Constraints for order parameters in analogical neural networks*

Adriano Barra [1] and Francesco Guerra [2]

[1] Dipartimento di Fisica, Sapienza Università di Roma
and Dipartimento di Matematica, Università di Bologna

[2] Dipartimento di Fisica, Sapienza Università di Roma
and Istituto Nazionale di Fisica Nucleare, Sezione di Roma1

Abstract. In this paper we study, via equilibrium statistical mechanics, the properties of the internal energy of an Hopfield neural network whose patterns are stored continuously (Gaussian distributed).

The model is shown to be equivalent to a bipartite spin glass in which one party is given by dichotomic neurons and the other party by Gaussian spin variables. Dealing with replicated systems, beyond the Mattis magnetization, we introduce two overlaps, one for each party, as order parameters of the theory: The first is a standard overlap among neural configurations on different replicas, the second is an overlap among the Gaussian spins of different replicas.

The aim of this work is to show the existence of constraints for these order parameters close to ones found in many other complex systems as spin glasses and diluted networks: we find a class of Ghirlanda-Guerra-like identities for both the overlaps, generalizing the well known constraints to the neural networks, as well as new identities where noise is involved explicitly.

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* *in memoriam*

[1] e-mail: adriano.barra@roma1.infn.it
[2] e-mail: francesco.guerra@roma1.infn.it

1 Introduction

Despite the several recent progresses in statistical mechanics of complex systems avoiding replica trick (see for instance [12][22]), the original Amit Gutfrund Sompolinsky theory (AGS) [3][4][5] for associative neural network is still nowadays lacking a complete rigorous mathematical backbone in this sense.
In fact, while the low storage memory case [3] has been largely understood and even some generalization considered (see for instance [18][19][20][21]), in the high storage limit (number of encoded memories linearly diverging with the number of working neurons) nor the existence of the thermodynamic limit (clearly understood for the paradigmatic Sherrington-Kirkpatrick model [16]) neither a complete description of the ergodic behavior of the system [9] have been obtained yet (even though whatever has been proved is in agreement with AGS picture obtained via the replica trick).
While attempting progresses in finding the critical line for ergodicity and a clear scenario for the replica symmetric regime, in this paper we investigate the existence, for these networks, of proper order parameter constraints, typical features of complex systems (see for instance [1][13][14] for discussions on linear constraints or [10][11] for higher order constraints).
In our framework, the analogical Hopfield neural network is thought of as a bipartite spin glass in suitably defined variables, in which the two parties interact one another via the memory kernel.
Consequently our constraints are satisfied by the following two overlaps: A standard overlap taking into account the similarity among replicas at the level of the neuronal configurations (first party), and an overlap weighting the similarity between the Gaussian spins (second party) among different replicas.
Due to the symmetry of the interaction among the two parties, the constraints the overlaps obey are symmetric with respect to their permutation too: remarkably the same symmetry was already found in the Random Overlap Structure framework [2], where, when analyzing the optimal structure [15], roughly speaking, two mean field spin glass models were made to interact and identities formally equivalent to our one were found [8].
The relations for these two overlaps coupled together are obtained with standard techniques: by avoiding divergencies in the response of the energy with respect to a change in the noise (which plays here the role of the temperature in material systems), and as consequences of the self-averaging of the internal energy.
Furthermore we show that the internal energy of the system can be completely described in term of a self-overlap among the Gaussian spins which we prove to be self-averaging β almost-everywhere.
The paper is structured as follows: In Section 2 the analogical neural network is intro-

duced and its related statistical mechanics framework defined.
Section 3 deals with the a detailed study of the internal energy: it is expressed in more ways in terms of our overlaps and the full self-average of the spin self-overlap is shown.
Section 4 deals with its β-streaming evaluation as well as its self-averaging properties: the whole set of identities is proven in this section.
Section 5 is left for outlook and conclusions.

2 Definition of the neural network model

The neural network model we use resembles several features of the original AGS one: It is a mean field fully connected network such that each neuron interacts with the whole neural community. Its memory kernel is stored into the synaptic matrix following the Hebb prescription [17] but, differently with respect to the original AGS theory [4], our memory variables are not dichotomic bit, while share the same continuous support, weighed by a standard distribution $\mathcal{N}[0,1]$.
Concretely we introduce a large network of N two-state neurons $\pm 1 \ni \sigma_i$, $i \in (1,..,N)$, which schematize the single neuronal dynamics [3] by matching the value -1 with a quiescent (or integrating) neuron and the value $+1$ with a spiking (or firing) neuron. They interact throughout the following synaptic matrix J_{ij} (defined accordingly the Hebb rule for learning),

$$J_{ij} = \sum_{\mu=1}^{k} \xi_i^\mu \xi_j^\mu, \tag{2.1}$$

where each random variable $\xi^\mu = \{\xi_1^\mu, .., \xi_N^\mu\}$ represents a pattern already stored by the network.
As far as we deal with equilibrium properties we are marginally concerned with the time scales involved in the dynamics, which however are postulated to live on, at least, three different time sectors: The spiking dynamics of each neuron, which happens on time scales much shorter than the others involved in the propagation of spikes trough the network, is thought effectively as instantaneous (spin-flip). In complete opposition the synaptic dynamics, where learning is stored into the memory kernel by updating the synaptic matrix, happens on time scales much slower with respect to the ones involved in the propagation of spikes into the network and consequently the synaptic matrix is frozen at the beginning such that no evolution for the memories is hallowed.
Between these time sectors lives the one for the equilibrium of the neural network, that is the object of our study.
The analysis of the network assumes that the system has already memorized k patterns

(no learning is investigated) and we will be interested in the case in which this number increases proportionally (linearly) to the system size (high storage level).
In standard literature these patters are usually taken at random with distribution $P(\xi_i^\mu) = (1/2)\delta_{\xi_i^\mu,+1} + (1/2)\delta_{\xi_i^\mu,-1}$, while we extend their support to be on the real axes weighted by a Gaussian probability distribution, i.e.

$$P(\xi_i^\mu) = \frac{1}{\sqrt{2\pi}} e^{-(\xi_i^\mu)^2/2}. \tag{2.2}$$

Of course, avoiding pathological case, in the high storage level and in the high temperature region, the results should show robustness with respect to the particular choice of the probability distribution and we should recover the standard AGS theory.
The Hamiltonian of the model is defined as follows

$$H_N(\sigma;\xi) = -\frac{1}{N}\sum_{\mu=1}^{k}\sum_{i<j}^{N}\xi_i^\mu\xi_j^\mu\sigma_i\sigma_j, \tag{2.3}$$

which, splitting the summations $\sum_{i<j}^N = \frac{1}{2}\sum_{ij}^N - \frac{1}{2}\sum_i^N \delta_{ij}$ enable us to write down the following partition function

$$\begin{aligned} Z_{N,p}(\beta;\xi) &= \sum_\sigma \exp\Big(\frac{\beta}{2N}\sum_{\mu=1}^{k}\sum_{ij}^{N}\xi_i^\mu\xi_j^\mu\sigma_i\sigma_j - \frac{\beta}{2N}\sum_{\mu=1}^{k}\sum_{i}^{N}(\xi_i^\mu)^2\Big) \\ &= \tilde{Z}_{N,p}(\beta;\xi)\exp\Big(\frac{-\beta}{2N}\sum_{\mu=1}^{k}\sum_{i=1}^{N}(\xi_i^\mu)^2\Big), \end{aligned} \tag{2.4}$$

where β, the inverse temperature in spin glass theory, denotes the level of noise in the network and we defined

$$\tilde{Z}_{N,p}(\beta;\xi) = \sum_\sigma \exp(\frac{\beta}{2N}\sum_{\mu=1}^{k}\sum_{ij}^{N}\xi_i^\mu\xi_j^\mu\sigma_i\sigma_j). \tag{2.5}$$

Notice that the last term at the r.h.s. of eq. (2.4) does not depend on the particular state of the network.
Consequently we focus just on $\tilde{Z}(\beta;\xi)$. Let us apply the Hubbard Stratonovich lemma to linearize with respect to the bilinear quenched memories carried by the $\xi_i^\mu\xi_j^\mu$; if we define the "Mattis magnetization" [3] m_μ as

$$m_\mu = \frac{1}{N}\sum_{i}^{N}\xi_i^\mu\sigma_i, \tag{2.6}$$

we can write

$$\tilde{Z}_{N,k}(\beta;\xi) = \sum_\sigma \exp(\frac{\beta N}{2}\sum_{\mu=1}^k m_\mu^2) \tag{2.7}$$
$$= \sum_\sigma \int \prod_{\mu=1}^k (\frac{dz_\mu \exp(-z_\mu^2/2)}{\sqrt{2\pi}}) \exp(\sqrt{\beta N}\sum_{\mu=1}^k m^\mu z_\mu).$$

In what follows, the following partition function, defining implicitly an effective Hamiltonian, will be used:

$$\tilde{Z}_{N,k}(\beta;\xi) = \sum_\sigma \int \prod_\mu^k d\mu(z_\mu) \exp\big(\sqrt{\frac{\beta}{N}}\sum_{\mu,i}\sigma_i\xi_{i\mu}z_\mu\big), \tag{2.8}$$

where $d\mu(z_\mu)$ is the Gaussian measure.
Note that, as we have mapped the neural network problem into a spin glass problem also the normalization factor of the effective Hamiltonian is changed coherently.
In fact, in the high storage case, this structure clearly reflects the interaction among the N dichotomic spin σ and the k Gaussian variables z through the random interaction matrix encoded by the patterns (in the low level of stored memories, where N goes to infinity but k remains finite such an equivalence breaks down).
Reflecting this "bipartite" nature of the Hopfield model expressed by eq. (2.8) we introduce two other order parameters beyond the "Mattis magnetization" (eq. (2.6)): the first is the standard overlap between the replicated neurons, defined as

$$q_{ab} = \frac{1}{N}\sum_{i=1}^N \sigma_i^a\sigma_i^b \tag{2.9}$$

and the second is the overlap between the spins of different replicas, defined as

$$p_{ab} = \frac{1}{k}\sum_{\mu=1}^k z_\mu^a z_\mu^b. \tag{2.10}$$

Taken F as a generic function of the neurons, we define the Boltzmann state $\omega_\beta(F)$ at a given level of noise β as

$$\omega_\beta(F) = \omega(F) = (\tilde{Z}_{N,k}(\beta;\xi))^{-1}\sum_\sigma \int \prod_\mu^k d\mu(z_\mu)F(\sigma)\exp\big(\sqrt{\frac{\beta}{N}}\sum_{\mu,i}\sigma_i\xi_{i\mu}z_\mu\big), \tag{2.11}$$

and often we will drop the subscript β for the sake of simplicity. The s-replicated Boltzmann measure is defined as $\Omega = \omega^1 \times \omega^2 \times ... \times \omega^s$ in which all the single Boltzmann states are independent states at the same noise level β^{-1} and share an identical distribution of quenched memories ξ.
The average over the quenched memories will be denoted by $\mathbb{E}$ and for a generic function of these memories $F(\xi)$ can be written as

$$\mathbb{E}[F(\xi)] = \int \prod_{\mu=1}^{p} \prod_{i=1}^{N} \frac{d\xi_i^\mu e^{-\frac{(\xi_i^\mu)^2}{2}}}{\sqrt{2\pi}} F(\xi) = \int F(\xi) d\mu(\xi), \tag{2.12}$$

of course $\mathbb{E}[\xi_i^\mu] = 0$ and $\mathbb{E}[(\xi_i^\mu)^2] = 1$.
We use the symbol $\langle . \rangle$ to mean $\langle . \rangle = \mathbb{E}\Omega(.)$.
In the thermodynamic limit, it is assumed

$$\lim_{N\to\infty} \frac{k}{N} = \alpha,$$

α being a given real number, parameter of the theory.
The standard quantity of interest is the intensive quenched pressure, defined as

$$A_{N,k}(\beta) = -\beta f_{N,k}(\beta) = \frac{1}{N} \mathbb{E} \ln \tilde{Z}_{N,k}(\beta; \xi), \tag{2.13}$$

where $f_{N,k}(\beta) = u_{N,k}(\beta) - \beta^{-1} s_{N,k}(\beta)$ is the free energy density, $u_{N,k}(\beta)$ the internal energy density and $s_{N,k}(\beta)$ the intensive entropy.
Assuming that the thermodynamic limit of the free energy and the internal energy exist, these quantities will be denoted as $A(\alpha, \beta)$, $u(\alpha, \beta)$.

3 Properties of the internal energy

In this section we study the properties of the internal energy: at first we evaluate it explicitly in terms of our overlaps. After showing that it can be expressed via the spin self-overlap $\langle p_{11} \rangle$, we prove also that $\langle p_{11} \rangle$ is completely self-averaging.
Let us start with the next

Theorem 3.1 *The following expressions for the internal energy density hold in the thermodynamic limit*

$$\lim_{N\to\infty} \frac{1}{N} \langle H_{N,k}(\sigma; \xi) \rangle = \frac{\alpha}{2(1-\beta)} \Big(1 - \langle q_{12} p_{12} \rangle \Big), \tag{3.1}$$

$$\lim_{N\to\infty} \frac{1}{N} \langle H_{N,k}(\sigma; \xi) \rangle = \frac{\alpha}{2\beta} \Big(\langle p_{11} \rangle - 1 \Big). \tag{3.2}$$

Proof
The proof, as well as several others through the paper, uses direct calculation and Wick theorem (see eq.(3.3)) as we deal with Gaussian distributed variables like the spins and the memories.
In fact we remember that for these quantities, considering $f(\xi)$ as a generic well behaved function of the memories, the following relation (integration by parts) holds:

$$\mathbb{E}\xi f(\xi)=\mathbb{E}\partial_{\xi}f(\xi). \tag{3.3}$$

So we can write

$$\begin{aligned}\langle H_{N,k}(\sigma;\xi)\rangle &= \partial_{\beta}\frac{1}{N}\mathbb{E}\log\sum_{\sigma}\exp\Big(\sqrt{\frac{\beta}{N}}\sum_{i,\mu}\xi_{i\mu}\sigma_i z_{\mu}\Big) \qquad (3.4)\\ &= \frac{1}{2N\sqrt{N\beta}}\sum_{i,\mu}\mathbb{E}\xi_{i\mu}\omega(\sigma_i z_{\mu})=\frac{1}{2N\sqrt{N\beta}}\sum_{i,\mu}\mathbb{E}\partial_{\xi_{i\mu}}\omega(\sigma_i z_{\mu})\\ &= \frac{1}{2N^2}\sum_{i,\mu}\mathbb{E}\Big(\omega(\sigma_i z_{\mu}\sigma_i z_{\mu})-\omega(\sigma_i z_{\mu})\omega(\sigma_i z_{\mu})\Big)\end{aligned}$$

which in the thermodynamic limit becomes

$$u(\alpha,\beta)=\frac{\alpha}{2}\Big(\langle p_{11}\rangle-\langle p_{12}q_{12}\rangle\Big). \tag{3.5}$$

Now it is enough to show that

$$(1-\beta)\langle p_{11}\rangle+\beta\langle p_{12}q_{12}\rangle=1, \tag{3.6}$$

and the proof is complete.
This can be achieved as follows

$$\begin{aligned}\langle p_{11}\rangle &\equiv \mathbb{E}\omega(\frac{1}{k}\sum_{\mu}z_{\mu}^2)=\frac{1}{k}\sum_{\mu}\tilde{Z}^{-1}\sum_{\sigma}\int\prod_{\mu}dz_{\mu}e^{-z_{\mu}^2/2}z_{\mu}^2e^{\sqrt{\frac{\beta}{N}}\sum_{i,\mu}\xi_{i\mu}\sigma_i z_{\mu}} \qquad (3.7)\\ &= \frac{1}{k}\tilde{Z}^{-1}\sum_{\mu}\sum_{\sigma}\int\prod_{\mu}dz_{\mu}(-\partial_{z_{\mu}}e^{-z_{\mu}^2/2})z_{\mu}e^{\sqrt{\frac{\beta}{N}}\sum_{i,\mu}\xi_{i\mu}\sigma_i z_{\mu}}\\ &= \frac{1}{p}\sum_{\mu}\Big(1+\sqrt{\frac{\beta}{N}}\sum_{i}\omega(\xi_{i\mu}\sigma_i z_{\mu})\Big)\\ &= \frac{1}{k}\sum_{\mu}\Big(1+\frac{\beta}{N}\sum_{i}\big(\omega(z_{\mu}^2)-\omega^2(z_{\mu}\sigma_i)\big)\Big)\\ &= 1+\beta\langle p_{11}\rangle-\beta\langle q_{12}p_{12}\rangle,\end{aligned}$$

and the thesis is proven. □

As we saw in eq. (3.2), we can express the internal energy via $\langle p_{11}\rangle$. The following theorem is therefore important.

Theorem 3.2 *In the thermodynamic limit,* β *almost everywhere, the self-overlap* p_{11} *completely self-averages:*

$$\begin{aligned}\lim_{N\to\infty}\Big(\mathbb{E}\omega^2(k^{-1}\sum_\mu z_\mu^2)\Big) &= \lim_{N\to\infty}\Big(\mathbb{E}\omega(k^{-1}\sum_\mu z_\mu^2)\Big)^2 = \lim_{N\to\infty}\Big(\mathbb{E}\omega(k^{-2}\sum_{\mu,\nu} z_\mu^2 z_\nu^2)\Big), \\ i.e.\quad \langle p_{11}p_{22}\rangle &= \langle p_{11}\rangle^2 = \langle p_{11}^2\rangle. \end{aligned} \tag{3.8}$$

Proof
The proof works by direct calculations and is split in two different steps, the former linking the first two terms of eq.(3.8), the latter linking the second with the last.
By looking at the self-averaging of the internal energy,

$$\lim_{N\to\infty}\langle\big(u_{N,k}(\beta)-\langle u_{N,k}(\beta)\rangle\big)^2\rangle = 0,$$

we show that $\mathbb{E}\omega^2(p_{11}) = (\mathbb{E}\omega(p_{11}))^2$ or in terms of overlaps $\langle p_{11}\rangle^2 = \langle p_{11}p_{22}\rangle$:
Squaring both the sides of eq.(3.2) we get

$$\lim_{N\to\infty}(\mathbb{E}\omega(u(N,k)(\beta)))^2 = \frac{\alpha^2}{4\beta^2}\Big(\langle p_{11}\rangle^2 - 2\langle p_{11}\rangle + 1\Big). \tag{3.9}$$

Now we must evaluate $\mathbb{E}\omega^2(p_{11})$:

$$\begin{aligned}\mathbb{E}\omega^2(p_{11}) &= \frac{k}{4\beta N^2}\mathbb{E}\sum_{\mu\nu}\Big(\omega(z_\mu^2)-1\Big)\Big(\omega(z_\nu^2)-1\Big) \\ &= \frac{1}{4\beta N^2}\mathbb{E}\sum_\mu\Big(\omega^2(z_\mu^2)-2\omega(z_\mu^2)+1\Big) \\ &= \frac{\alpha^2}{4\beta}\Big(\langle p_{11}p_{22}\rangle - 2\langle p_{11}\rangle + 1\Big). \end{aligned} \tag{3.10}$$

Subtracting eq. (3.10) to eq.(3.9) we obtained the first part of eq. (3.8).
To obtain the missing relation, i.e. $\langle p_{12}^2\rangle = \langle p_{11}p_{22}\rangle$, we must work out the β-derivative of the internal energy. It will involve a polynomial in the overlaps multiplied by a factor k.

By avoiding its k-divergency (as we are in the high storage memory case when $N \to \infty$ also $k \to \infty$, linearly with N) we obtain the other relation.

$$
\begin{aligned}
\partial_\beta \langle p_{11} \rangle &= \frac{d}{d\beta} \mathbb{E} \frac{\sum_\sigma \int d\mu(z_\mu)(k^{-1}\sum_\mu z_\mu^2)\exp(\sqrt{\frac{\beta}{N}}\sum_{i\mu}\xi_{i\mu}\sigma_i z_\mu)}{\sum_\sigma \int d\mu(z_\mu)\exp(\sqrt{\frac{\beta}{N}}\sum_{i\mu}\xi_{i\mu}\sigma_i z_\mu)} \qquad (3.11)\\
&= \frac{1}{2k\sqrt{N\beta}} \sum_{\mu,\nu,i} \mathbb{E}\Big(\omega(z_\mu^2 \xi_{i\nu}\sigma_i z_\nu) - \omega(z_\mu^2)\omega(\xi_{i\nu}\sigma_i z_\nu) \Big) \\
&= \frac{1}{2k\sqrt{\beta N}} \sum_{\mu,\nu,i} \mathbb{E}\partial_{\xi_{i\nu}} \Big(\omega(z_\mu^2 \xi_{i\nu}\sigma_i z_\nu) - \omega(z_\mu^2)\omega(\xi_{i\nu}\sigma_i z_\nu) \Big) \\
&= \frac{1}{2k\sqrt{\beta N}} \sum_{\mu,\nu,i} \mathbb{E}\Big(\sqrt{\frac{\beta}{N}} \big(\omega(z_\mu^2 z_\nu^2) - \omega(z_\mu^2 \sigma_i z_\nu)\omega(\sigma_i z_\nu) \big) - \partial_{\xi_{i\nu}} \big(\omega(z_\mu^2)\omega(\sigma_i z_\nu) \big) \Big),
\end{aligned}
$$

where

$$
\begin{aligned}
\partial_{\xi_{i\mu}} \big(\omega(z_\mu^2)\omega(\sigma_i z_\nu) \big) &= \sqrt{\frac{\beta}{N}}\omega(z_\mu^2 \sigma_i z_\nu)\omega(\sigma_i z_\nu) - \sqrt{\frac{\beta}{N}}\omega(z_\mu^2)\omega^2(\sigma_i z_\nu) \\
&+ \sqrt{\frac{\beta}{N}}\omega(z_\mu^2)\omega(z_\nu^2) - \sqrt{\frac{\beta}{N}}\omega(z_\mu^2)\omega^2(\sigma_i z_\nu). \qquad (3.12)
\end{aligned}
$$

Pasting eq. (3.12) into (3.11) we get

$$
\begin{aligned}
\partial_\beta \langle p_{11} \rangle = \frac{1}{2kN} \sum_{\mu,\nu,i} \mathbb{E} \quad \Big(\quad & \omega(z_\mu^2 z_\nu^2) - \omega(z_\mu^2 \sigma_i z_\nu)\omega(\sigma_i z_\nu) - \omega(z_\mu^2 \sigma_i z_\nu)\omega(\sigma_i z_\nu) \\
+ \quad & \omega(z_\mu^2)\omega^2(\sigma_i z_\nu) + \omega(z_\mu^2)\omega(z_\nu^2) + \omega(z_\mu^2)\omega^2(\sigma_i z_\nu) \Big)
\end{aligned}
$$

which gives

$$
\partial_\beta \langle p_{11} \rangle = \frac{k}{2}\Big(\langle p_{11}^2 \rangle - \langle p_{11} p_{22} \rangle \Big). \qquad (3.13)
$$

As we are in the high stored pattern limit ($k \to \infty$), in the thermodynamic limit we get $\langle p_{11}^2 \rangle = \langle p_{11} p_{22} \rangle$, and the proof is ended. $\Box$

Let us call $\bar{p}(\beta)$ the value taken by all overlaps p_{aa} in the infinite volume limit, and by η_{aa} the rescaled fluctuations

$$
\eta_{aa} = \sqrt{k}(p_{aa} - \bar{p}). \qquad (3.14)
$$

Then we have the following.

Corollary 3.3 *In the ergodic regime, defined by the line $\beta = 1/(1+\sqrt{\alpha})$, where the intensive free energy is given by $A(\alpha,\beta) = \ln 2 - \frac{1}{2}\alpha \ln(1-\beta)$ [3][9], the value of the* *overlap $\bar{p}$ and the k-rescaled fluctuations have the following behavior*

$$\bar{p}(\beta) = \frac{1}{1-\beta}, \tag{3.15}$$

$$\langle \eta_{11}^2 \rangle = \frac{2}{(1-\beta)^2}. \tag{3.16}$$

Proof
From the relation $A(\alpha,\beta) = \ln 2 - (\alpha/2)\ln(1-\beta)$ we get

$$\frac{\partial A(\alpha,\beta)}{\partial \beta} = \frac{\alpha}{2}\frac{1}{1-\beta} \equiv \frac{\alpha}{2\beta}(\bar{p}-1), \tag{3.17}$$

from which immediately we get eq.(3.15).
Then we write

$$\frac{\partial \bar{p}(\beta)}{\partial \beta} = \frac{1}{(1-\beta)^2} \equiv \frac{k}{2\beta}(\langle p_{11}^2\rangle - \langle p_{11}p_{22}\rangle) - \frac{1}{\beta}\bar{p}(\beta) \tag{3.18}$$

by which immediately we get

$$\frac{1}{\beta(1-\beta)^2} = \frac{1}{2\beta}\Big(\langle \eta_{11}^2\rangle - \langle \eta_{11}\eta_{22}\rangle\Big). \tag{3.19}$$

Now, noticing that, at least in the ergodic region, in the thermodynamic limit $\langle \eta_{11}\eta_{22}\rangle \to 0$, we get the result. $\square$

Note that this corollary automatically implies $\langle q_{12}p_{12}\rangle = 0$ in the ergodic regime, as it should be.

4 Constraints

Now we turn to the constraints: Starting with the linear identities we state the following

Proposition 4.1 *In the thermodynamic limit, and β almost-everywhere, the following generalization of the linear overlap constraints holds for the analogical neural network*

$$\langle q_{12}^2 p_{12}^2\rangle - 4\langle q_{12}p_{12}q_{23}p_{23}\rangle + 3\langle q_{12}p_{12}q_{34}p_{34}\rangle = 0. \tag{4.1}$$

Proof
Let us address our task by looking at the β streaming of the internal energy density, once expressed via $\langle q_{12}p_{12}\rangle$:

$$\partial_\beta\langle q_{12}p_{12}\rangle = \frac{1}{Nk}\sum_{\mu,i}\mathbb{E}\partial_\beta\omega^2(z_\mu\sigma_i) = \frac{1}{Nk}\sum_{\mu,i}\mathbb{E}2\omega(z_\mu\sigma_i)\partial_\beta\omega(z_\mu\sigma_i) \tag{4.2}$$

$$= \frac{2}{Nk}\sum_{\mu,i}\mathbb{E}\omega(z_\mu\sigma_i)\xi_{i\nu}\Big(\omega(z_\mu\sigma_i z_\nu\sigma_j) - \omega(z_\mu\sigma_i)\omega(z_\nu\sigma_j)\Big), \tag{4.3}$$

now we use Wick theorem on ξ to get

$$\begin{aligned}\partial_\beta\langle q_{12}p_{12}\rangle &= \frac{2}{N^2k^2}\sum_{\mu,\nu,i,j}\Big(\big(\omega(z_\mu\sigma_i z_\nu\sigma_j) - \omega(z_\mu\sigma_i)\omega(z_\nu\sigma_j)\big)\big(\omega(z_\mu\sigma_i z_\nu\sigma_j) - \\ &+ \omega(z_\mu\sigma_i)\omega(z_\nu\sigma_j)\big) + \omega(z_\mu\sigma_i)\{\omega(z_\mu\sigma_i\sigma_j z_\nu z_\nu\sigma_j) - \omega(z_\mu\sigma_i z_\nu\sigma_j)\omega(z_\nu\sigma_j) \\ &- \omega(z_\mu\sigma_i)\omega(z_\mu\sigma_i z_\nu\sigma_j)\omega(z_\nu\sigma_j) + \omega(z_\mu\sigma_i)\omega(z_\nu\sigma_j)\omega(z_\nu\sigma_j)\omega(z_\mu\sigma_i) \\ &- \omega(z_\mu\sigma_i)\omega(z_\mu\sigma_i)\omega(z_\nu\sigma_j z_\nu\sigma_j) + \omega(z_\mu\sigma_i)\omega(z_\nu\sigma_j)\omega(z_\nu\sigma_j)\omega(z_\mu\sigma_i)\}\Big).\end{aligned}$$

Introducing the overlaps we have

$$\begin{aligned}\partial_\beta\langle q_{12}p_{12}\rangle &= k\Big(\langle p_{12}^2q_{12}^2\rangle - \langle p_{12}q_{12}p_{13}q_{13}\rangle \\ &- \langle p_{12}q_{12}p_{13}q_{13}\rangle + \langle p_{12}q_{12}p_{34}q_{34}\rangle + \langle\bar{p}q_{12}p_{12} - \langle p_{12}q_{12}p_{13}q_{13}\rangle \\ &- \langle p_{12}q_{12}p_{13}q_{13}\rangle + \langle p_{12}q_{12}p_{34}q_{34}\rangle - \langle\bar{p}q_{12}p_{12}\rangle + \langle p_{12}q_{12}p_{34}q_{34}\rangle.\end{aligned} \tag{4.4}$$

The several cancelations leave the following remaining terms

$$\partial_\beta\langle q_{12}p_{12}\rangle = k\Big(\langle q_{12}^2p_{12}^2\rangle - 4\langle q_{12}p_{12}q_{23}p_{23}\rangle + 3\langle q_{12}p_{12}q_{34}p_{34}\rangle\Big) \tag{4.5}$$

and, again in the thermodynamic limit, in the high storage case, the thesis is proved. □

Theorem 4.2 *In the thermodynamic limit, for almost all values of β, the following generalization of the quadratic Ghirlanda-Guerra relations holds for the analogical neural network*

$$\langle q_{12}p_{12}q_{23}p_{23}\rangle = \frac{1}{2}\langle q_{12}^2p_{12}^2\rangle + \frac{1}{2}\langle q_{12}p_{12}\rangle^2, \tag{4.6}$$

$$\langle q_{12}p_{12}q_{34}p_{34}\rangle = \frac{1}{3}\langle q_{12}^2p_{12}^2\rangle + \frac{2}{3}\langle q_{12}p_{12}\rangle^2. \tag{4.7}$$

Proof
Starting from

$$\mathbb{E}(u_N^2(\beta)) = \frac{1}{4\beta N^2 N}\sum_{\mu,i}\sum_{\nu,j}\xi_{i\mu}\xi_{j\nu}\omega(\sigma_i z_\mu)\omega(\sigma_j z_\nu),$$

with a calculation perfectly analogous of the one performed in the proof of Proposition 4.1 we obtain the following expression

$$\lim_{N\to\infty}\mathbb{E}(u_N^2(\beta)) = \frac{\alpha^2}{4}\Big(\langle(\bar{p}-q_{12}p_{12})^2\rangle + 6\langle q_{12}p_{12}q_{34}p_{34}\rangle - 6\langle q_{12}p_{12}q_{23}p_{23}\rangle\Big), \tag{4.8}$$

which must be compared with the square of the r.h.s. of eq.(3.5) that is equal to

$$\mathbb{E}(u_N^2(\beta)) = \frac{\alpha^2}{4}\Big(\bar{p}^2 - 2\bar{p}\langle q_{12}p_{12}\rangle + \langle q_{12}^2 p_{12}^2\rangle\Big). \tag{4.9}$$

As a consequence, subtracting eq.(4.9) to eq.(4.8) and taking into account also eq. (4.1) (that we rewrite for simplicity) we get the linear system

$$0 = \langle q_{12}^2 p_{12}^2\rangle + 6\langle q_{12}p_{12}q_{34}p_{34}\rangle - 6\langle q_{12}p_{12}q_{23}p_{23}\rangle - \langle q_{12}p_{12}\rangle^2 \tag{4.10}$$

$$0 = \langle q_{12}^2 p_{12}^2\rangle - 4\langle q_{12}p_{12}q_{23}p_{23}\rangle + 3\langle q_{12}p_{12}q_{34}p_{34}\rangle \tag{4.11}$$

whose solutions gives exactly the expressions reported in Theorem 4.2. □

Theorem 4.3 *For the analogical neural network, a new class of identities, which involve explicit dependence on the noise of the network, holds in the thermodynamic limit; examples of which are*

$$1 = (1-\beta)\bar{p} + \beta\langle q_{12}p_{12}\rangle, \tag{4.12}$$

$$0 = (1+\beta\bar{p}-\bar{p})\langle q_{12}^2\rangle + -2\beta\langle q_{12}p_{12}q_{13}^2\rangle + \beta\langle q_{13}^2 p_{12}\rangle. \tag{4.13}$$

Proof
The proof of eq.(4.12) is simply the explicit calculation of the quantity $\langle p_{11}\rangle = \mathbb{E}\omega(k^{-1}\sum_\mu z_\mu^2)$, as established in the derivation of eq.(3.7) and that in the thermodynamic limit, remembering Theorem 3.2, $\langle p_{11}\rangle = \bar{p}$.
The proof of eq.(4.13) works exactly on the line of the proof of eq.(4.12) by simply working out explicitly the term $\mathbb{E}\omega(k^{-2}\sum_{\mu,\nu} z_\mu^2 z_\nu^2)$ (and so on for higher order relations). □

5 Summary

In this paper we analyzed the properties of the internal energy of an analogical neural network:
At first we mapped the problem into a bipartite spin glass and evaluate its internal energy by introducing two order parameters able to fulfil our task: a standard spin glass overlap comparing neural configurations (first party) on different replicas and an overlap among the Gaussian spins (second party) on different replicas.
We showed that the internal energy density can be expressed via the spin self-overlap and proved its full self-average.
Furthermore, for these overlaps, we investigate the presence of constraints, founding both the linear and the quadratic identities, as expected, being the analogical neural network a well known complex system.
These constraints appear with a clear symmetric structure with respect to the two overlaps, which interact together in both the families. Ultimately, this symmetry reflects the bipartite nature of the neural networks by which interaction among the k Gaussian spins and the N dichotomic variables is encoded in the memory patterns ξ.
Future works will be developed toward the analysis of the critical line for the ergodicity, extending previous results up to that line [9] and to the study of the still rather obscure (at the mathematical level) retrieval of the replica symmetric regime.

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