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On the mean-field spin glass transition

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Abstract. In this paper we analyze two main prototypes of disordered mean-field systems, namely the Sherrington-Kirkpatrick (SK) and the Viana-Bray (VB) models, to show that, in the framework of the cavity method, the transition from the annealed regime to a broken replica symmetry phase can be thought of as the failure of the *saturability* property (detailed explained along the paper) of the overlap fluctuations which act as the order parameters of the theory. We show furthermore how this coincides with the lacking of the commutativity of the infinite volume limit with respect to a, suitably chosen, vanishing perturbing field inducing the transition as prescribed by standard statistical mechanics. This is another step towards a complete theory of disordered systems. As a well known consequence it turns out that the annealed and the replica symmetric regions must coincide, implying that the averaged overlap is zero in this phase. Within our framework the finding of the values of the critical point for the SK and line for the VB becomes available straightforwardly and the method is of a large generality and applicable to several other mean field models

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1 Introduction

In simple second order paramagnetic-ferromagnetic transitions (i.e. the paradigmatic Curie-Weiss (CW) model with pairwise Ising interactions [1,2]) the order parameter is of immediate physical meaning and it is straightforward to introduce the proper external field to drive the system into the desired pure state. Then we can check the resulting variation in the macroscopic state of the system by looking at the behavior of the order parameter itself (i.e. the values taken by the magnetization [2]), by which the free energy can be expressed.

In disordered systems, where the external field triggering the transition is not known, much effort has been necessary to describe the "spin glass" transition in terms of a symmetry breaking; this has been achieved within the replica formalism by the well known Parisi RSB scheme [3], recently proved by Guerra [4] and Talagrand [5].

In this paper we show that without using the "replica trick", but dealing only with the simpler global gauge symmetry $\sigma_i \to \sigma_i \sigma_{N+1}$ [6], it is possible to show that the "spin glass" transition can be depicted by the lacking of commutativity of the infinite volume limit (known to hold for a large class of models [7]) against a vanishing perturbation, as in standard statistical mechanics [8]. More sharply we are going to show that it is possible to formulate a property for the overlap monomials, as the order parameters, that we call *saturability*, which is a direct consequence of the gauge symmetry of the models, that stops holding at the transition point. So investigating this property is investigating the transition.

As an ultimate consequence the response to such a field turns out to be the order parameter, again accordingly with standard statistical mechanics.

The work is structured as follows: in Section 2 models and fundamental definitions are introduced, in Sections 3, 4 the general framework and tools for the main theorem are recalled; Sections 5, 6 deal with the main results while Section 7 is left for outlook and conclusions.

2 Models of disordered systems

In what follows we will focus mainly on two well known examples of disordered systems: the first is the SK model, which is a fully connected network of Ising spins $\sigma: i \to \sigma_i = \pm 1$, interacting via a two body Hamiltonian through interaction matrices J_{ij} distributed according to i.i.d. Gaussian $\mathcal{N}(0,1)$ [3]. The second is the VB model, in which the network among the spins is the Poissonian Erdos-Renyi graph built accordingly to P_ζ, that is a Poisson random variable with mean ζ; the coupling J_ν

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are taken to be ± 1 with symmetric distribution and an additional degree of freedom α for connectivity [9,10] is introduced.

- $H_N^{SK}(\sigma; J) = -\frac{1}{\sqrt{N}} \sum_{i<j}^{1,N} J_{ij}\sigma_i\sigma_j$
- $H_N^{VB}(\sigma, \alpha; J) = -\sum_{\nu=1}^{P_{\alpha N}} J_\nu \sigma_{i_\nu}\sigma_{j_\nu}$.

The quantities encoding the thermodynamic properties of the models are the overlaps, which are defined on several configurations (*replicas*) $\sigma^{(1)}, \ldots, \sigma^{(n)}$ by

$$q_{1\cdots n} = \frac{1}{N}\sum_{i=1}^{N} \sigma_i^{(1)} \cdots \sigma_i^{(n)}. \tag{1}$$

In the SK model only the two-replica overlap appears; this is due to the Gaussian nature of the couplings [3].

The expectation with respect to all the (*quenched*) random variables defined so far will be denoted by $\mathbf{E}$, and the Gibbs expectation of an observable $\mathcal{O}(\sigma)$ at inverse temperature β with respect to these Hamiltonian will be denoted by ω and defined as

$$\omega(\mathcal{O}) = Z_N(\beta)^{-1} \sum_\sigma \mathcal{O}(\sigma) \exp(-\beta H_N(\sigma)), \tag{2}$$

and depends clearly on α (for the VB) and β (both for VB and SK), while the normalization factor $Z_N(\beta) = \sum_\sigma \exp(-\beta H_N(\sigma))$ is the partition function. We define $\Omega = \omega_1 ... \omega_n$ the Boltzmann product state among n independent replicas (of course if we deal with just one replica $n = 1$ and $\Omega = \omega$) and the total average $\langle\cdot\rangle = \mathbf{E}\Omega(\cdot)$. The pressure, i.e. minus β times the free energy, is

$$P_N(\alpha, \beta) = \frac{1}{N}\mathbf{E} \ln \sum_\sigma \exp(-\beta H_N(\sigma, \alpha)), \tag{3}$$

and the whole thermodynamics can be obtained once this is explicitly evaluated.

3 Perturbing the measure randomly

For the simpler CW model, the proper way to look at the transition is to menage the infinite volume limit and a vanishing magnetic field, which has the scope of making the system collapse into a minima of the *broken ergodicity* phase. The two operations do not commute in that region. At the contrary, in the ergodic phase, where there is just one minimum of the pressure, given by zero magnetization, the two operations do commute. The breaking of the commutativity can then be used to fix the transition.

In the *broken ergodicity* phase, even though the Hamiltonian shows invariance with respect to a class of transformations (i.e. the $\mathbf{Z}_2$ group [1]), in the presence of at least an infinitesimal external field, the Boltzmann state shares no longer this symmetry and just a subgroup of the original $\mathbf{Z}_2$ group survives (i.e. the linear invariance driven by the deterministic magnetic field).

Dealing with spin glasses and replica trick, the Parisi RSB scheme [3] offers a picture in which a symmetry (the replica symmetry) becomes broken at the critical point and, thanks to the peculiar property of the Parisi ansatz, such breaking can be iterated several times, paving the way to ultrametricity. In other words, the subgroup obtained with a step of replica symmetry breaking has the same structure of the original group itself [3,11].

Incidentally and in opposition to the CW model, in our way of looking at the transition, the perturbation applied to obtain the correct order parameter and behavior at criticality, shares the same property of the Hamiltonian (i.e. for the SK is a Gaussian perturbation). So while trying to formulate in a classical sense the spin glass transition (breaking of the commutativity of the limits as discussed) we modify the *standard* external field inspired, also if in a complete different approach, by the Parisi scheme.

The key idea of our speculation is considering the addition of a new spin, thanks to the randomness of the coupling, as an external random field vanishing in the thermodynamic limit. By the interpolation method [7] the $N+1$ spin can be added to the N-spin system smoothly via a properly defined cavity function $\Psi(t)$, $t \in [0, 1]$, which reads off as:

$$\Psi^{SK}(t) = \mathbf{E} \ln \omega \left(e^{\beta\sqrt{\frac{t}{N}}\sum_i^N J_i\sigma_i} \right) \tag{4}$$

$$\Psi^{VB}(t) = \mathbf{E} \ln \omega \left(e^{\beta \sum_\nu^{P_{2\alpha t}} J_\nu \sigma_{i_\nu}} \right). \tag{5}$$

In the next section, with the use of this function, we will split the free energy of an $N+1$ system in the one of an N system plus the contribution of the remaining spin: we can build the thermodynamic state of $N+1$ particles by performing the limit of $\sqrt{t} \to 1$ and applying the gauge symmetry by considering the transformation $\sigma_i \to \sigma_i\sigma_{N+1}$ (which leaves both the Hamiltonian invariant [12,13]).

Due to the gauge symmetry the above cavity function turns out to be proper two body interaction, as in the original Hamiltonian, and the sum on $N+1$ spins in the partition function is trivially twice the sum on N spins because the $N+1$ plays the role of an hidden variable (the consequent factor 2, after the logarithm is taken, takes into account the high temperature entropy).

With F as a generic function of the spins, we can in fact define a generalized Boltzmann measure (denoted by the subscript $\langle.\rangle_t$) respectively for the SK and for the VB as,

$$\omega_t(F)_{SK} = \frac{\omega\left(Fe^{\beta\sqrt{\frac{t}{N}}\sum_{i=1}^N J_i\sigma_i}\right)}{\omega\left(e^{\beta\sqrt{\frac{t}{N}}\sum_{i=1}^N J_i\sigma_i}\right)}, \tag{6}$$

$$\omega_t(F)_{VB} = \frac{\omega\left(Fe^{\beta\sum_{\nu=1}^{P_{2\alpha t}} J_\nu\sigma_{i_\nu}}\right)}{\omega\left(e^{\beta\sum_{\nu=1}^{P_{2\alpha t}} J_\nu\sigma_{i_\nu}}\right)}. \tag{7}$$

Note that in the $\sqrt{t} = 0$ case we always recover the unperturbed Boltzmann measure of an N-spin system and in the $\sqrt{t} = 1$ case we recover the unperturbed Boltzmann measure of an $N+1$-spin system with a little shift, respectively in the temperature for the SK and in the connectivity for the VB, which becomes negligible in the thermodynamic limit.

As they will be of precious help soon, let us now briefly describe the property of stochastic stability for a large class of averaged overlap correlation functions (*OCF*) $\langle q_{2n}\rangle_t$ (proven in [6,12,14]) by splitting them in two main categories, with two different peculiarities: filled *OCF*, showing *robustness* with respect to a stochastic perturbation, and fillable *OCF*, showing *saturability* with respect to the same perturbation.

Definition 1. *Filled OCF are overlap monomials among s replicas such that each replica appears an even number of times.*
Examples are $\langle q_{12}^2\rangle$, $\langle q_{1234}^2\rangle$ *or* $\langle q_{12}q_{23}q_{13}\rangle$.

Definition 2. *Fillable OCF are overlap monomials among s replicas which become filled when multiplied by an overlap among exactly the lacking replicas to be filled.*
Examples are $\langle q_{12}\rangle$, $\langle q_{1234}\rangle$ *or* $\langle q_{12}q_{13}\rangle$.

Remark 1. For completeness we stress that there exist also non-fillable *OCF* (but they do not play any role here): in the SK, due to the Gaussian nature of the couplings, *OCF* as $\langle q_{12}q_{34}\rangle$ which are fillable for the VB model, are non-fillable (being filled by q_{1234} which is zero for the SK model). Furthermore all the *OCF* built by different numbers of replicas are non-fillable (i.e. $\langle q_{12}q_{123}\rangle$), but, working with zero external field as we do, they are zero by symmetry, being zero all the multi-overlaps built by an odd number of replicas.

Proposition 1. *In the thermodynamic limit the averaged filled OCF are stable with respect to the perturbing field induced by the cavity function. We refer to this property as robustness*

$$\partial_t\langle filled\ OCF\rangle_t = \partial_t\langle filled\ OCF\rangle = 0.$$

Proposition 2. *In the thermodynamic limit, in the whole region of temperature where the gauge symmetry holds (i.e. in the ergodic phase), the fillable OCF become filled*

$$\lim_{N\to\infty}\langle fillable\ OCF\rangle_t = \langle filled\ OCF\rangle.$$

We refer to this last property as saturability.

Examples are $\lim_{N\to\infty}\langle q_{12}\rangle_t = \langle q_{12}^2\rangle$, $\lim_{n\to\infty}\langle q_{1234}\rangle_t = \langle q_{1234}^2\rangle$.

We just sketch the proof of the above propositions and we remaind to [6,12,14] for a detailed discussion and proofs.

Sketched proof. Let us show how the fillable OCF turn out to be filled OCF in the $N\to\infty$ limit, then the stability of the filled OCF will be a straightforward consequence: defining Q_{ab} as a fillable OCF and using Q_{ij} for the product of the filled replicas inside Q_{ab}, and leaving a,b as the non filled replicas we have that

$$\langle Q_{ab}\rangle_t = \left\langle\left(\sum_{ij}\left(\sigma_i^a\sigma_j^b\right)/N^2\right)Q_{ij}(\sigma)\right\rangle_t.$$

Factorizing the state Ω we obtain

$$\langle Q_{ab}\rangle_t = \frac{\mathbf{E}}{N^2}\left(\sum_{ij}\Omega_t\left(\sigma_i^a\sigma_j^bQ_{ij}(\sigma)\right)\right) \tag{8}$$

$$= \frac{\mathbf{E}}{N^2}\left(\sum_{ij}\omega_t\left(\sigma_i^a\right)\omega_t\left(\sigma_j^b\right)\Omega_t(Q_{ij})\right). \tag{9}$$

Now we rewrite the last expression for $\sqrt{t}=1$ and use the hypothesis of preserved gauge symmetry: by applying this symmetry ($\sigma_i\to\sigma_i\sigma_{N+1}$), the states acting on the replicas a and b are $\omega_{t=1}(\sigma_i^a)\to\omega(\sigma_i^a\sigma_{N+1}^a)+O(N^{-1})$ while the remaining product state Ω_t continue to work on a even number of replicas and is not modified (giving rise to the saturability of the filled OCF). Putting all the replicas in a unique product state we have:

$$\omega\left(\sigma_i^a\sigma_{N+1}^a\right)\omega\left(\sigma_i^b\sigma_{N+1}^b\right)\Omega(Q_{ij}) = \Omega\left(\sigma_i^a\sigma_j^b\sigma_{N+1}^a\sigma_{N+1}^bQ_{ij}\right). \tag{10}$$

By the gauge symmetry again we can think of the index $N+1$ as a dumb hidden variable k and multiplying by $1=N^{-1}\sum_{k=1}^N$ in the thermodynamic limit we have the proof.

4 A simple application of the fundamental theorem of calculus

As preannounced in the previous section, once defined the perturbing parameter it is always possible, via the fundamental theorem of calculus, to relate the free energy with its derivative with respect to the chosen parameter, the missing term being the cavity function, so to obtain.

Theorem 1. *In the thermodynamic limit the following relations hold*

$$P_{SK}(\beta)+\frac{\beta}{2}(\partial P_{SK}/\partial\beta) = \ln 2+\Psi_{SK}(t=1,\beta) \tag{11}$$

$$P_{VB}(\alpha)+\alpha(\partial P_{VB}/\partial\alpha) = \ln 2+\Psi_{VB}(t=1,\alpha). \tag{12}$$

Sketched proof. Focusing just on the VB (as the SK case can be obtained by the infinite connectivity limit), it was proven in [15] that

$$P_{VB}(\alpha) = \lim_N\Bigg[\mathbf{E}\ln\Omega\Bigg(\sum_{\sigma_{N+1}}\exp\Bigg(\beta\sum_{\nu=1}^{P_{2\alpha}}J'_\nu\sigma_{k_\nu}\sigma_{N+1}\Bigg)\Bigg) - \mathbf{E}\ln\Omega\left(\exp-\beta\left(H'_N(\alpha/N)\right)\right)\Bigg] \tag{13}$$

where the quenched variables in H' are independent of those in Ω, just like for the first term in the right hand side. The second term of the right hand side is easy to compute, and it is the derivative of P_{VB} multiplied by α, because

$$\frac{\mathbf{E}}{N}\ln\Omega(\exp-\beta(H'_N(\alpha/N)))=$$
$$P_{VB}(\alpha(1+1/N))-P_{VB}(\alpha). \quad (14)$$

This leads to the result to prove, as the gauge invariance of Ω allows to take out the sum over σ_{N+1} as $\ln 2$, and therefore the first term in the right hand side of (13) is precisely Ψ.

One may note that there is not a complete symmetry between the version of Theorem (1) for the SK and for the VB, because in the first the term multiplying the derivative of the pressure with respect to β is $\beta/2$ while the corresponding term for the VB is α and not $\alpha/2$. This is because the simplest version of the Viana-Bray Hamiltonian has a Poisson variable per bond as $H_N = -\sum_{ij}\sum_{\nu=0}^{P_{\alpha/N}} J_{ij}^{\nu}\sigma_i\sigma_j$, while in our VB Hamiltonian each link gets a bond with probability close to α/N for large N (the probabilities of getting two, three, ..., bonds scale as $1/N^2, 1/N^3, \ldots$ so can be neglected in the thermodynamic limit).

Working with directed links (as we do) the probability of having a bond on any undirected link is twice larger as large as for directed link (i.e. $2\alpha/N$) and so, for large N, each site has therefore an average of 2α bonds connecting to it and this explains the reason. We stress however that in this way we allow self-loop but they add just σ-independent constant to H_N and are irrelevant, but we take the advantage of dealing with an H_N which is the sum of i.i.d.

Back to Theorem 1, let us work out the explicit expression of the derivatives with respect to the interpolated parameter (β,α) as this is achievable analytically:

$$dP_{SK}/d\beta = \beta^2\left(1-\langle q_{12}^2\rangle\right)/2 \quad (15)$$
$$dP_{VB}/d\alpha = 2\alpha\sum_n^{\infty}\left(\tanh^{2n}(\beta)/2n\right)\left(1-\langle q_{1\ldots 2n}^2\rangle\right) \quad (16)$$

and let us consider the same derivatives performed on the cavity functions

$$\partial_t\Psi^{SK}(t) = \beta^2(1-\langle q_{12}\rangle_t)/2 \quad (17)$$
$$\partial_t\Psi^{VB}(t) = 2\alpha\sum_n^{\infty}\left(\tanh^{2n}(\beta)/2n\right)(1-\langle q_{1\ldots 2n}\rangle_t). \quad (18)$$

The two operation give similar results but as the derivatives of the free energy (Eqs. (15), (16)) offer only filled *OCF*, the derivatives of the cavity function (Eqs. (17), (18)) offer only fillable *OCF*. The two being related by *saturability*. Furthermore we can expand both the overlap and the multi overlaps via the streaming equations [12,14], in terms of filled *OCF* obtaining as first steps

– SK: $\langle q_{12}\rangle_t = \langle q_{12}^2\rangle\beta^2 t + \langle q_{12}q_{23}q_{13}\rangle\beta^4 t^2/2 + \ldots$

– VB: $\langle q_{1234}\rangle_t = \langle q_{1234}^2\rangle 2\alpha\tanh^2(\beta)t$

$$-3\langle q_{1234}q_{12}q_{34}\rangle 4\alpha^2\tanh^4(\beta)t^2/2 + \ldots$$

and so on. All the *OCF* of the right hand sides show *robustness* (i.e. they are filled or stochastically stable) such that, once pasted into the derivative expression of the cavity functions, the integration is straightforward as it turns out to be polynomial.

5 Validity and failure of saturability

In this section we want to show how by applying blindly the saturability property we are not able to go beyond the replica symmetric region. This can be observed as follows: If we consider the expressions of the cavity functions (4, 5) and apply saturability we fill the *OCF* and the two terms, cavity function and free energy derivative, can be pasted together allowing us to rewrite Theorem 1 as

$$P_{SK}(\beta) = \ln 2 + (\beta/2)(\partial P_{SK}(\beta)/\partial\beta) \quad (19)$$
$$P_{VB}(\alpha,\beta) = \ln 2 + \alpha(\partial P_{VB}(\alpha,\beta)/\partial\alpha). \quad (20)$$

It is straightforward to check that the solutions of these simple ODE are the well known annealed solutions.

This implies that the *saturability* property becomes broken in the broken replica symmetric phase. Not surprisingly this property is ultimately related to the gauge symmetry of the model which is preserved in the annealed or replica symmetric region and broken otherwise.

To obtain a behavior of the free energy below the critical point we must bypass *saturability* by expanding the order parameters via the filled *OCM* around the values $q_{2n}\sim 0$. Defining $\tau = 2\alpha\tanh^2(\beta)$ we obtain

$$P_{SK}(\beta) = \ln 2 + (\beta^2/4)\left[1-(1-\beta^2)\langle q_{12}^2\rangle\right] + (\beta^6/3)\langle q_{12}q_{23}q_{13}\rangle + O\left(\langle q_{ij}^4\rangle\right) \quad (21)$$

$$P_{VB}(\alpha,\beta) = \ln 2 + \left[\frac{\tau}{2}+\frac{\tau}{4}(1-\tau)\langle q_{12}^2\rangle\right] + \frac{\tau^3}{3}\langle q_{12}q_{23}q_{13}\rangle + (\langle q_{ij}^4\rangle + (2\alpha)^{-1}\left[\frac{\tau}{4}+\frac{\tau}{8}\left(1-\tau(2\alpha)^{-1}\right)\langle q_{1234}^2\rangle\right] - \left(3\tau^3/4\right)\langle q_{1234}q_{12}q_{34}\rangle + O\left(\langle q_{ijkl}^4\rangle\right). \quad (22)$$

The stochastic stability of the filled monomials, holding at every temperature, makes the above expansion meaningful also where the previous fails. Is in fact immediate to recognize the expansions of the broken replica symmetry regime [12,13].

Remark 2. Using the above expansions (21, 22) for the free energy we can argue immediately that the critical point for the SK must be $\beta_c = 1$ and the critical line for the VB must be $2\alpha\tanh^2(\beta)=1$: this can be seen as follows: let us note that $A_{SK}(\beta) = (\beta^2/4)(1-\beta^2)$ is the coefficient of the second order of the expansion in power of the order parameter q_2 for the SK model (and let us restrict our argument to it as there is complete analogy to the VB). In the ergodic phase (with preserved symmetry) the minimum of the free energy corresponds to a zero order parameter (i.e. $q^2=0$). This implies that $A(\beta)\geq 0$. Anyway, immediately below the critical point values of the order parameter different from zero are possible if and only if $A(\beta) \leq 0$ and consequently at the critical point $A(\beta)$ must be zero.

This identifies the critical point for the SK and the critical line for the VB.

6 Lacking of the volume limit commutativity

Let us explain in other terms the concept of the previous section: we are going to prove now that the transition annealed-RSB phase can be thought of as the lacking of the commutativity of the two operations of fixing the modulus of the external strength ($\sqrt{t} = 1$) and letting $\lim_{N\to\infty}$. Let us rewrite Theorem 1 as

$$P_{SK}(\beta) = \ln 2 + \frac{\beta^2}{2}(1 - \langle q_{12}^2 \rangle) + \frac{1}{2}\int_0^\infty dt(1 - \langle q_{12}\rangle_t)$$
$$P_{VB}(\alpha,\beta) = \ln 2 + \sum_n \frac{\alpha}{n}\tanh^{2n}(\beta)(1 - \langle q_{2n}^2\rangle) + \int_0^\infty \sum_n \frac{\alpha}{n}\tanh^{2n}(\beta)(1 - \langle q_{12}\rangle_t) dt) \quad (23)$$

when performing the infinite volume limit first, we have that all the fillable OCF turn out to be filled OCF so $\langle q_{2n}\rangle_t \to \langle q_{2n}^2\rangle$ and there is nor streaming to be applied neither integration to be done because higher order OCF are killed, being $\langle q_{2n}^2\rangle$ filled; setting $\sqrt{t} \to 1$ to free the measure we get the replica symmetric solution for both the system

$$P_{SK}(\beta) = \ln 2 + \frac{\beta^2}{4}\left(1 - \langle q_{12}^2\rangle\right) \quad (24)$$
$$P_{VB}(\alpha,\beta) = \ln 2 + \alpha\sum_{n=1}^\infty \frac{1}{2n}\tanh^{2n}(\beta)(1 - \langle q_{2n}\rangle). \quad (25)$$

In the region where this solution holds, *saturability* holds too, as it is used to get the solution: this implies $\langle q_{2n}\rangle = 0$ because, always in the large N-limit $\langle q_{2n}\rangle_t = \langle q_{2n}^2\rangle$ such that the only possible solution is $\langle q_{2n}\rangle = 0$ as $\langle q_{2n}(\beta = 0)\rangle = 0$ and this result must be stable in the whole β-region where saturability holds, such that the replica symmetric solution coincides with the annealed expression (i.e. $\ln 2 + \beta^2/4$ for the SK).

6.1 The order parameter

As the last remark, focusing just on the SK for the sake of simplicity, we want to show that the order parameter coupled with our perturbing field is exactly the overlap between two replicas. Let us think at the cavity field as to an external field of strength $h \equiv \sqrt{t/N}$. This is not surprising as the tuning parameter $\sqrt{t}$ can be thought of as the amplitude of the Gaussian perturbation carried by the J_i, in the interesting regime in which it vanishes in the thermodynamic limit thanks to the $\sqrt{N}$ as its denominator.

Let us see what the response to this external field is:

$$\left(\partial_{\sqrt{t}}\left(\beta^{-1}\mathbf{E}\ln\sum_\sigma e^{-\beta H_{SK}(\sigma;J) + \sqrt{\beta\frac{t}{N}}\sum_i J_i\sigma_i}\right)\right)_{t=1} = \left(\frac{\mathbf{E}\sqrt{t}\sum_i}{\beta N}(1 - \omega^2(\sigma_i))\right)_{t=1} = \beta^{-1}(1 - \langle q_{12}\rangle). \quad (26)$$

So, again as in standard statistical mechanics [8], the response to our kind of perturbation is the order parameter, which, in order to bound $(1 - \langle q_{12}\rangle) \in [0,1]$ must live on $[0,1]$ too, according to Parisi theory.

7 Conclusions and outlook

In this paper we analyzed the stability of the quenched-Boltzmann state, for mean field spin glasses, with respect to a fundamental invariance of a huge class of Hamiltonian: the global gauge symmetry.

We tried to perform an approach as close as possible to the standard receipt of statistical mechanics but the natural choice for the external field has been a random field with the same symmetry of the coupling of the Hamiltonian: this is due in our framework to the equivalence of stochastic perturbation and cavity field, which are two concept directly interchangeable for gauge-invariant models. This seems a generalization for the phase transition triggered by a vanishing external field.

We tested our procedure focusing of the fully connected Gaussian model (Sherrington-Kirkpatrick) and on the Poissonian diluted model (Viana-Bray) obtaining a complete agreement with replica-theory [3,9] and previous rigorous results [5,10,12].

We proved that the transition from the annealed regime to a broken replica phase coincide with the breaking of the this invariance by the quenched Boltzmann state encoded in the failure of a suitable property of the order parameters: *saturability*. As a consequence the annealing and the replica symmetric solution must be the same and in this phase the order parameters $\langle q_{2n}\rangle$ must be zero.

As a first result the transition can be managed dealing with the infinite volume limit and the strength of the external perturbation obtaining the RS scenario or the RSB one depending on when N is send to infinity. As a second result the derivative of the free energy with respect to the strength of this perturbation gives the overlap, as the coupled order parameter. Both the results are standard in non disorder statistical mechanics, shorting our understanding of disordered systems by such a viewpoint.

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