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J. Stat. Mech. (2007) P08025

(http://iopscience.iop.org/1742-5468/2007/08/P08025)

Journal of Statistical Mechanics: Theory and Experiment
An IOP and SISSA journal

Stability properties and probability distributions of multi-overlaps in dilute spin glasses

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Received 24 April 2007
Accepted 30 July 2007
Published 21 August 2007

Online at stacks.iop.org/JSTAT/2007/P08025
doi:10.1088/1742-5468/2007/08/P08025

Abstract. We prove that the Aizenman–Contucci relations, well known for fully connected spin glasses, hold in diluted spin glasses as well. We also prove more general constraints in the same spirit for multi-overlaps, systematically confirming and expanding previous results. The strategy that we employ makes no use of self-averaging, and allows us to generate hierarchically all such relations within the framework of random multi-overlap structures. The basic idea is to study, for these structures, the consequences of the closely related concepts of stochastic stability, quasi-stationarity under random shifts, factorization of the trial free energy. The very simple technique allows us to prove also the phase transition for the overlap: it remains strictly positive (on average) below the critical temperature if a suitable external field is first applied and then removed in the thermodynamic limit. We also deduce, from a cavity approach, the general form of the constraints on the distribution of multi-overlaps found within quasi-stationary random multi-overlap structures.

Keywords: rigorous results in statistical mechanics, cavity and replica method, spin glasses (theory)

ArXiv ePrint: cond-mat/0612041

Contents

1. Introduction

Dilute spin glasses are studied mostly for two reasons: their finite connectivity makes them in a certain sense close to finite-dimensional systems, while retaining a mean field character; and they are mathematically equivalent to some important random optimization problems (such as X-OR-SAT and K-SAT [15]). The proper setting for the study of mean field dilute spin glasses are the random multi-overlap structures (RaMOSt), and the whole physics behavior of dilute spin glasses is carried by the probability distribution of the multi-overlaps [7], which play the same role as the 2-overlap does for fully connected models. In the case of the latter, it is known that the Ghirlanda–Guerra identities [12] allow for the computation of the critical exponents governing the critical behavior of the 2-overlap [1], and guarantee that the 2-overlap is positive below the critical temperature [15]. The relations due to Aizenman and Contucci [2], on the other hand, imply [1] that the expectation of the 2-overlap is strictly positive below the critical temperature (due to a phase transition triggered by an external field). Ghirlanda–Guerra identities are a consequence of the self-averaging of the energy density, and extend to dilute spin glasses, where one can also find more general relations for multi-overlaps [9]. By contrast, Aizenman–Contucci (AC) relations are a consequence of stochastic stability [2, 5], but they also follow from a certain kind of self-averaging, as shown by Franz *et al* [10], who extended the AC relations to dilute spin glasses and multi-overlaps. In this paper we provide a new proof of the AC relations for dilute spin glasses, and of their generalized version for multi-overlaps. We emphasize that stochastic stability (and similar concepts) are intimately related to self-averaging properties. Moreover, both approaches can be used with observable of various forms, not just the with the energy. The joint use of the techniques developed in [1] within the approach developed in [7] is at the basis of the present work. The latter is organized as follows. The next two sections introduce the model and our notation, and illustrate the cavity perspective which the RaMOSt approach relies on. In section 4 we show that simple symmetry arguments within the cavity method lead to the proof of the phase transition of the expectation of the overlap: below its critical temperature the overlap remains strictly positive, if an external (cavity) field is applied and then removed in the thermodynamic limit. Section 5 is devoted to a proof that AC relations hold in dilute glasses, along with relations in the same spirit for multi-overlaps. We have already stressed that our proof is radically different from the one hinted at in [10]. Section 6 presents the form of the derivative with respect to a perturbing parameter of the expectation of a generic function of some replicas. The result makes it possible to develop systematically the constraints on multi-overlaps, whose critical behavior control can be here improved as compared to section 4 (although the critical exponents are not found yet). The straightforward but tedious and long calculations needed in some expansions are reported in the appendices, preceded by concluding remarks.

2. Model and notation

Consider N points, indexed by italic letters i, j, etc, with an Ising spin attached to each of them, so to have spin configurations

$$\sigma : \{1, \dots, N\} \ni i \to \sigma_i = \pm 1.$$

Hence we may consider $\sigma \in \{-1, +1\}^N$. Let P_ζ be a Poisson random variable of mean ζ, let $\{J_\nu\}$ be independent identically distributed copies of a random variable J with symmetric distribution. For the sake of simplicity we will assume $J = \pm 1$, without loss of generality [13]. We want to consider randomly chosen points, we therefore introduce $\{i_\nu\}, \{j_\nu\}$ as independent identically distributed random variables, with uniform distribution over $1, \dots, N$. Assuming there is no external field, the Hamiltonian of the Viana–Bray (VB) model for dilute mean field spin glass is the following symmetric random variable

$$H_N(\sigma, \alpha; \mathcal{J}) = -\sum_{\nu=1}^{P_{\alpha N}} J_\nu \sigma_{i_\nu} \sigma_{j_\nu}, \qquad \alpha \in \mathbb{R}_+.$$

$\mathbb{E}$ will be the expectation with respect to all the (quenched) variables, i.e. all the random variables except the spins, collectively denoted by $\mathcal{J}$. The non-negative parameter α is called *degree of connectivity*. The Gibbs measure ω is defined by

$$\omega(\varphi) = \frac{1}{Z} \sum_\sigma \exp(-\beta H(\sigma)) \varphi(\sigma)$$

for any observable $\varphi : \{-1, +1\}^N \to \mathbb{R}$, and clearly

$$Z_N(\beta) = \sum_\sigma \exp(-\beta H_N(\sigma)),$$

which is the well known partition function. When dealing with more than one configuration, the product Gibbs measure is denoted by Ω, and various configuration taken from the each product space are called 'replicas'. As already done above, we will often omit the dependence on β and on the size of the system N of various quantities. In general, we will commit some slight notational abuses to lighten the expressions when there is no risk of confusion. The free energy density f_N is defined by

$$-\beta f_N(\beta) = \frac{1}{N}\mathbb{E}\ln Z_N(\beta).$$

The whole physical behavior of the model is encoded by [7] the even multi-overlaps $q_{1\cdots 2n}$, which are functions of several configurations $\sigma^{(1)}, \sigma^{(2)}, \ldots$ defined by

$$q_{1\cdots 2n} = \frac{1}{N}\sum_{i=1}^{N}\sigma_i^{(1)}\cdots\sigma_i^{(2n)}.$$

3. Cavity approach and random multi-overlap structures

The thermodynamic limit of the free energy density exists if and only if the sequence of the increments (due to the addition of a particle to the system) is convergent in the Cesàro sense (indicated by a boldface $\mathbf{C}$):

$$\lim_{M\to\infty}\frac{1}{M}\mathbb{E}\ln Z_M \equiv \lim_{M\to\infty}\frac{1}{M}\sum_{n=0}^{M-1}\mathbb{E}\ln\frac{Z_{n+1}}{Z_n} \equiv \mathbf{C}\lim_{M\to\infty}\mathbb{E}\ln\frac{Z_{M+1}}{Z_M}.$$

The idea at the basis of the cavity approach is in fact to measure the effect on the free energy of the addition of one spin to the system (see [4] for a beautiful summary). Let us denote the given M spins by τ, as we want to save the symbol σ for the added spin(s). Now, following [7], we can write, in distribution,

$$-H_{M+1}(\tau,\sigma;\alpha) \sim \sum_{\nu=1}^{P_{\alpha(M^2/(M+1))}} J_\nu\tau_{k_\nu}\tau_{l_\nu} + \sum_{\nu=1}^{P_{\alpha(2M/(M+1))}} \tilde{J}_\nu\tau_{m_\nu}\sigma_{i_\nu}, \tag{1}$$

where we have neglected a term which does not contribute when M is large [7], $\{\tilde{J}_\nu\}$ are independent copies of J; $\{k_\nu\}$, $\{m_\nu\}$, and $\{l_\nu\}$ are independent random variables all uniformly distributed over $\{1,\ldots,M\}$; $\{i_\nu\}$ are independent random variables uniformly distributed over the set $\{1,\ldots,N\equiv 1\}$, consisting of $\{1\}$ only. So $\sigma_{i_\nu}\equiv\sigma_1$. Notice that we can also write, in distribution,

$$H_{M+1}(\tau,\sigma;\alpha) \sim H_M(\tau;\alpha') + \tilde{h}_\tau\sigma_1 \tag{2}$$

where

$$\alpha' = \alpha\frac{M}{M+1}, \qquad \tilde{h}_\tau = -\sum_{\nu=1}^{P_{2\alpha'}}\tilde{J}_\nu\tau_{k_\nu}.$$

Notice also that similarly

$$H_M(\tau;\alpha) = H_M(\tau;\alpha',\mathcal{J}) + H_M(\tau;\alpha'/M,\hat{\mathcal{J}}), \tag{3}$$

thanks to the additivity property of Poisson variables, and the two Hamiltonians in the right-hand side have independent quenched random variables $\mathcal{J},\hat{\mathcal{J}}$. Hence, if we call

$$H_M(\tau;\alpha'/M,\hat{\mathcal{J}}) = \hat{H}_\tau(\alpha') = -\sum_{\nu=1}^{P_{\alpha'}}\hat{J}_\nu\tau_{k_\nu}\tau_{l_\nu},$$

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then

$$\mathbb{E}\ln\frac{Z_{M+1}}{Z_M}=\mathbb{E}\ln\frac{\sum_{\tau,\sigma}\xi_\tau\exp(-\beta\tilde{h}_\tau\sigma)}{\sum_\tau\xi_\tau\exp(-\beta\hat{H}_\tau)},$$

with

$$\xi_\tau=\exp(-\beta H_M(\tau;\alpha')).$$

As elegantly explained in [4], this equation expresses the incremental contribution to the free energy in terms of the mean free energy of a particle (a spin) added to a reservoir whose internal state is described by (τ,ξ_τ), corrected by an inverse-fugacity term $\hat{H}$, which encodes a connectivity shift. The latter may be thought of as the free energy of a 'place holder': the *cavity* into which the $(M+1)$st particle is added. One may note that the addition of a particle to the reservoir of M particles has an effect on the state of the reservoir. For $M\gg 1$, the value of the added spin, σ, does not affect significantly the field which would exist for the next increment in M. Hence, for the next addition of a particle we may continue to regard the state of the reservoir as given by just the configuration τ. However, the weight of the configuration (which is still to be normalized to yield its probability) undergoes the change:

$$\xi_\tau\to\xi_\tau\mathrm{e}^{-\beta\tilde{h}_\tau\sigma}.$$

This transformation is called *cavity dynamics*.

When we add more particles to the system, they do not interact, as there will just be copies of the cavity fields $\tilde{h}^i$ acting paramagnetically on each added spin (see [7] for details). Therefore if we add infinitely many particles (to an already infinite reservoir), we can replace the initial complicated model with a simpler (at least in principle) paramagnet. The reasoning just illustrated thus paves the way to the proper concept to introduce for the computation of the free energy [7].

Definition 1. *Given a probability space* $\{\Omega,\mu(\mathrm{d}\omega)\}$*, a Random multi-overlap structure* $\mathcal{R}$ *is a triple* $(\Sigma,\{\tilde{q}_{2n}\},\xi)$ *where*

- Σ *is a discrete space;*
- $\xi:\Sigma\to\mathbb{R}_+$ *is a system of random weights, such that* $\sum_{\gamma\in\Sigma}\xi_\gamma\le\infty$ μ*-almost surely;*
- $\tilde{q}_{2n}:\Sigma^{2n}\to\mathbb{R}, n\in\mathbb{N}$ *is a positive semi-definite* multi-overlap kernel *(equal to 1 on the diagonal of* Σ^{2n}*, so that by Schwartz inequality* $|\tilde{q}|\le 1$*).*

By looking at the properties of $\tilde{h},\hat{H}$ in (2) and (3), we know that when many particles (say N) are added to the system, we need [7] in general $N+1$ random variables $\{\tilde{h}^i_\gamma(\alpha;\tilde{J})\}_{i=1}^N$ and $\hat{H}_\gamma(\alpha;\hat{J})$, $\gamma\in\Sigma$, such that

$$\frac{\mathrm{d}}{\mathrm{d}\alpha}\mathbb{E}\ln\sum_{\gamma\in\Sigma}\xi_\gamma\exp(-\beta\tilde{h}^i_\gamma)=2\sum_{n>0}\frac{1}{2n}\tanh^{2n}(\beta)(1-\langle\tilde{q}_{2n}\rangle)\tag{4}$$

$$\frac{\mathrm{d}}{\mathrm{d}\alpha}\mathbb{E}\ln\sum_{\gamma\in\Sigma}\xi_\gamma\exp(-\beta\hat{H}_\gamma)=\sum_{n>0}\frac{1}{2n}\tanh^{2n}(\beta)(1-\langle\tilde{q}^2_{2n}\rangle).\tag{5}$$

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These are the fields to plug into the 'trial pressure' $(-\beta f_N(\beta))$

$$G_N(\mathcal{R}) = \frac{1}{N}\mathbb{E}\ln\frac{\sum_{\gamma,\sigma}\xi_\gamma\exp(-\beta\sum_{i=1}^N\tilde{h}^i_\gamma\sigma_i)}{\sum_\gamma\xi_\gamma\exp(-\beta\hat{H}_\gamma)}. \tag{6}$$

The Boltzmann RaMOSt [7] is the one we started from, constructed by thinking of a reservoir of M spins τ

$$\Sigma = \{-1,1\}^M \ni \tau, \qquad \xi_\tau = \exp(-\beta H_M(\tau)), \qquad \tilde{q}_{1\cdots 2n} = \frac{1}{M}\sum_{k=1}^M \tau_k^{(1)}\cdots\tau_k^{(2n)}$$

with

$$\tilde{h}^i_\tau(\alpha) = \sum_{\nu=1}^{P_{2\alpha}}\tilde{J}^i_\nu\tau_{k^i_\nu}, \qquad \hat{H}_\tau(\alpha N) = -\sum_{\nu=1}^{P_{\alpha N}}\hat{J}_\nu\tau_{k_\nu}\tau_{l_\nu}$$

and all the $\hat{J}$'s are independent copies of J, independent of any other copy.

The next theorem will not be used, but it justifies the whole machinery described so far.

Theorem 1 (Extended Variational Principle). *Taking the infimum for each N separately the trial function $G_N(\mathcal{R})$ over the whole RaMOSt space, the resulting sequence tends to the limiting pressure $-\beta f$ of the VB model as N tends to infinity:*

$$-\beta f = \lim_{N\to\infty}\inf_{\mathcal{R}} G_N(\mathcal{R}).$$

A RaMOSt $\mathcal{R}$ is said to be optimal if $G(\mathcal{R}) = -\beta f(\beta)\forall\beta$. We will denote by Ω the measure associated with the RaMOSt weights ξ as well.

Is it possible to show [7] that optimal RaMOSts enjoy the same factorization property enjoyed by the Boltzmann RaMOSt, described in the following.

Theorem 2 (factorization of optimal RaMOSts). *In the whole region where the parameters are uniquely defined, the following Cesàro limit is linear in N and $\bar{\alpha}$*

$$\mathbf{C}\lim_M \mathbb{E}\ln\Omega_M\{c_1\cdots c_N\exp[-\beta\hat{H}(\bar{\alpha})]\} = N(-\beta f + \alpha A) + \bar{\alpha}A,$$

where $c_i = 2\cosh(\beta\tilde{h}^i)$,

$$A = \sum_{n=1}^\infty \frac{1}{2n}\tanh^{2n}(\beta)(1 - \langle q_{2n}^2\rangle), \tag{7}$$

and the averages in both sides of the equation are assumed to taken by means of weights at connectivity α.

As we extended the use of Ω from the Gibbs measure to any RaMOSt measure, we are clearly extending to any RaMOSt the notation $\mathbb{E}\Omega(\cdot) = \langle\cdot\rangle$ too. We will use only part of the previous theorem, or more precisely a modification of the part involving the inverse fugacity only. Notice that in the definition (6) of the trial pressure G the part with the cavity fields and the part with the inverse fugacity are taken already factorized (the inverse fugacity appears at the denominator). If we therefore focus on the fugacity part only in the theorem above, by setting all the cavity fields $\tilde{h}^i$ to zero, the property described in theorem 2 becomes what is often called *stochastic stability* (of the measure Ω with respect to the perturbation $\hat{H}$), which we will prove and exploit in section 5. It should be now clear, from the construction described about equations (2) and (3), that this perturbation can be either due to the addition of N particles, or due to a connectivity shift, and leads to a linear response of the free energy. Hence the theorem above combines two invariance properties of the optimal RaMOSt measure Ω: the one of the cavity part with respect to the cavity dynamics, and the one of the fugacity part with respect to connectivity shifts. The invariance with respect to the cavity dynamics is a special case of *quasi-stationarity*, i.e. the invariance up to a correcting factor under random shifts (see [3, 4] for a detailed introduction). The Parisi ultrametric ansatz, both for dilute and for fully connected Gaussian models [15], is based on hierarchical random probability cascades, which exhibit the quasi-stationarity of the generalized random energy model [4, 15]. A very stimulating conjecture is that random probability cascades include all the quasi-stationary structures.

As an aside remark, we point out that for stochastically stable systems, a dynamical order parameter can be defined and related to the static order parameter. Ultrametricity in the dynamics implies static ultrametricity, which is in turn implied by so-called *separability*, and connected to the idea of *overlap equivalence*. We refer to [11, 14, 9] for details, here we just wish to stress that all these concepts and the one of self-averaging are intimately related, and have deep physical meanings.

4. Non-negativity of the average of multi-overlaps

We know from [8, 13] that above the critical temperature β_c all the multi-overlaps (including the 2-overlap) are identically zero, as the replica symmetric solution holds. We also know, from [13], that the (rescaled) 2-overlap shows diverging fluctuations at the critical temperature β_c where the replica symmetry is broken, while the (rescaled) multi-overlaps of more than two replicas do not exhibit diverging fluctuations at this inverse temperature. If the expression of the fluctuations of the rescaled multi-overlaps given in [13] could be proven to be valid down to suitable lower temperatures, we would have what is physically a common belief [16], i.e. that the critical temperature $\beta_c^{(2n)}$ at which the fluctuations of $\sqrt{N}q_{2n}$ diverge is given by

$$\tanh^{2n}(\beta_c^{(2n)}) = \frac{1}{2\alpha}, \qquad \alpha > \frac{1}{2}.$$

So that q_{2n} would be zero up to $\beta_c^{(2n)}$, where it would start its concave increase toward 1 as $\beta \to \infty$.

We want to show that the 2-overlap exhibit the same phase transition as in the Gaussian SK model [1]: if we apply an external field and then remove it in the thermodynamic limit, the 2-overlap remains strictly positive below its critical temperature, where its variance becomes non-zero. The same is expected to hold for all multi-overlaps.

Let us introduce the following notation: $\omega_{\tilde{\alpha}}(\cdot), \langle\cdot\rangle_{\tilde{\alpha}}$ denote the usual expectations except for a perturbation in the *Boltzmannfaktor*, which is assumed here to be

$$\exp\left[-\beta\left(H_N(\sigma;\alpha) - \sum_{i=1}^{N}\left(\sum_{\nu=1}^{P^i_{\tilde{\alpha}}} \tilde{J}^i_\nu\right)\sigma_i\right)\right],$$

doi:10.1088/1742-5468/2007/08/P08025

i.e. the initial *Boltzmannfaktor* is perturbed with independent copies of an external field $\tilde{h}(\bar{\alpha}) = \sum_{\nu=1}^{P_{\bar{\alpha}}} \tilde{J}_\nu$ modulated by $\bar{\alpha}$.

This section is devoted to the next

Theorem 3. *The following holds:*

(1) for any inverse temperature β

$$\lim_{N\to\infty} \langle q_{2n} \rangle_{\bar{\alpha}=2\alpha/N} \geq 0;$$

(2) for any $\beta > \beta_c^{(2)}$, *defined by* $2\alpha \tanh^2(\beta_c^{(2)}) = 1$,

$$\lim_{N\to\infty} \langle q_{12} \rangle_{\bar{\alpha}=2\alpha/N} > 0.$$

We will see that our method would imply immediately the analogous statement for all multi-overlaps, should the formula for their fluctuation be proven to hold at lower temperatures as well.

The theorem will be a simple consequence of two lemmas, which require a definition as well.

Lemma 1. *Consider the set of indices* $\{i_1, \ldots, i_r\}$, *with* $r \in [1, N]$. *Then*

$$\lim_{\bar{\alpha}\to 2\alpha/N} \omega_{N,\bar{\alpha}}(\sigma_{i_1} \cdots \sigma_{i_r}) = \omega_{N+1}(\sigma_{i_1} \cdots \sigma_{i_r} \sigma_{N+1}^r) + \mathrm{O}\left(\frac{1}{N}\right),$$

where r *is an exponent (not a replica index) and we have made explicit the dependence of* ω *on the size of the system.*

This lemma is a consequence of the fact that with the chosen $\bar{\alpha}$ the presence of the external field is equivalent to the introduction of an additional particle, labelled $N + 1$. This should be clear from (1) and (2) above, and from the gauge symmetry with respect to the transformation $\sigma_i \to \sigma_i \sigma_{N+1}$, but the reader may want to refer to lemma 1 in [1] for details. Notice that in the case of monomials with r even the previous lemma tells us that the presence of the external field has no influence in the thermodynamic limit. This will be important in the rest, and we hence proceed with the next

Definition 2. *A polynomial function of some overlaps is called*

- filled *if every replica appears an even number of times in it;*
- fillable *if it can be made filled by multiplying it by exactly one multi-overlap of appropriately chosen replicas.*

The exponent r is always even for filled polynomials, whose average is therefore not altered by the external field, so to have, for instance

$$\langle q_{12} q_{23} q_{13} \rangle_{\bar{\alpha}} = \langle q_{12} q_{23} q_{13} \rangle.$$

From the previous lemma we will deduce the following.

Proposition 1. *Let* $Q_{1\cdots 2n}$ *be a fillable polynomial of the overlaps, such that* $q_{1\cdots 2n} Q_{1\cdots 2n}$ *is filled. Then*

$$\lim_{N\to\infty} \langle Q_{1\cdots 2n} \rangle_{\bar{\alpha}=2\alpha/N} = \langle q_{1\cdots 2n} Q_{1\cdots 2n} \rangle,$$

where the right-hand side is understood to be evaluated in the thermodynamic limit.

Proof. Let us assume for a generic overlap correlation function Q, of s replicas, the following representation

$$Q = \prod_{a=1}^{s} \sum_{i_l^a} \prod_{l=1}^{n^a} \sigma^a_{i_l^a} I(\{i_l^a\})$$

where a labels the replicas, the internal product takes into account the spins (labelled by l) which contribute to the a part of the overlap $q_{a,a'}$ and runs to the number of time that the replica a appears in Q, the external product takes into account all the contributions of the internal one and the I factor fixes the constraints among different replicas in Q; so, for example, $Q = q_{12}q_{23}$ can be decomposed in this form noting that $s = 3$, $n^1 = n^3 = 1, n^2 = 2$, $I = N^{-2}\delta_{i_1^1,i_1^3}\delta_{i_1^2,i_2^3}$, where the δ functions fixes the links between replicas $1, 2 \to q_{1,2}$ and $2, 3 \to q_{2,3}$. The averaged overlap correlation function is

$$\langle Q \rangle = \sum_{i_l^a} I(\{i_l^a\}) \prod_{a=1}^{s} \omega_\alpha \left(\prod_{l=1}^{n^a} \sigma^a_{i_l^a} \right).$$

Now if Q is a fillable polynomial, and we evaluate it at $\bar{\alpha} = 2\alpha/N$, let us decompose it, using the factorization of the ω state on different replica, as

$$\langle Q \rangle = \sum_{i_l^a, i_l^b} I(\{i_l^a\}, \{i_l^b\}) \prod_{a=1}^{u} \omega_a \left(\prod_{l=1}^{n^a} \sigma^a_{i_l^a} \right) \prod_{b=u}^{s} \omega_b \left(\prod_{l=1}^{n^b} \sigma^b_{i_l^b} \right),$$

where u stands for the number of the unfilled replicas inside the expression of Q. So we split the measure Ω into two different subset ω_a and ω_b: in this way the replica belonging to the b subset are always in even number, while the ones in the a subset are always odds. Applying the gauge $\sigma_i^a \to \sigma_i^a \sigma_{N+1}^a, \forall i \in (1, N)$ the even measure is unaffected by this transformation ($\sigma_{N+1}^{2n} \equiv 1$) while the odd measure takes a σ_{N+1} inside the Boltzmann measure (lemma 1).

$$\langle Q \rangle = \sum_{i_l^a, i_l^b} I(\{i_l^a\}, \{i_l^b\}) \prod_{a=1}^{u} \omega \left(\sigma^a_{N+1} \prod_{l=1}^{n^a} \sigma^a_{i_l^a} \right) \prod_{b=u}^{s} \omega \left(\sigma^b_{N+1} \prod_{l=1}^{n^b} \sigma^b_{i_l^b} \right).$$

At the end we can replace in the last expression the subindex $N + 1$ of σ_{N+1} by k for any $k \neq \{i_l^a\}$ and multiply by one as $1 = N^{-1}\sum_{k=0}^{N}$. Up to orders $O(1/N)$, which go to zero in the thermodynamic limit, we have the proof. □

At this point the first part of theorem 3 is simply a corollary of this proposition in the case $Q = q$. The second part follows from the fact that all multi-overlaps (including the 2-overlap) are zero above the critical temperature $1/\beta_c^{(2)}$, and the 2-overlap starts fluctuating below, so that $\langle q_2^2 \rangle$ is strictly positive and coincides (in the limit) with $\langle q_2 \rangle$ by proposition 1. The fact that $\langle q_2^2 \rangle$ is strictly positive for $\beta > \beta_c^{(2)}$ is obvious from the fact that the replica symmetric solution does not hold in this region [13].

We will discuss the generalization of theorem 3 to multi-overlaps in [6].

5. Stability relations from quasi-stationarity

In this section we want to prove the following.

Theorem 4. *The consequences of stochastic stability in fully connected models extend to dilute spin glasses, and constraints analogous to those found for overlaps of two replicas only hold for multi-overlaps. More precisely,*

(1) the Aizenman–Contucci relations hold in dilute spin glasses. A first example is

$$\langle q_{12}^2 q_{13}^2 \rangle = \tfrac{1}{4}\langle q_{12}^4 \rangle + \tfrac{3}{4}\langle q_{12}^2 q_{34}^2 \rangle;$$

(2) further relations for multi-overlaps hold in dilute spin glasses. A first example is

$$\langle q_{1234}^2 q_{15}^2 \rangle = \tfrac{3}{8}\langle q_{1234}^2 q_{12}^2 \rangle + \tfrac{5}{8}\langle q_{1234}^2 q_{56}^2 \rangle.$$

We start addressing the proof of the theorem by proving a lemma that gives the explicit form of the contribution to the free energy of a connectivity shift.

Lemma 2. *Let $\Omega, \langle\cdot\rangle$ be the usual Gibbs and quenched Gibbs expectations at inverse temperature β, associated with the Hamiltonian $H_N(\sigma,\alpha;\mathcal{J})$. Then, in the whole region where the parameters are uniquely defined*

$$\lim_{N\to\infty} \mathbb{E}\ln\Omega\exp\left(\beta'\sum_{\nu=1}^{P_{\alpha'}} J'_\nu \sigma_{i'_\nu}\sigma_{j'_\nu}\right) = \alpha'\sum_{n=1}^{\infty}\frac{1}{2n}\tanh^{2n}(\beta')(1-\langle q_{2n}^2\rangle), \tag{8}$$

where the random variables $P_{\alpha'}, \{J'_\nu\}, \{i'_\nu\}, \{j'_\nu\}$ are independent copies of the analogous random variables appearing in the Hamiltonian in Ω.

Notice that, in distribution

$$\beta\sum_{\nu=1}^{P_{\alpha N}} J_\nu\sigma_{i_\nu}\sigma_{j_\nu} + \beta'\sum_{\nu=1}^{P_{\alpha'}} J'_\nu\sigma_{i'_\nu}\sigma_{j'_\nu} \sim \beta\sum_{\nu=1}^{P_{(\alpha+\alpha'/N)N}} J''_\nu\sigma_{i_\nu}\sigma_{j_\nu} \tag{9}$$

where $\{J''_\nu\}$ are independent copies of J with probability $\alpha N/(\alpha N+\alpha')$ and independent copies of $J\beta'/\beta$ with probability $\alpha'/(\alpha N+\alpha')$. In the right-hand side above, the quenched random variables will be collectively denoted by $\mathcal{J}''$. Notice also that the sum of Poisson random variables is a Poisson random variable with mean equal to the sum of the means, and hence we can write

$$A_t \equiv \mathbb{E}\ln\Omega\exp\left(\beta'\sum_{\nu=1}^{P_{\alpha' t}} J'_\nu\sigma_{i'_\nu}\sigma_{j'_\nu}\right) = \mathbb{E}\ln\frac{Z_N(\alpha_t;\mathcal{J}'')}{Z_N(\alpha;\mathcal{J})}, \tag{10}$$

where we defined, for $t\in[0,1]$,

$$\alpha_t = \alpha + \alpha'\frac{t}{N} \tag{11}$$

so that $\alpha_t \to \alpha\ \forall\ t$ as $N\to\infty$.

Proof. Let us compute the t-derivative of A_t, as defined in (10)

$$\frac{\mathrm{d}}{\mathrm{d}t}A_t = \mathbb{E}\sum_{m=1}^{\infty}\frac{\mathrm{d}}{\mathrm{d}t}\pi_{\alpha' t}(m)\ln\sum_\sigma\exp\Big(\beta'\sum_{\nu=1}^{m} J'_\nu\sigma_{i'_\nu}\sigma_{j'_\nu}\Big).$$

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Using the following elementary property of the Poisson measure

$$\frac{\mathrm{d}}{\mathrm{d}t}\pi_{t\zeta}(m) = \zeta(\pi_{t\zeta}(m-1) - \pi_{t\zeta}(m)) \tag{12}$$

we get

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}A_t &= \alpha'\mathbb{E}\sum_{m=0}^{\infty}[\pi_{\alpha' t}(m-1) - \pi_{\alpha' t}(m)]\ln\sum_{\sigma}\exp\left(\beta'\sum_{\nu=1}^{m}J'_\nu\sigma_{i'_\nu}\sigma_{j'_\nu}\right)\\ &= \alpha'\mathbb{E}\ln\sum_{\sigma}\exp(\beta' J'\sigma_{i'_m}\sigma_{j'_m})\exp\left(\beta'\sum_{\nu=1}^{P_{\alpha' t}}J'_\nu\sigma_{i'_\nu}\sigma_{j'_\nu}\right)\\ &\quad -\alpha'\mathbb{E}\ln\sum_{\sigma}\exp\left(\beta'\sum_{\nu=1}^{P_{\alpha' t}}J'_\nu\sigma_{i'_\nu}\sigma_{j'_\nu}\right)\\ &= \alpha'\mathbb{E}\ln\Omega_t\exp(\beta' J'\sigma_{i'_m}\sigma_{j'_m}),\end{aligned}$$

where we included the t-dependent weights in the average Ω_t. Now use the following identity

$$\exp(\beta' J'\sigma_i\sigma_j) = \cosh(\beta' J') + \sigma_i\sigma_j\sinh(\beta' J')$$

to get

$$\frac{\mathrm{d}}{\mathrm{d}t}A_t = \alpha'\mathbb{E}\ln\Omega_t[\cosh(\beta' J')(1+\tanh(\beta' J')\sigma_{i'_m}\sigma_{j'_m})].$$

It is clear that

$$\mathbb{E}\omega_t^{2n}(\sigma_{i_m}\sigma_{j_m}) = \langle q_{2n}^2\rangle_t,$$

so we now expand the logarithm in power series and see that, in the limit of large N, as $\alpha_t \to \alpha$ the result does not depend on t, everywhere the measure $\langle\cdot\rangle_t$ is continuous as a function of the parameter t. From the comments that preceded the current proof, formalized in (9)–(11), this is the same as assuming that Ω is regular as a function of α, because $J'' \to J$ in the sense that in the large N limit J'' can only take the usual values ± 1 since the probability of being $\pm\beta'/\beta$ becomes zero. Therefore integrating over t from 0 to 1 is the same as multiplying by 1. Due to the symmetric distribution of J, the expansion of the logarithm yields the right-hand side of (8), where the odd powers are missing. □

Remark 1. The same result holds for the hierarchical Parisi trial structure [8]. In general, what we study in this section relies only on (8), that holds for quasi-stationary RaMOSts, to which our results therefore extend.

Proof of theorem 4. Consider once again

$$\hat{H} = -\sum_{\nu=1}^{P_{\alpha'}}J'_\nu\sigma_{i'_\nu}\sigma_{j'_\nu}. \tag{13}$$

We will let again Ω be the infinite volume Gibbs measure associated with the VB Hamiltonian at connectivity α and inverse temperature β.

Due to the symmetry of Ω, we have [5]

$$\mathbb{E}\ln\Omega\exp(\beta'\hat{H}) = \tfrac{1}{2}\mathbb{E}\ln\Omega\exp(-\beta'(\hat{H}-\hat{H}')),$$

where we 'replicated' $\hat{H}' = \hat{H}(\sigma')$.

For the left-hand side of (8), a tedious expansion yields

$$\begin{aligned}
&\tfrac{1}{2}\ln\Omega\exp(-\beta'(\hat{H}-\hat{H}'))\\
&\quad= \beta'^2\tfrac{1}{4}[2\Omega(\hat{H}^2) - 2\Omega^2(\hat{H})] + \beta'^4\tfrac{1}{24}[\Omega(\hat{H}^4) - 4\Omega(\hat{H}^3)\Omega(\hat{H}) - 3\Omega^2(\hat{H}^2)\\
&\qquad + 12\Omega(\hat{H}^2)\Omega^2(\hat{H}) - 6\Omega^4(\hat{H})] + \beta'^6\left[\frac{1}{6!}\Omega(\hat{H}^6) - \frac{1}{5!}\Omega(\hat{H}^5)\Omega(\hat{H})\right.\\
&\qquad - \tfrac{1}{48}\Omega(\hat{H}^4)\Omega(\hat{H}^2) - \tfrac{1}{72}\Omega^2(\hat{H}^3) + \tfrac{1}{6}\Omega(\hat{H})\Omega(\hat{H}^2)\Omega(\hat{H}^3) + \tfrac{1}{24}\Omega^3(\hat{H}^2)\\
&\qquad + \tfrac{1}{24}\Omega^2(\hat{H})\Omega(\hat{H}^4) - \tfrac{1}{6}\Omega(\hat{H}^3)\Omega^3(\hat{H}) - \tfrac{3}{8}\Omega^2(\hat{H})\Omega^2(\hat{H}^2)\\
&\qquad \left. + \tfrac{1}{2}\Omega(\hat{H}^2)\Omega^4(\hat{H}) - \tfrac{1}{6}\Omega^6(\hat{H})\right] + \mathrm{O}(\beta'^8),
\end{aligned}$$

of which we have to take the quenched expectation $\mathbb{E}$, using the formulas in appendix A. For the right-hand side of (8), the expansion of the hyperbolic tangent, performed explicitly for convenience in appendix A, leads to

$$\begin{aligned}
&\alpha'\sum_{n>0}\frac{1}{2n}\tanh^{2n}(\beta')(1-\langle q^2_{1\cdots 2n}\rangle)\\
&\quad= \beta'^2\alpha'(\tfrac{1}{2} - \tfrac{1}{2}\langle q^2_{12}\rangle) + \beta'^4\alpha'(-\tfrac{1}{12} + \tfrac{1}{3}\langle q^2_{12}\rangle - \tfrac{1}{4}\langle q^2_{1234}\rangle)\\
&\qquad + \beta'^6\alpha'(\tfrac{2}{90} - \tfrac{17}{90}\langle q^2_{12}\rangle + \tfrac{1}{3}\langle q^2_{1234}\rangle - \tfrac{1}{6}\langle q^2_{123456}\rangle) + \mathrm{O}(\beta'^8).
\end{aligned}$$

Recall that the averages $\langle\cdot\rangle$ do not depend on β', so we now have two power series in β' that we can equate term by term. The order zero and the odd orders are absent on both sides. Let us consider the second order. Taking into account the formulas given in appendix A, we have two identical (constant) monomials in α'

$$\tfrac{1}{2}(1-\langle q^2_{12}\rangle) = \tfrac{1}{2}(1-\langle q^2_{12}\rangle)$$

and we gain no information. Let us move on to order four: using again the formulas given in appendix A, we have the equality of two polynomials of degree two in α'. There is no constant term, and no second power in the right-hand side. Equating term by term we get a trivial identity for the linear part in α' (and hence no information). From the quadratic term in α', on the other hand, we obtain the following relation

$$\langle q^2_{12}q^2_{13}\rangle = \tfrac{1}{4}\langle q^4_{12}\rangle + \tfrac{3}{4}\langle q^2_{12}q^2_{34}\rangle \tag{14}$$

which is the first AC relation we wanted to prove.

Notice that in the linear part in α' a multi-overlap is present: q_{1234}, but it cancels. By contrast, in the quadratic part only 2-overlaps are present.

Let us now focus on the sixth order. Again we get a useless identity from the linear part in α', from which 4-overlaps and 6-overlaps get canceled out, and we gain no information. From the quadratic part, where 4-overlaps are present (but no 6-overlaps

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are there), $\langle q_{12}^2\rangle$ disappears, while the other terms with 2-overlaps only cancel because of the previous AC relation. In fact, from the expansion we have

$$\frac{15}{6!}-\frac{15}{5!}\langle q_{12}^2\rangle-\frac{7}{48}-\frac{1}{6}\langle q_{12}^4\rangle-\frac{15}{72}\langle q_{12}^2\rangle+\frac{7}{6}\langle q_{12}^2\rangle+\frac{4}{3}\langle q_{12}^2q_{23}^2\rangle+\frac{1}{8}+\frac{1}{2}\langle q_{12}^4\rangle+\frac{1}{24}\langle q_{12}^2\rangle$$
$$+\tfrac{1}{3}\langle q_{12}^2q_{23}^2\rangle+\tfrac{1}{4}\langle q_{12}^2\rangle-\tfrac{1}{2}\langle q_{1234}^2\rangle-2\langle q_{12}^2q_{34}^2\rangle-\tfrac{9}{8}\langle q_{12}^2\rangle-3\langle q_{12}^2q_{23}^2\rangle$$
$$-\tfrac{3}{2}\langle q_{12}^2q_{1234}^2\rangle+\tfrac{1}{2}\langle q_{1234}^2\rangle+3\langle q_{12}^2q_{34}^2\rangle+4\langle q_{12}^2q_{2345}^2\rangle-\tfrac{15}{6}\langle q_{12}^2q_{3456}^2\rangle=0.$$

So we are left, after a few trivial calculations, with the following new relation for multi-overlaps

$$\langle q_{1234}^2q_{15}^2\rangle=\tfrac{3}{8}\langle q_{1234}^2q_{12}^2\rangle+\tfrac{5}{8}\langle q_{1234}^2q_{56}^2\rangle \tag{15}$$

announced in the statement of the theorem.

In the cubic part in α' only overlaps of two replicas are present, and only the monomials of order six remain, as the ones of lower degree cancel out directly. The remaining relation is

$$\tfrac{1}{12}\langle q_{12}^6\rangle-\langle q_{12}^2q_{23}^4\rangle+\tfrac{3}{4}\langle q_{12}^2q_{34}^4\rangle-\tfrac{1}{3}\langle q_{12}^2q_{23}^2q_{31}^2\rangle+\langle q_{12}^2q_{13}^2q_{14}^2\rangle+3\langle q_{12}^2q_{23}^2q_{34}^2\rangle$$
$$-6\langle q_{12}^2q_{23}^2q_{45}^2\rangle+\tfrac{5}{2}\langle q_{12}^2q_{34}^2q_{56}^2\rangle=0.$$

Proceeding further, the expansion generates all the identities due to stochastic stability, in full agreement with the self-averaging identities found in [10, 9].

We will provide a more general and systematic form of the relations in the next section.

6. Stability relations from the cavity streaming equation

In this section we study a family of constraints on the distribution of the overlaps. To address this task we will consider the quenched expectation of a generic function of s replicas, with respect to the perturbed measure with weights

$$\exp\left(-\beta H_N(\sigma;\alpha)+\beta'\sum_{\nu=1}^{P'_{2\alpha t}}\tilde{J}_\nu\sigma_{i_\nu}\right), \tag{16}$$

whose use will be indicated with a subscript t in the expectations. Once again, $\{\tilde{J}_\nu\}$ are independent copies of J.

From now on let us put $\theta=\tanh(\beta')$, and, assuming $J=\pm1$, we have $\theta^{2n}=\mathbb{E}\tanh^{2n}(\beta'J)$, $\tanh^{2n+1}(\beta'J)=J\theta^{2n+1}\ \forall\ n\in\mathbb{N}$. Let us also just put $\omega_t=\omega_t(\sigma)$, with a slight abuse of notation.

Proposition 2. *Let Φ be a function of s replicas. Then the following cavity streaming equation holds*

$$\frac{\mathrm{d}\langle\Phi\rangle_t}{\mathrm{d}t}=-2\alpha\langle\Phi\rangle_t+2\alpha\mathbb{E}\Bigg[\Omega_t\Phi\Bigg\{1+J\sum_{a}^{1,s}\sigma_{i_1}^a\theta+\sum_{a<b}^{1,s}\sigma_{i_1}^a\sigma_{i_1}^b\theta^2+J\sum_{a<b<c}^{1,s}\sigma_{i_1}^a\sigma_{i_1}^b\sigma_{i_1}^c\theta^3+\cdots\Bigg\}$$
$$\times\left\{1-sJ\theta\omega_t+\frac{s(s+1)}{2!}\theta^2\omega_t^2-\frac{s(s+1)(s+2)}{3!}J\theta^3\omega_t^3+\cdots\right\}\Bigg]\ \forall\ \theta. \tag{17}$$

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Proof. Let us explicitly perform the calculation of the derivative, using (12).

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\,\mathbb{E}Z_t^{-1}\sum_{\sigma}\Phi\exp\left(\sum_{a=1}^{s}\left(\beta\sum_{\nu}^{P'_{\alpha N}}J_\nu\sigma^a_{i_\nu}\sigma^a_{j_\nu}+\beta'\sum_{\nu}^{P'_{2\alpha t}}\tilde{J}_\nu\sigma^a_{i_\nu}\right)\right)\\
&= 2\alpha\mathbb{E}\frac{\Omega_t[\Phi\exp(\beta'\sum_{a=1}^{s}J\sigma^a_{i_1})]}{\Omega_t\exp(\beta'\sum_{a=1}^{s}J\sigma^a_{i_1})}-2\alpha\langle\Phi\rangle_t\\
&= 2\alpha\mathbb{E}\frac{\Omega_t[\Phi\prod_{a=1}^{s}(\cosh(\beta'J)+\sigma^a_{i_1}\sinh(\beta'J))]}{\Omega_t[\prod_{a=1}^{s}(\cosh(\beta'J)+\sigma^a_{i_1}\sinh(\beta'J))]}-2\alpha\langle\Phi\rangle_t\\
&= 2\alpha\mathbb{E}\frac{\Omega_t[\Phi\prod_{a}^{s}(1+J\theta\sigma^a_{i_1})]}{(1+\theta\omega)^s}-2\alpha\langle\Phi\rangle_t.
\end{aligned}$$

Now note that

$$\frac{1}{(1+J\theta\omega_t)^s}=1-Js\theta\omega_t+\frac{s(s+1)}{2!}\theta^2\omega_t^2-J\frac{s(s+1)(s+2)}{3!}\theta^3\omega_t^3+\frac{s(s+1)(s+2)(s+3)}{4!}\theta^4\omega_t^4-\cdots$$

and that

$$\prod_{a=1}^{s}(1+J\theta\sigma^a_{i_1})=1+J\sum_{a}^{1,s}\sigma^a_{i_1}\theta+\sum_{a<b}^{1,s}\sigma^a_{i_1}\sigma^b_{i_1}\theta^2+J\sum_{a<b<c}^{1,s}\sigma^a_{i_1}\sigma^b_{i_1}\sigma^c_{i_1}\theta^3+\cdots.$$

The theorem follows immediately. □

In the limit $\alpha\to\infty$ as $\beta_{\mathrm{SK}}=2\alpha\theta^2$ is kept constant, the powers of θ higher than two are killed, and we recover the equation for the Gaussian SK model [5]

$$\frac{\mathrm{d}\langle\Phi\rangle_t}{\mathrm{d}t}=\left\langle\Phi\left(\sum_{a<b}^{1,s}q_{ab}-s\sum_{a}^{1,s}q_{a,s+1}+\frac{s(s+1)}{2}q_{s+1,s+2}\right)\right\rangle_t. \tag{18}$$

We know from sections 4 and 5 that if in the previous theorem we take Φ to be a filled polynomial the left-hand side of (17) is zero in the thermodynamic limit, and we have a polynomial in θ (and hence in β') on the right-hand side that can be equated to zero term by term (we do not need to re-expand in β' and equate the new coefficients to zero). Now if, in each term of the expansion that we equate to zero in this case, we additionally take $\beta'=\beta$, then we also guarantee that the fillable polynomials get filled, thanks to proposition 1. In other words

Proposition 3. *The generator of the constraints on the distribution of the overlap is:*

$$\lim_{N\to\infty}\partial_t\langle\Phi\rangle=0$$

where Φ is filled and $\beta'=\beta$.

Let us consider the first simple example, $\Phi=q_{12}^2$. Proposition 2 then yields

$$\begin{aligned}
\lim_{N\to\infty}\partial_t\langle q_{12}^2\rangle_t &= \lim_{N\to\infty}\langle q_{12}^3-4q_{12}^2q_{23}+3q_{12}^2q_{34}\rangle_t\theta^2+\mathrm{O}(\theta^4)\\
&= \lim_{N\to\infty}\langle q_{12}^3-4q_{12}^2q_{23}+3q_{12}^2q_{34}\rangle_t\beta'^2+\mathrm{O}(\beta'^4)=0\\
&\Rightarrow \langle q_{12}^4-4q_{12}^2q_{23}^2+3q_{12}^2q_{34}^2\rangle=0,
\end{aligned}$$

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which is understood to be taken in the thermodynamic limit and is just the Aizenman–Contucci relation we have found already in the previous section.

If we choose instead $\Phi = q_{1234}^2$, we obtain

$$\lim_{N\to\infty} \partial_t \langle q_{1234}^2 \rangle_t = \langle \theta^2 (3q_{1234}^2 q_{12}^2 - 8q_{1234}^2 q_{15}^2 + 10 q_{1234}^2 q_{15^2}) + \theta^4 (q_{1234}^4 - 16 q_{1234}^2 q_{1235}^2 + 60 q_{1234}^2 q_{1256}^2 - 80 q_{1234}^2 q_{1567}^2 + 35 q_{1234}^2 q_{5678}^2)\rangle + \mathrm{O}(\theta^6) = 0.$$

From the order two in β' we have the coefficient of θ^2, and equating to zero gives

$$\langle q_{1234}^2 q_{15}^2 \rangle = \tfrac{3}{8} \langle q_{1234}^2 q_{12}^2 \rangle + \tfrac{5}{8} \langle q_{1234}^2 q_{56}^2 \rangle.$$

At the order four in β' we have leftover term from the order two, which thus vanish, and the coefficient of θ^4:

$$\langle q_{1234}^4 \rangle = \langle 16 q_{1234}^2 q_{1235}^2 - 60 q_{1234}^2 q_{1256}^2 + 80 q_{1234}^2 q_{1567}^2 - 35 q_{1234}^2 q_{5678}^2 \rangle.$$

In a similar way we can obtain all the other relations, from the higher orders, simply by equating to zero all the coefficients of the expansion in powers of θ, with no need to expand in β'.

When $\Phi = q_{1\cdots s}^2$, notice that the relation we obtain from the lowest order in (17) is formally identical to equation (18), with zero on the left-hand side, without the limit $\alpha' \to \infty$. Hence, using the invariance with respect to permutations of replicas, we have the general form of the constraint of which (14) and (15) are two special cases. In general, for a suitable function $\Phi_{1\cdots s}$ of s replicas, from proposition 2 we can state the following.

Theorem 5. *Given an even integer s, the AC relation*

$$\langle \Phi_{1\cdots s} q_{1,s+1}^2 \rangle = \frac{s-1}{2s} \langle \Phi_{1\cdots s} q_{1,2}^2 \rangle + \frac{s+1}{2s} \langle \Phi_{1\cdots s} q_{s+1,s+2}^2 \rangle$$

holds.

Subtracting the equation above from the well known Ghirlanda–Guerra identity [12]

$$\langle \Phi_{1\cdots s} q_{1,s+1}^2 \rangle = \frac{1}{s} \langle \Phi_{1\cdots s} \rangle \langle q_{12}^2 \rangle + \frac{s-1}{s} \langle \Phi_{1\cdots s} q_{s+1,s+2}^2 \rangle$$

we get the other well known relation

$$\langle \Phi_{1\cdots s} q_{s+1,s+2}^2 \rangle = \frac{2}{s+1} \langle \Phi_{1\cdots s} \rangle \langle q_{12}^2 \rangle + \frac{s-1}{s+1} \langle \Phi_{1\cdots s} q_{12}^2 \rangle. \tag{19}$$

While the Ghirlanda–Guerra identities are a consequence of self-averaging, in the thermodynamic limit, of the energy density $H_N/N \equiv K$

$$\langle K \Phi_s \rangle = \langle K \rangle \langle \Phi_s \rangle,$$

the AC relations are a consequence of stochastic stability; but they can also be deduced from a self-averaging relation:

$$\mathbb{E}\Omega(K\Phi_s) = \mathbb{E}\Omega(K)\Omega(\Phi_s).$$

Clearly the third relation (19) is hence a consequence of

$$\mathbb{E}\Omega(K)\Omega(\Phi_s) = \langle K \rangle \langle \Phi_s \rangle.$$

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We stress that not only the energy density can be used to get the various relations, but several other quantities (as long as self-averaging is preserved) would do as well.

If we consider 4-overlaps, proposition 2 gives

$$\frac{(s-1)(s-2)(s-3)}{4!}\langle \Phi_{1\cdots s} q^2_{1,2,3,4}\rangle - \frac{s(s-1)(s-2)}{3!}\langle \Phi_{1\cdots s} q^2_{1,2,3,s+1}\rangle + \frac{(s-1)s(s+1)}{4}\langle \Phi_{1\cdots s} q^2_{1,2,s+1,s+2}\rangle - \frac{s(s+1)(s+2)}{3!}\langle \Phi_{1\cdots s} q^2_{1,s+1,s+2,s+3}\rangle + \frac{(s+1)(s+2)(s+3)}{4!}\langle \Phi_{1\cdots s} q^2_{s+1,s+2,s+3,s+4}\rangle = 0$$

which again can be deduced from a self-averaging relation too [10, 9] and should be compared with the generalization of Ghirlanda–Guerra relations

$$\frac{s(s-1)(s-2)(s-3)}{3!}\langle q_{1,2,3,4}\Phi\rangle - \frac{s(s-1)(s-2)}{2}\langle q_{1,2,3,s+1}\Phi\rangle + \frac{(s-1)s(s+1)}{2!}\langle q_{1,2,s+1,s+2}\Phi\rangle - \frac{s(s+1)(s+2)}{3!}\langle q_{1,s+1,s+2,s+3}\Phi\rangle + \langle q_{1234}\rangle\langle\Phi\rangle = 0,$$

which has been found in [9], as a consequence of the self-averaging of the energy density.

We do not write explicitly the general form of the constraints deduced from proposition 2, as it is a very simple but tedious computation, which shows that the relations are in agreement with [10].

6.1. Revisiting the positivity of multi-overlaps

We here hint at how to gain a better control of the phase transition discussed in section 4, using the expansion of the cavity streaming equation, to justify from a different perspective what is proven in [13]: the fluctuations of the multi-overlaps diverge at lower temperatures as the number of replicas increases. This is a first step in the calculation of the critical exponents of the critical behavior of the multi-overlaps. We only sketch the arguments, which proceed along the lines described in [1].

We are going to prove that the first contribution to the average of the 2-overlap in its $\tanh(\beta') = \theta$ expansion is of order two, while it is of order four for the 4-overlap, and so on for higher order multi-overlaps, as intuitively expected.

Let us write the streaming equation for $\langle q_{12}\rangle_t$, with $\beta' = \beta$, $\alpha' = \alpha$

$$\partial_t\langle q_{12}\rangle_t = \alpha\theta^2\langle q^2_{12} - 4q_{12}q_{23} + 3q_{12}q_{34}\rangle_t + \mathrm{O}(\theta^4).$$

But $\langle q^2_{12}\rangle_t = \langle q^2_{12}\rangle$ because q^2_{12} is a filled monomial, and it can be integrated offering

$$\langle q_{12}\rangle_t = \alpha\theta^2\langle q^2_{12}\rangle t + \alpha\theta^2\int_0^1 \mathrm{d}t(-4\langle q_{12}q_{23}\rangle_t + 3\langle q_{12}q_{34}\rangle_t) + \mathrm{O}(\theta^4).$$

We now prove that the terms inside the integral are of higher order in θ. It is enough to notice that such terms are fillable but not filled, so we can expand them using the streaming equation to evaluate the leading order at which they contribute, which is θ^3 as can be deduced from the expansions given in appendix B. The same approach can be

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used for the 4-overlap; in fact we can write

$$\partial_t \langle q_{1234} \rangle_t = \alpha\theta^2 \langle 10 q_{1234} q_{56} - 16 q_{1234} q_{15} + 6 q_{1234} q_{12} \rangle_t + \alpha\theta^4 \langle q_{1234}^2 - 16 q_{1234} q_{1235} + 60 q_{1234} q_{1256} - 80 q_{1234} q_{1567} - 35 q_{1234} q_{5678} \rangle_t + \mathrm{O}(\theta^6).$$

It can be readily seen that the only contribution at the fourth order is due to $\langle q_{1234}^2 \rangle$, given that this is the only filled monomial. The calculations in appendix B show that the three contributions from the second order in θ (i.e. $q_{1234}q_{56}, q_{1234}q_{15}, q_{1234}q_{12}$) contribute at orders higher than four, so the first term in the four-replica multi-overlap expansion is positive (being a square), in agreement with what we showed in section 4. We notice that, while the first contribution to the 2-overlap is of order two in θ, the first contribution to the 4-overlap is of order four, and this result extends analogously to the higher order multi-overlaps. So it is not surprising that at the point where the 2-overlap fluctuations start diverging, the fluctuations of the 4-overlap do not, and so on for higher orders, in agreement with what is proven in [13].

7. Conclusion and outlook

We have proven the validity of Aizenman–Contucci relations for dilute spin glasses and exhibited further relations for multi-overlaps. Some more general relations can be found with the same stochastic stability methods for internal energy, but also from the cavity part of the RaMOSt trial function, by means of a control of the response of the average of generic observables with respect to the change of a perturbing parameter. We also showed that the multi-overlaps undergo the same transition the 2-overlap exhibits in fully connected models, i.e. they remain strictly positive, below the critical temperature, if we apply an external field and then remove it in the thermodynamic limit. The external field is properly modulated, in diluted systems, by the degree of connectivity of the perturbation.

The further natural development is the study of the extension to multi-overlaps of the self-averaging identities (known as Ghirlanda–Guerra in fully connected models) to prove that even multi-overlaps are non-negative with probability one in dilute spin glasses (the identities will be derived in [9]). This would extend to odd spin interactions the replica bounds so far rigorously valid only for even interactions, and such a result would be important for the application of dilute spin glasses to optimization problems like the K-SAT.

Another development of the current work is the calculation of the critical exponents of the multi-overlaps, which has been gained for fully connected models in [1] with the same techniques here shown to be fruitful in dilute models too. We will report on critical exponents in dilute spin glasses soon, in [6].

Acknowledgments

The authors are extremely grateful to Peter Sollich for reviewing the manuscript, and to an anonymous referee for pointing out a weak point in a previous version of this paper.

Appendix A. Formulas for the RaMOSt expansions

Let us report for convenience the well known expansion

$$\tanh(x) \sum \frac{2^{2n}}{2n!}(2^{2n}-1)B_{2n}x^{2n-1}$$

where B_n and the Bernoulli numbers, defined by

$$\frac{x}{e^x - 1} = \sum \frac{B_n x^n}{n!}$$

so that

$$\tanh(\beta) = \beta - \frac{1}{3}\beta^3 + \frac{2}{15}\beta^5 - \frac{17}{315}\beta^7 + \frac{62}{2835}\beta^9 \cdots.$$

We also report here the following results of the computations we used in the expansions of the previous sections.

Order two:

$$\mathbb{E}\Omega(\hat{H}^2) = \alpha'$$
$$\mathbb{E}\Omega^2(\hat{H}) = \alpha'\langle q_{12}^2\rangle.$$

Order four:

$$\mathbb{E}\Omega(\hat{H}^4) = \alpha' + \alpha'^2 3$$
$$\mathbb{E}\Omega(\hat{H}^3)\Omega(\hat{H}) = \alpha'\langle q_{12}^2\rangle + \alpha'^2 3\langle q_{12}^2\rangle$$
$$\mathbb{E}\Omega^2(\hat{H}^2) = \alpha' + \alpha'^2(1 + 2\langle q_{12}^4\rangle)$$
$$\mathbb{E}\Omega(\hat{H}^2)\Omega^2(\hat{H}) = \alpha'\langle q_{12}^2\rangle + \alpha'^2(\langle q_{12}^2\rangle + 2\langle q_{12}^2 q_{13}^2\rangle)$$
$$\mathbb{E}\Omega^4(\hat{H}) = \alpha'\langle q_{1234}^2\rangle + \alpha'^2 3\langle q_{12}^2 q_{34}^2\rangle.$$

Order six:

$$\mathbb{E}\Omega(\hat{H}^6) = \alpha' + \alpha'^2 15 + \alpha'^3 15$$
$$\mathbb{E}\Omega(\hat{H}^5)\Omega(\hat{H}) = \alpha'\langle q_{12}^2\rangle + \alpha'^2 15\langle q_{12}^2\rangle + \alpha'^3 15\langle q_{12}^2\rangle$$
$$\mathbb{E}\Omega(\hat{H}^4)\Omega(\hat{H}^2) = \alpha' + \alpha'^2(7 + 8\langle q_{12}^4\rangle) + \alpha'^3(3 + 12\langle q_{12}^4\rangle)$$
$$\mathbb{E}\Omega^2(\hat{H}^3) = \alpha'\langle q_{12}^2\rangle + \alpha'^2 15\langle q_{12}^2\rangle + \alpha'^3(9\langle q_{12}^2\rangle + 6\langle q_{12}^6\rangle)$$
$$\mathbb{E}\Omega(\hat{H})\Omega(\hat{H}^2)\Omega(\hat{H}^3) = \alpha'\langle q_{12}^2\rangle + \alpha'^2(7\langle q_{12}^2\rangle + 8\langle q_{12}^2 q_{23}^2\rangle) + \alpha'^3(3\langle q_{12}^2\rangle + 6\langle q_{12}^2 q_{13}^2\rangle + 6\langle q_{12}^2 q_{13}^4\rangle)$$
$$\mathbb{E}\Omega^3(\hat{H}^2) = \alpha' + \alpha'^2(3 + 12\langle q_{12}^4\rangle) + \alpha'^3(1 + 6\langle q_{12}^4\rangle + 8\langle q_{12}^2 q_{23}^2 q_{31}^2\rangle)$$
$$\mathbb{E}\Omega(\hat{H}^4)\Omega^2(\hat{H}) = \alpha'\langle q_{12}^2\rangle + \alpha'^2(7\langle q_{12}^2\rangle + 8\langle q_{12}^2 q_{23}^2\rangle) + \alpha'^3(3\langle q_{12}^2\rangle + 12\langle q_{12}^2 q_{23}^2\rangle)$$
$$\mathbb{E}\Omega^3(\hat{H})\Omega(\hat{H}^3) = \alpha'\langle q_{1234}^2\rangle + \alpha'^2(3\langle q_{1234}^2\rangle + 12\langle q_{12}^2 q_{34}^2\rangle) + \alpha'^3(9\langle q_{12}^2 q_{34}^2\rangle + 6\langle q_{12}^2 q_{23}^2 q_{34}^2\rangle)$$
$$\mathbb{E}\Omega^2(\hat{H})\Omega^2(\hat{H}^2) = \alpha'\langle q_{12}^2\rangle + \alpha'^2(3\langle q_{12}^2\rangle + 8\langle q_{12}^2 q_{23}^2\rangle + 4\langle q_{12}^2 q_{1234}^2\rangle) + \alpha'^3(\langle q_{12}^2\rangle + 2\langle q_{12}^2 q_{34}^4\rangle + 4\langle q_{12}^2 q_{23}^2\rangle + 8\langle q_{12}^2 q_{23}^2 q_{34}^2\rangle)$$
$$\mathbb{E}\Omega(\hat{H}^2)\Omega^4(\hat{H}) = \alpha'\langle q_{1234}^2\rangle + \alpha'^2(\langle q_{1234}^2\rangle + 6\langle q_{12}^2 q_{34}^2\rangle + 8\langle q_{12}^2 q_{2345}^2\rangle) + \alpha'^3(3\langle q_{12}^2 q_{34}^2\rangle + 12\langle q_{12}^2 q_{23}^2 q_{45}^2\rangle)$$
$$\mathbb{E}\Omega^6(\hat{H}) = \alpha'\langle q_{123456}^2\rangle + \alpha'^2 15\langle q_{12}^2 q_{3456}^2\rangle + \alpha'^3 15\langle q_{12}^2 q_{34}^2 q_{56}^2\rangle.$$

Appendix B. Formulas for the cavity expansions

In this appendix we report, to facilitate the reader who wants to perform the calculations in detail, the streaming equations for the expansion of section 6.

$$\partial_t\langle q_{12}\rangle_t = \alpha\theta^2\langle q_{12}^2 - 4q_{12}q_{23} + 3q_{12}q_{34}\rangle_t + O(\theta^4)$$

$$\partial_t\langle q_{12}q_{23}\rangle_t = \alpha\theta^2\langle 6q_{12}q_{23}q_{45} - 6q_{12}q_{23}q_{14} - 3q_{12}q_{23}q_{24} + q_{12}q_{23}q_{13} + 2q_{12}^2q_{23}\rangle + \alpha\theta^4\langle 15q_{12}q_{23}q_{4567} - 20q_{12}q_{23}q_{1456} - 10q_{12}q_{23}q_{2456} + 12q_{12}q_{23}q_{1245} + 6q_{12}q_{23}q_{1345} - 3q_{12}q_{23}q_{1234}\rangle_t + \mathrm{O}(\theta^6)$$

$$\partial_t\langle q_{12}q_{34}\rangle_t = \alpha\theta^2\langle 10q_{12}q_{34}q_{56} - 16q_{12}q_{34}q_{15} + 2q_{12}^2q_{34} + 4q_{12}q_{34}q_{13}\rangle_t + \alpha\theta^4\langle 35q_{12}q_{34}q_{5678} - 80q_{12}q_{34}q_{1567} + 20q_{12}q_{34}q_{1256} + 40q_{12}q_{34}q_{1356} - 16q_{12}q_{34}q_{1235} + q_{12}q_{34}q_{1234}\rangle_t + \mathrm{O}(\theta^6)$$

$$\partial_t\langle q_{1234}\rangle_t = \alpha\theta^2\langle 10q_{1234}q_{56} - 16q_{1234}q_{15} + 6q_{1234}q_{12}\rangle_t + \alpha\theta^4\langle q_{1234}^2 - 16q_{1234}q_{1235} + 60q_{1234}q_{1256} - 80q_{1234}q_{1567} - 35q_{1234}q_{5678}\rangle_t + \mathrm{O}(\theta^6)$$

$$\partial_t\langle q_{1234}q_{12}\rangle_t = \alpha\theta^2\langle 10q_{1234}q_{12}q_{56} - 8q_{1234}q_{12}q_{15} - 8q_{1234}q_{12}q_{35} + q_{1234}q_{12}^2 + 5q_{1234}q_{12}q_{13}\rangle_t + \alpha\theta^4\langle 35q_{1234}q_{12}q_{5678} - 40q_{1234}q_{12}q_{1567} - 40q_{1234}q_{12}q_{3567} + 10q_{1234}q_{12}q_{1256} + 40q_{1234}q_{12}q_{1356} + 10q_{1234}q_{12}q_{3456} - 16q_{1234}q_{12}q_{1235} + q_{1234}^2q_{12}\rangle_t + \mathrm{O}(\theta^6)$$

$$\partial_t\langle q_{1234}q_{15}\rangle_t = \alpha\theta^2\langle 15q_{1234}q_{15}q_{67} - 15q_{1234}q_{15}q_{26} - 10q_{1234}q_{15}q_{56} + 5q_{1234}q_{15}q_{12} + 4q_{1234}q_{15}q_{23} + q_{1234}q_{15}^2\rangle_t + \alpha\theta^4\langle 70q_{1234}q_{15}q_{6789} - 105q_{1234}q_{15}q_{2678} - 70q_{1234}q_{15}q_{1678} + 15q_{1234}q_{15}q_{1567} + 90q_{1234}q_{15}q_{1267} + 45q_{1234}q_{15}q_{2367} + 20q_{1234}q_{15}q_{1235} - 25q_{1234}q_{15}q_{1246} - 5q_{1234}q_{15}q_{2346} + 4q_{1234}q_{15}q_{1236} + q_{1234}^2q_{15}\rangle_t + \mathrm{O}(\theta^6)$$

$$\partial_t\langle q_{1234}q_{56}\rangle_t = \alpha\theta^2\langle 21q_{1234}q_{56}q_{78} - 24q_{1234}q_{56}q_{17} - 12q_{1234}q_{56}q_{57} + 6q_{1234}q_{56}q_{12} + 8q_{1234}q_{56}q_{57} + q_{1234}q_{56}^2\rangle + \alpha\theta^4\langle 126q_{1234}q_{56}q_{7890} - 112q_{1234}q_{56}q_{5789} - 224q_{1234}q_{56}q_{1789} + 21q_{1234}q_{56}q_{5678} + 168q_{1234}q_{56}q_{1578} + 126q_{1234}q_{56}q_{1378} - 24q_{1234}q_{56}q_{3567} + 14q_{1234}q_{56}q_{1235} + q_{1234}^2q_{56}\rangle_t + \mathrm{O}(\theta^6)$$

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J. Stat. Mech. (2007) P08025