin glass literature as a simplified, correlation-neglecting [13], model.

In this paper our task is to apply some of these techniques (mainly interpolations [7, 8, 14] and smooth cavity fields [5, 9]) to these models so to close our treatment of simple systems.

The paper is structured as follows: in Section 2 we introduce the p-spin ferromagnetic model; we prove the existence of the thermodynamic limit and analyze its thermodynamics with the standard approach of statistical mechanics [15]. In Section 3, focusing on one of the two Hamiltonian we present, we introduce and adapt the smooth cavity field technique [5] to this model and we show some general properties for the order parameter (i.e. self-averaging constraints and self-consistency equation), which, as in the standard Ising model, is found to be the magnetization [4, 11]. In Section 4 we focus on the second Hamiltonian and we study its thermodynamics, obtaining self-averaging and self-consistency for the order parameter (again the magnetization) within a framework proper of analytical mechanics. Section 5 is left for the adaptation of the REM to a theory of ferromagnetism. Section 6 generalizes these approaches to the disordered case. A discussion follows at the end.

2. THE FERROMAGNETIC EVEN P-SPIN MODEL

2.1. *Definition of the model*

The Hamiltonian of the even p-spin Ising model is defined on N spin configurations $\sigma : i \to \sigma_i = \pm 1$, labeled by $i = 1 \dots N$, and we will deal mainly with the following two versions:

$$H_N(\sigma) = -\frac{(p-1)!}{N^{p-1}} \sum_{1 \leqslant i_1 < i_2 \cdots < i_p \leqslant N} \sigma_{i_1}\sigma_{i_2}\dots\sigma_{i_p} \tag{1}$$

$$H_N(\sigma) = -\frac{p!}{2N^{p-1}} \sum_{1 \leqslant i_1 < i_2 \cdots < i_p \leqslant N} \sigma_{i_1}\sigma_{i_2}\dots\sigma_{i_p} \tag{2}$$

Both the Hamiltonians above are suitable for a good thermodynamic limit (see Section 2.2) and offer a correct thermodynamics; hence, they are thought of as extension to several interacting variables of the two-body Ising–Curie–Weiss model for ferromagnetism. However, we stress that as we will deal also with the $p \to \infty$ at the end, the Hamiltonian 2 is suitable for this purpose and consequently will be analyzed later, close to the infinitely interacting particle limit.

We assume throughout the paper that, without explicit indications, there is no external field and p is an even natural number. The thermodynamic of the model is carried by the free energy density $f_N(\beta) = F_N(\beta)/N$, which is related to the Hamiltonian via

$$e^{-\beta F_N(\beta)} = Z_N(\beta) = \sum_{\sigma} e^{-\beta H_N(\sigma)} \tag{3}$$

$Z_N(\beta)$ being the partition function. For the sake of convenience, we will not deal with $f_N(\beta) = N^{-1} F_N(\beta)$ but with the thermodynamic *pressure* $\alpha(\beta)$ defined via

$$\alpha(\beta) = \lim_{N\to\infty} \alpha_N(\beta) = \lim_{N\to\infty} -\beta f_N(\beta) = \lim_{N\to\infty} \frac{1}{N} \ln Z_N(\beta) \tag{4}$$

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A key role will be played by the magnetization m, its fluctuations and its momenta, and hence let us introduce it as

$$m_N = \frac{1}{N}\sum_{1\leqslant i\leqslant N}\sigma_i, \quad \langle m_N\rangle = \frac{\sum_\sigma m_N e^{-\beta H_N(\sigma)}}{\sum_\sigma e^{-\beta H_N(\sigma)}} \tag{5}$$

2.2. *Infinite volume limit*

In this subsection we want to apply the Guerra–Toninelli interpolation scheme [10, 16] to the model, to prove the existence of the thermodynamic limit.

In a nutshell the idea is two steps:

- show that the free energy can be bounded in the system size N;
- show that the free energy is sub-additive in the system size N.

Combining the two steps together the existence of the thermodynamic limit follows automatically, as explained, for instance, in [17].

To obtain the first point, it is enough to note that, when dealing with the simple ferromagnetic models, it is possible to obtain a bound in the size for the free energy simply by aligning all the spins among themselves and we have for the two outlined Hamiltonians, respectively, (1) and (2)

$$Z_N(\beta) = \sum_\sigma e^{\frac{\beta(p-1)!}{(N)^{p-1}}\sum_{i_1\ldots i_p}\sigma_{i_1}\ldots\sigma_{i_p}} \leqslant \sum_\sigma e^{\frac{\beta(p-1)!}{N^{p-1}}\frac{N^p}{p!}} \leqslant 2^N e^{\beta N/p} \tag{6}$$

$$\alpha_N(\beta) = \frac{1}{N}\ln Z_N(\beta) \leqslant \ln 2 + \frac{\beta}{p}\left(1-\frac{1}{N}\right) \Rightarrow \alpha(\beta) \leqslant \ln 2 + \frac{\beta}{p} \tag{7}$$

$$Z_N(\beta) = \sum_\sigma e^{\frac{\beta p!}{2N^{p-1}}\sum_{i_1\ldots i_p}\sigma_{i_1}\ldots\sigma_{i_p}} \leqslant \sum_\sigma e^{\frac{\beta p!}{2N^{p-1}}\frac{N^p}{p!}} \leqslant 2^N e^{\beta N/2} \tag{8}$$

$$\alpha_N(\beta) = \frac{1}{N}\ln Z_N(\beta) \leqslant \ln 2 + \frac{\beta}{2}\left(1-\frac{1}{N}\right) \Rightarrow \alpha(\beta) \leqslant \ln 2 + \frac{\beta}{2} \tag{9}$$

Both the equations reduce to the well-known high-temperature expression of the Curie–Weiss model if $p=2$ [4, 11].

In the next step we must show subadditivity of the model with respect to the system size: to address this task we split the system built by N spins in two subsystems of N_1 and N_2 spins, respectively, such that $N_1+N_2=N$ and by interpolating among a partition function $Z_N(\beta)$ of N spins and the product of two partition functions $Z_{N_1}(\beta)Z_{N_2}(\beta)$ of N_1 and N_2 spins we obtain the result.

To this purpose, note that the Hamiltonian scales as $\langle H_N(\sigma)\rangle \sim N\langle m^p\rangle$ and consider the interpolating parameter $0\leqslant t\leqslant 1$, and the auxiliary partition function

$$Z_N(t) = \sum_{\{\sigma\}} \exp(\beta g_k(p)(Ntm^p(\sigma) + N_1(1-t)m_1^p(\sigma) + N_2(1-t)m_2^p(\sigma))) \tag{10}$$

where the function $g_k(p)$ takes into account the different (but irrelevant in this context) dependence on p of Equations (1) and (2), (i.e. for $k=1$ we consider the Hamiltonian (1) and $g_1(p)=p^{-1}$;

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otherwise $g_2(p)=\frac{1}{2}$). Of course, for the boundary values $t=0,1$ one has

$$-\frac{1}{N\beta}\ln Z_N(1)=f_N(\beta) \tag{11}$$

$$-\frac{1}{N\beta}\ln Z_N(0)=\frac{N_1}{N}f_{N_1}(\beta)+\frac{N_2}{N}f_{N_2}(\beta) \tag{12}$$

and, taking the derivative with respect to t, we obtain

$$-\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{N\beta}\ln Z_N(t)=-g_k(p)\left\langle m^p(\sigma)-\frac{N_1}{N}m_1^p(\sigma)-\frac{N_2}{N}m_2^p(\sigma)\right\rangle_t \tag{13}$$

where $\langle\ \rangle_t$ denotes the Boltzmann–Gibbs thermal average with the extended weight encoded in the t-dependent partition function (10) and we defined $m_1(\sigma)$, $m_2(\sigma)$ the magnetizations corresponding to the subsystems, i.e.

$$m_1(\sigma)=\frac{1}{N_1}\sum_{i=1}^{N_1}\sigma_i,\quad m_2(\sigma)=\frac{1}{N_2}\sum_{i=N_1+1}^{N}\sigma_i \tag{14}$$

Then one sees that $m(\sigma)$ is a convex linear combination of $m_1(\sigma)$ and $m_2(\sigma)$:

$$m(\sigma)=\frac{N_1}{N}m_1(\sigma)+\frac{N_2}{N}m_2(\sigma) \tag{15}$$

and since the function $x\rightarrow x^p$ is convex, one has

$$Z_N(\beta)\leqslant\sum_{\{\sigma\}}\exp(\beta(N_1m_1^p(\sigma)+N_2m_2^p(\sigma)))=Z_{N_1}(\beta)Z_{N_2}(\beta) \tag{16}$$

Therefore, integrating Equation (13) back in t between 0 and 1, and recalling the boundary conditions (11), (12), the super-additivity property (17) is revealed:

$$Nf_N(\beta)=-\frac{1}{\beta}\ln Z_N(\beta)\geqslant N_1f_{N_1}(\beta)+N_2f_{N_2}(\beta) \tag{17}$$

and we can state that the infinite volume limit for $\alpha_N(\beta)$ does exist and equals its sup for both the Hamiltonians:

$$\lim_{N\rightarrow\infty}\alpha_N(\beta)=\sup_N\alpha_N(\beta)\equiv\alpha(\beta) \tag{18}$$

2.3. Standard approach in thermodynamics

In this section we will focus as a pedagogical example of standard statistical mechanics by analyzing the Hamiltonian (1) but the whole procedure applies identically also to (2). The idea at the basis of the mean field approximation is that the probability distribution function $P(\sigma)$ at equilibrium can be factorized in the product of independent probabilities on the lattice sites, so as to have

$$P(\sigma)=P_i(\sigma_i) \tag{19}$$

DOI: 10.1002/mma

where

$$P_i(\sigma_i) = \frac{1+\langle m\rangle}{2}\delta(\sigma_i - 1) + \frac{1-\langle m\rangle}{2}\delta(\sigma_i + 1) \tag{20}$$

such that we neglect correlation among spins.

We are interested in the averaged free energy density $\langle f_N(\beta)\rangle = -\beta^{-1}\langle \alpha_N(\beta)\rangle = \langle U(\beta)\rangle - \beta^{-1}\langle S(\beta)\rangle$ where

$$\langle U(\beta)\rangle = \frac{\lim_{N\to\infty}\langle H_N(\beta)\rangle}{N} = -\frac{\langle m^p\rangle}{p} \tag{21}$$

is the internal energy and

$$\langle S(\beta)\rangle = \left(\frac{1+\langle m\rangle}{2}\right)\ln\left(\frac{1+\langle m\rangle}{2}\right) + \left(\frac{1-\langle m\rangle}{2}\right)\ln\left(\frac{1-\langle m\rangle}{2}\right) \tag{22}$$

is the entropy density.

Combining together the two expressions above we find

$$\alpha_N(\beta) = -\beta\frac{\langle m^p\rangle}{p} + \left(\frac{1+\langle m\rangle}{2}\right)\ln\left(\frac{1+\langle m\rangle}{2}\right) + \left(\frac{1-\langle m\rangle}{2}\right)\ln\left(\frac{1-\langle m\rangle}{2}\right) \tag{23}$$

from which, imposing the stationarity for the magnetization, we obtain

$$0 = \frac{\partial\alpha(\beta)}{\partial\langle m\rangle} = -\beta\langle m^{p-1}\rangle + \frac{1}{2}\ln\left(\frac{1+\langle m\rangle}{1-\langle m\rangle}\right) \Rightarrow \langle m\rangle = \tanh(\beta\langle m^{p-1}\rangle) \tag{24}$$

which is the generalization of the well-known self-consistency equation for the Ising model (recovered for $p=2$).

These equations (varying p) have a non-trivial phase in which the Z_2 group becomes broken and a simpler ergodic phase. The system belong to one phase or to another one depending on the (i.e. graphical) solutions of the self-consistency equation, undergoing at a given temperature, a phase transition.

Concerning the nature of the transition, things behave quite different depending on p. For general p the transition is first order (there is a discontinuous jump in the values of the magnetization when crossing the critical temperature), while $p=2$ is a special case. The transition is second order and the order parameter behaves continuously with the temperature.

This can be understood by analyzing the stability of the free energy with respect to a particular solution encoded in the value of the magnetization, by exploring the second derivative:

$$\partial_{\langle m\rangle}\left(-\beta\langle m^{p-1}\rangle + \frac{1}{2}\ln\left(\frac{1+\langle m\rangle}{1-\langle m\rangle}\right)\right) = -\beta(p-1)\langle m^{p-2}\rangle + \frac{1}{(1-\langle m^2\rangle)} \tag{25}$$

from which we see that (neglecting the case $p=0,1$, which are trivial), only when $p=2$ we immediately have that $\langle m\rangle \sim (1-1/\beta)^{1/2}$, with $\beta\in(1,\infty)$, offering a critical behavior. For general $p\geqslant 3$ this is no longer the case and the model behaves differently.

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3. THERMODYNAMICS VIA CAVITY FIELD TECHNIQUES

3.1. Pasting a spin

The interpolating technique can be very naturally implemented in the cavity method; let us consider again the Hamiltonian (1) and its the partition function for a system made by $N+1$ spins:

$$\begin{aligned} Z_{N+1}(\beta) &= \sum_{\sigma} \mathrm{e}^{-\beta H_{N+1}(\sigma)} \\ &= \sum_{\sigma_{N+1}=\pm 1} \sum_{\sigma} \mathrm{e}^{\frac{\beta(p-1)!}{(N+1)^{p-1}} \sum_{1\leqslant i_1<\cdots<i_p\leqslant N} \sigma_{i_1}\cdots\sigma_{i_p}} \\ &\quad \times \mathrm{e}^{\frac{\beta(p-1)!}{(N+1)^{p-1}} \sum_{1\leqslant i_1<\cdots<i_{p-1}\leqslant N} \sigma_{i_1}\cdots\sigma_{i_{p-1}}\sigma_{N+1}} \end{aligned} \tag{26}$$

By applying the gauge transformation $\sigma_i \to \sigma_i \sigma_{N+1}$ (which is a symmetry of the Hamiltonian) we obtain

$$Z_{N+1}(\beta) = 2 Z_N(\beta^*) \tilde{\omega}\left(\mathrm{e}^{\frac{\beta(p-1)!}{(N+1)^{p-1}} \sum_{1\leqslant i_1<\cdots<i_{p-1}\leqslant N} \sigma_{i_1}\cdots\sigma_{i_{p-1}}}\right) \tag{27}$$

where $\tilde{\omega}$ is the Boltzmann state at the inverse temperature $\beta^* = \beta(1/(1+N^{-(p-1)}))$ (note that in the thermodynamic limit higher is p faster is the convergence of β^* to β). Let us reverse the temperature shift and apply the logarithm to both the sides of Equation (27) to obtain

$$\ln Z_{N+1}(\beta^*) = \ln 2 + \ln Z_N(\beta) + \ln \omega_N\left(\mathrm{e}^{\frac{\beta(p-1)!}{N^{p-1}} \sum_{1\leqslant i_1<\cdots<i_p\leqslant N} \sigma_{i_1}\cdots\sigma_{i_{p-1}}}\right) \tag{28}$$

Equation (28) tells us that via the third term of its r.h.s. we can bridge an Ising system with N particles at an inverse temperature β to an Ising system with $N+1$ particles at a shifted inverse temperature $\beta^* = \beta(1/(1+N^{-(p-1)}))$. Focusing on such a term let us define an extended partition function $Z_N(\beta, t)$ as

$$Z_N(\beta, t) = \sum_{\sigma} \mathrm{e}^{-\beta H_N(\sigma)} \mathrm{e}^{\frac{t(p-1)!}{N^{p-1}} \sum_{1\leqslant i_1<\cdots<i_{p-1}\leqslant N} \sigma_{i_1}\cdots\sigma_{i_{p-1}}} \tag{29}$$

Note that the above partition function, at $t=\beta$, turns out to be, via the global gauge symmetry $\sigma_i \to \sigma_i \sigma_{N+1}$, a partition function for a system of $N+1$ spins at a shifted temperature β^* apart from a constant term. On the same line, we define the generalized Boltzmann state $\langle\ \rangle_t$ as

$$\langle F(\sigma)\rangle_t = \frac{\langle F(\sigma) \mathrm{e}^{\frac{t(p-1)!}{N^{p-1}} \sum_{1\leqslant i_1<\cdots<i_{p-1}\leqslant N} \sigma_{i_1}\cdots\sigma_{i_{p-1}}}\rangle}{\langle \mathrm{e}^{\frac{t(p-1)!}{N^{p-1}} \sum_{1\leqslant i_1<\cdots<i_{p-1}\leqslant N} \sigma_{i_1}\cdots\sigma_{i_{p-1}}}\rangle} \tag{30}$$

$F(\sigma)$ being a generic function of the spins: Furthermore, we need to introduce, respectively, as fillable and filled monomials the odd and even momenta of the magnetization weighted by the extended Boltzmann measure such that

- $\langle m_N^{2n+1}\rangle_t$ with $n \in \mathbf{N}$ is fillable;
- $\langle m_N^{2n}\rangle_t$ with $n \in \mathbf{N}$ is filled.

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3.2. Stability with respect to deterministic perturbation

When considering the generalized Boltzmann state, there are peculiar properties of both the filled and the fillable monomials that we have to exploit: In the thermodynamic limit, the first class does not depend on the perturbation induced by the cavity field and, at $t=\beta$, the latter (via the $\sigma_i \to \sigma_i \sigma_{N+1}$ symmetry) is projected into the first class. Let us show these concept in some details.

Theorem 3.1
In the $N\to\infty$ limit the averages $\langle m_N^{2n}\rangle$ of the filled monomials are t-independent for almost all values of β, such that

$$\lim_{N\to\infty} \partial_t \langle m_N^{2n}\rangle_t = 0$$

Proof
The proof is a straightforward application of Lemma 3.3. □

Theorem 3.2
Let $\langle M\rangle_t$ be a fillable monomial of the magnetization (this means that $\langle mM\rangle$ is filled). We have

$$\lim_{N\to\infty} \lim_{t\to\beta} \langle M\rangle_t = \langle mM\rangle \tag{31}$$

Proof
The proof is a straightforward application of Lemma 3.3. □

For a clearer statement of the lemma we take the freedom of pasting the volume dependence of the averages as a subscript close to the perturbing tuning parameter t.

Lemma 3.3
Let $\langle\ \rangle_N$ and $\langle\ \rangle_{N,t}$ be the states defined, on a system of N spins, respectively, by the canonical partition function $Z_N(\beta)$ and by the extended one $Z_N(\beta,t)$; if we consider the ensemble of indexes $\{i_1\dots i_r\}$ with $r\in[1,N]$, then for $t=\beta$, where the two measures become comparable, thanks to the global gauge symmetry (i.e. the substitution $\sigma_i\to\sigma_i\sigma_{N+1}$) the following relation holds

$$\omega_{N,t=\beta}(\sigma_{i_1}\dots\sigma_{i_r}) = \omega_{N+1}(\sigma_{i_1}\dots\sigma_{i_r}\sigma_{N+1}^r) + O\left(\frac{1}{N}\right) \tag{32}$$

where r is an exponent; hence, if r is even $\sigma_{N+1}^r=1$, while if it is odd $\sigma_{N+1}^r=\sigma_{N+1}$.

Proof
Let us write $\omega_{N,t}$ for $t=1$, defining for the sake of simplicity $\pi=\sigma_{i_1}\dots\sigma_{i_r}$:

$$\omega_{N,t=\beta}(\pi) = \left[\sum_\sigma \frac{1}{Z_N(\beta)} \mathrm{e}^{\frac{\beta(p-1)!}{N^{p-1}}\sum_{1\leqslant i_1<\cdots<i_p\leqslant N}\sigma_{i_1}\dots\sigma_{i_p} + \frac{\beta p!}{N^{p-1}}\sum_{i_1\dots i_{p-1}}\sigma_{i_1}\dots\sigma_{i_{p-1}}}\pi\right] \tag{33}$$

Introducing first a sum over σ_{N+1} at the numerator and at the denominator (which is the same as multiply and divide for 2^N because there is still no dependence to σ_{N+1}) and making the transformation $\sigma_i\to\sigma_i\sigma_{N+1}$, the variable σ_{N+1} appears at the numerator and it is possible to build the status at $N+1$ particles with the little temperature shift that vanishes in the thermodynamic

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limit:

$$\omega_{N,t=\beta}(\pi) = \omega_{N+1}(\pi\sigma^r_{N+1}) + O\left(\frac{1}{N}\right) \tag{34}$$

where r is an exponent; hence, $\sigma^r_{N+1} = \sigma_{N+1}$ if r is odd and we obtained a filled monomial starting from a fillable, or $\sigma^r_{N+1} = 1$ if r is even, and remaining with a filled term as we started. □

3.3. Self-consistency and self-averaging

Now we want to use the introduced machinery to obtain the two fundamental relations for the thermodynamics of the model: an equation that describes the system by varying the temperature in terms of an order parameter (a self-consistency equation) and an equation that states that the order parameter is a *good* order parameter, i.e. a self-averaging equation.

To this task let us consider the stream with respect to the perturbed Boltzmann measure: when a generic well-defined function of the spins $F(\sigma)$ is considered, the following streaming equation holds:

$$\frac{\partial \langle F_N(\sigma)\rangle_t}{\partial t} = \langle F_N(\sigma) m_N^{p-1}\rangle_t - \langle F_N(\sigma)\rangle_t \langle m_N^{p-1}\rangle_t \tag{35}$$

It is possible to prove straightforwardly Equation (35) by direct simple derivation.

In order to obtain self-consistency and self-averaging we should have to deal with variances of suitable order parameters. It is immediate to find that the streaming of $\langle m_N^{p-1}\rangle_t$ fulfill our task obeying the following differential equation:

$$\partial_t \langle m_N^{p-1}\rangle_t = \langle m_N^{2(p-1)}\rangle_t - \langle m_N^{p-1}\rangle_t^2 \tag{36}$$

which, thanks to Theorem (3.2), becomes trivial in the thermodynamic limit. In fact, calling $m = \lim_{N\to\infty} m_N$ and skipping the subscript t on $\lim_{N\to\infty}\langle m_N^{2(p-1)}\rangle_t$ (because it is filled and consequently in the thermodynamic limit does not depend on t) we obtain

$$\frac{1}{\langle m^{2(p-1)}\rangle}\partial_t \langle m^{p-1}\rangle_t = 1 - \left(\frac{\langle m^{p-1}\rangle_t^2}{\langle m^{2(p-1)}\rangle}\right)$$

which is easily solved by splitting the variables and the solution is

$$\langle m^{p-1}\rangle_t = \sqrt{\langle m^{2(p-1)}\rangle}\tanh(\beta t\sqrt{\langle m^{2(p-1)}\rangle}) \tag{37}$$

Once assumed for now self-averaging (i.e. $\sqrt{\langle m^{2p}\rangle} = \langle m^p\rangle$) and evaluated Equation (37) by using the gauge at $t=\beta$ (i.e. $\langle m^{2(p-1)}\rangle_{t=\beta} = \langle m^{2p}\rangle$) we obtain

$$\langle m\rangle = \tanh(\beta\langle m^{p-1}\rangle) \tag{38}$$

and self-consistency (Equation (24)) is recovered.

We still have to proof the validity of the assumption we made concerning self-averaging, which, within our framework, is a straightforward application of Theorems (3.1) and (3.2); in fact, given natural numbers n, k the following relations hold:

$$\langle m^{2n} m^p\rangle = \langle m^{2n}\rangle\langle m^p\rangle \tag{39}$$

$$\langle m^p\rangle = \langle m^{p-k}\rangle\langle m^k\rangle \tag{40}$$

These equations can be proved simply deriving via the streaming equation (35) a filled monomial, then evaluating in the thermodynamic limits the result and then equating this result to zero as the fillable terms do not depend on t.

As an example consider $0=\lim_{N\to\infty}\partial_t\langle m^p\rangle$ with even p of course. By the streaming we know that $\partial_t\langle m^p\rangle=\langle m^p m^{p-1}\rangle_t-\langle m^p\rangle_t\langle m^{p-1}\rangle_t$ and in the thermodynamic limit we obtain $\langle m^{2p}\rangle=\langle m^p\rangle^2$.

4. THERMODYNAMICS VIA MECHANICAL TECHNIQUES

4.1. *The Hamilton–Jacobi streaming*

In this section we consider the Hamiltonian (2) and we want to derive its thermodynamics using two interpolating parameters, (t,x). In this manner it is possible to work out an Hamilton–Jacobi-like equation in the space of these parameters [4, 18] (where t plays a role of a *ficticious* time and x of a *ficticious* space) building by hands the effective potential, paying attention to obtain for this potential the variance of some power of the magnetization so as to obtain the exact solution by solving directly the free field problem.

The idea is as follows: Let us introduce an extended free energy density $\alpha_N(t,x)$ as

$$\alpha_N(t,x)=\frac{1}{N}\ln\sum_\sigma\exp\left(\frac{-tp!}{2N^{p-1}}\sum_{1\leqslant i_1\dots i_p\leqslant N}\sigma_{i_1}\dots\sigma_{i_p}+\frac{x\left(\frac{p}{2}!\right)}{N^{p/2-1}}\sum_{1\leqslant i_1\dots i_{p/2}\leqslant N}\sigma_{i_1}\dots\sigma_{i_{p/2}}\right)\tag{41}$$

It is straightforward to check that

$$\frac{\mathrm{d}\alpha_N(t,x)}{\mathrm{d}t}=-\frac{1}{2}\langle m^p\rangle,\qquad \frac{\mathrm{d}\alpha_N(t,x)}{\mathrm{d}x}=\langle m^{p/2}\rangle\tag{42}$$

Such that the following Hamilton–Jacobi equation holds:

$$\frac{\mathrm{d}\alpha_N(t,x)}{\mathrm{d}t}+\frac{1}{2}\left(\frac{\mathrm{d}\alpha_N(t,x)}{\mathrm{d}x}\right)^2+V(t,x)=0\tag{43}$$

where the potential $V(t,x)$ is exactly

$$V(t,x)=\tfrac{1}{2}(\langle m^p\rangle-\langle m^{p/2}\rangle^2)\tag{44}$$

4.2. *Free field solution*

Hence, if we solve the free field problem (i.e. $V(t,x)=0$) the solution we get automatically implies self-averaging for the order parameter and is, at least in principle, much simpler than the whole problem. Let us move in this manner.

From standard analytical mechanics, we know that the solution of the Hamilton–Jacobi equation is given by a solution in a particular point plus the integral of the Lagrangian in time.

As a particular point we choose of course $(t,x)\Rightarrow(0,x_0)$ because usually the term with higher-order coupling is the most difficult and we reject it by choosing $t=0$. The Lagrangian, as the

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potential is imposed to be zero, is simply the kinetic energy

$$\pounds(t,x)=\tfrac{1}{2}(\partial_x\alpha(t,x))^2 \tag{45}$$

such that, at the end, we obtain

$$\alpha(t,x)=\alpha(0,x_0)+\int_0^t \pounds(t',x)\,\mathrm{d}t' \tag{46}$$

Thermodynamics will be recovered as a particular point in this space (t,x) by choosing at the end $(t,x)\Rightarrow(-\beta,0)$.

Let us solve the two terms offering the solution: At first note that as the potential is absent the trajectories $x(t)$ are Galilean lines such that $x(t)=x_0+vt$ where the velocity field is $v=\partial_x\alpha(t,x)=\langle m^{p/2}\rangle$, than start moving solving for $\alpha(0,x_0)$:

$$\begin{aligned}\alpha(0,x_0)&=\frac{1}{N}\ln\sum_\sigma\exp\left(\frac{x_0\left(\frac{p}{2}!\right)}{2N^{p-1}}\sum_{1\leqslant\sigma_{i_1}\ldots\sigma_{i_{p/2}}\leqslant N}\sigma_{i_1}\ldots\sigma_{i_{p/2}}\right)\\&=\frac{1}{N}\ln\sum_\sigma\exp\left((x-\langle m^{p/2}\rangle t)\frac{p}{2}\langle m^{p/2-1}\rangle\sum_i\sigma_i\right)\\&=\ln 2+\ln\cosh\left(-\frac{pt}{2}\langle m^{p/2}\rangle\langle m^{p/2-1}\rangle\right)\\&=\ln 2+\ln\cosh\left(-\frac{pt}{2}\langle m^{p-1}\rangle\right)\end{aligned} \tag{47}$$

where in the last passage we used the factorization property $\langle m^{p/2}\rangle\langle m^{p/2}\rangle\to\langle m^p\rangle$ allowed by neglecting the potential.

Turning to the Lagrangian, we stress that, again, as the potential is zero the (kinetic) energy is a constant of motion and hence does not depend on time, such that

$$\int_0^t \pounds(x,t')\,\mathrm{d}t'=\int_0^t \pounds(x)\,\mathrm{d}t'=\pounds(x)t=\frac{1}{2}\langle m^{p/2}\rangle^2 t=\frac{1}{2}\langle m^p\rangle t \tag{48}$$

where in the last passage we used the self-averaging induced by neglecting the potential.

4.3. *Self-consistency and self-averaging*

Combining together the two pieces we obtain

$$\alpha(t,x)=\ln 2+\ln\cosh\left(-\frac{pt}{2}\langle m^{p-1}\rangle\right)+\frac{1}{2}\langle m^p\rangle t \tag{49}$$

and we recover $\alpha(\beta)$ by choosing $(t,x)\to(-\beta,0)$ such that the extended partition function implicitly defined in (41) turns out to be a Boltzmann partition function

$$\alpha(\beta)=\ln 2+\ln\cosh\left(\frac{p}{2}\beta\langle m^{p-1}\rangle\right)-\frac{1}{2}\langle m^p\rangle\beta \tag{50}$$

DOI: 10.1002/mma

by which, imposing the stationarity condition with respect to m, we obtain the self-consistency equation for the magnetization:

$$\langle m\rangle = (p-1)\tanh\left(\beta\frac{p}{2}\langle m^{p-1}\rangle\right) \tag{51}$$

which recover the well-known Ising–Curie–Weiss case for $p=2$.

Within this technique self-averaging is already obtained as we studied the free field solution, such that $V(t,x)=0, \forall(t,x)\in\mathbb{R}$ and of course in the point $(t=-\beta, x=0)$.

5. INFINITELY MANY INTERACTING VARIABLE

As a last remark on simple systems we want to develop the ordered counterpart of the REM [12] and its generalization (the GREM) [19], which, in this ferromagnetic context, should be thought of as deterministic energy model.

The REM is a really useful model in spin glass theory [20]. In a nutshell, concerning technicalities, in this system the correlations between energy levels are absent and the latter are exponentially distributed. Concerning the physics behind the computations it is well known that the REM can be thought of as the $\lim_{p\to\infty}$ of the p-spin model of spin glasses which, for $p=2$ coincides with the SK model [13]. Its Gaussian reformulation (GREM) has been very useful to develop the Ruelle probability cascades for Parisi theory [19]. Reminding to bibliography for a proper introduction we turn our attention to the same structure in the simpler ferromagnetic context.

If we consider the Hamiltonian (2), remembering that $\langle m^p\rangle\in[0,1]$, we see that

$$\lim_{p\to\infty} N^{-1}H_N(\sigma) = \lim_{p\to\infty}\frac{1}{2}\langle m^p\rangle = \frac{1}{2}\left(\prod_i^N \delta(\sigma_i-1)+\prod_i^N\delta(\sigma_i+1)\right) = \frac{1}{2}J_\sigma \tag{52}$$

Let us define the following free energy

$$-\beta f_N(\beta) = \alpha_N(\beta) = \frac{1}{N}\ln\sum_\sigma e^{\beta N J_\sigma} \tag{53}$$

in which the variables J_σ behave as follows:

$$J_\sigma = \begin{cases} 1 & \text{if } \sigma_i=1\ \forall i \text{ or } \sigma_i=-1\ \forall i \\ 0 & \text{otherwise}\end{cases}$$

It is straightforward to check that

$$\lim_{N\to\infty}\alpha_N(\beta) = \lim_{N\to\infty}\frac{1}{N}\ln(2e^{\beta N}+(2^N-2)) = \begin{cases}\ln 2 & \text{if } \beta<\ln 2\\ \beta & \text{if } \beta>\ln 2\end{cases} \tag{54}$$

This structure defines the analogous of the REM for non-random systems. The GREM extension counterpart here can be summarized as follows: Let us divide $N=N_1+N_2+\cdots+N_s$ such that the following ratio are well defined in the thermodynamic limit:

$$\frac{N_1}{N}\to\alpha_1, \frac{N_2}{N}\to\alpha_2\ldots\frac{N_s}{N}\to\alpha_s$$

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and introduce a collection of $J_{\sigma,i}$, $i\in(1\dots s)$ such that

$$J_{\sigma,1}=\begin{cases}1 & \text{if } \sigma_i=\cdots=\sigma_{N_1}=1\\ 0 & \text{otherwise}\end{cases}$$

$$\vdots$$

$$J_{\sigma,s}=\begin{cases}1 & \text{if } \sigma_i=\cdots=\sigma_{N_s}=1\\ 0 & \text{otherwise}\end{cases}$$

Introducing two sets of real numbers $\{\alpha_i\},\{a_i\}$, $i\in(1\dots s)$ such that $\alpha_1+\alpha_2+\cdots+\alpha_s=1$, $a_1+a_2+\cdots+a_s=1$, and defining the free energy as

$$\alpha_N(\beta)=\mathrm{e}^{\beta N\sum_i a_i J_{\sigma,i}} \tag{55}$$

again it is straightforward checking that

$$\lim_{N\to\infty}\alpha_N(\beta)=\sup_{\{a_i,\alpha_i\}}(\beta(a_1+\cdots+a_s),\beta(a_1+\cdots+a_{s-1})+\alpha_s\ln 2\dots\ln 2) \tag{56}$$

that is the analogous of the GREM for non-random systems.

6. A QUICK LOOK AT DERRIDA'S P-SPIN MODEL

In this section we want just to sketch how the methods explained along the paper can be translated to the disordered counterpart, which are still nowadays considered mathematically a challenge. We will not go throughout a real analysis of these models as it would require essentially another paper but just show the simplicity by which these ideas can be translated to complex systems.

It is very natural to generalize the ferromagnetic p-spin Hamiltonian, by letting the spins interact through random couplings. The resulting Hamiltonian (in zero external field) is

$$H_N^{(p)}(\sigma)=-\sqrt{\frac{p!}{2N^{p-1}}}\sum_{1\leqslant i_1<\cdots<i_p\leqslant N}J_{i_1\dots i_p}\sigma_{i_1}\dots\sigma_{i_p} \tag{57}$$

We assume the couplings $J_{i_1\dots i_p}$ to be independent identically distributed centered Gaussian random variables, with unit variance. One can easily show, along the lines of Section 2.2, that the normalization factor $\sqrt{p!/2N^{p-1}}$ is actually the one required to yield a good thermodynamic limit.

Dealing with an external average over the random couplings we need to introduce replicas of the states by defining a product Boltzmann state Ω as

$$\Omega(.)=\omega^a(.)\times\omega^b(.)\dots\omega^s(.) \tag{58}$$

where s is a generic natural number and we need to indicate with $\mathbb{E}$ the averages over the noise. The order parameter in this context is not the magnetization, which due to both positive and negative signs in the coupling does not play a particular role but the overlap, defined as

$$q_{ab}=\frac{1}{N}\sum_i\sigma_i^a\sigma_i^b,\quad \langle q_{ab}\rangle=\mathbb{E}\Omega\left(\frac{1}{N}\sum_i\sigma_i^a\sigma_i^b\right) \tag{59}$$

Note that the Gaussian random variables $H_N^{(p)}(\sigma)$ have a very simple covariance structure, which depends only on the mutual overlap of the configurations σ, σ':

$$\mathbb{E}(H_N^{(p)}(\sigma^1)H_N^{(p)}(\sigma^b))-(\mathbb{E}H_N^{(p)}(\sigma^a)\mathbb{E}H_N^{(p)}(\sigma^b))=\frac{N}{2}q_{ab}^p+O(1) \tag{60}$$

This system, known as p-spin model, was first introduced by Derrida [12], and later has been widely studied, both in the theoretical and in the mathematical physics literature (see, for instance, [21–25]).

The contribution of our paper to this model is showing, in a nutshell, some features of the technique applied to the ferromagnetic counterpart to a network with Gaussian interactions.

For such a model, still the investigation of the free energy and its decomposition in terms of a cavity function and the energy is available as well as the perturbative streaming.

The symmetry we want to use, $\sigma_i \to \sigma_i \sigma_{N+1}$, allows us to express the following cavity function:

$$\Psi(\beta, t)=\mathbb{E}\ln\Omega e^{\sqrt{(t/N^{(p-1)})}\sum_{i_1\ldots i_{p-1}} J_{i_1\ldots i_{p-1}}\sigma_{i_1}\ldots\sigma_{i_{p-1}}}$$

such that

$$\partial_t\Psi(\beta, t)=\frac{1}{2(p-1)!}(1-\langle q_{12}^{p-1}\rangle_t)$$

It is also immediate, by direct derivation over the temperature first and integration by parts over the noise later, that the energy can be expressed as

$$\frac{\langle H\rangle}{N}=-\frac{\mathrm{d}\langle\alpha\rangle}{\mathrm{d}\beta}=-\frac{\beta}{2}(1-\langle q_{12}^p\rangle)$$

and the cavity function can be evaluated via the p-spin streaming equation

$$\partial_t\langle F\rangle_t=\left\langle F\left(\sum_{a,b}^{s}q_{a,b}^{p-1}-s\sum_{a}^{2}q_{a,s+1}^{p-1}+\frac{s(s-1)}{2}q_{s+1,s+2}^{p-1}\right)\right\rangle_t$$

being F_s a function of s replicas.

Concerning the Hamilton–Jacobi structure again it is possible to work out such an equation for $S(t,x)=\alpha(t,x)-t/4-x/2$ by considering

$$\alpha(t,x)=\frac{\mathbb{E}}{N}\ln\sum_{\sigma}e^{(\sqrt{\frac{tp!}{2N^{p-1}}}\sum_{i_1\ldots i_p}J_{i_1\ldots i_p}\sigma_{i_1}\ldots\sigma_{i_p}+\sqrt{\frac{x\sqrt{2}(p/2)!}{N^{p/2-1}}}\sum_{i_1\ldots i_{p/2}}J_{i_1\ldots i_{p/2}}\sigma_{i_1}\ldots\sigma_{i_{p/2}})} \tag{61}$$

such that

$$\frac{\partial S(t,x)}{\partial t}=\frac{1}{4}\langle q_{12}^p\rangle, \quad \left(\frac{\partial S(t,x)}{\partial x}\right)^2=\frac{1}{2}\langle q_{12}^{p/2}\rangle^2 \tag{62}$$

Note that in the limit of $p=2$, from all the formulas above, we recover the Sherrington–Kirkpatrick picture [5, 25], being the general strategy preserved.

However, the structure of the disordered models is much more complicate to be managed, first of all because the overlap is not self-averaging in a considerable part of the phase space and hence the technique of neglecting the potential in the Hamilton–Jacobi method gives us a solution not

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always correct, but we remind deepening our understanding of these techniques in this context to future works.

7. DISCUSSION

In this paper we showed some features of the p-spin model within the classical theory of ferromagnetism. We showed how recent mathematical breakthrough in the spin glass counterpart can be, once properly adapted, applied to this phenomenology, emphasizing the importance of the techniques themselves for general mean field models.

By introducing two possible toy-Hamiltonians we tested two different methods, the former being the interpolating cavity field technique and the latter being the Hamilton–Jacobi framework.

In the first case we formulated the deterministic version of the so-called *stochastic stability* in the spin glass theory and we used it to show self-consistency and self-averaging properties of the order parameter (the magnetization) in a very simple way.

In the second case we extended the Hamilton–Jacobi technique for two-body interactions to p-spin interaction allowing to obtain self-consistency for the order parameter by imposing its self-averaging.

To close this general discussion on fully connected mean field models without frustration on a lattice a formulation of the ferromagnetic version of the REM has been presented.

At the end a brief introduction of the adaptation of these techniques to disordered models has been outlined.

Future works, still in the field of simple systems, should consider diluted network for the spin interactions.

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